On the Ubiquity of the Sine Wave

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Linear wave propagation is usually analyzed in terms of harmonic waves, characterized by a time factor $\exp(-i\omega t)$. Textbooks on wave physics always stress the importance of such harmonic waves, but the reason for their importance is not stated in any elementary text of which we are aware. It is obvious enough that harmonic analysis is often mathematically convenient, but any other set of expansion functions may be used with equal validity. The harmonic wave takes on a special significance when dispersion is present; dispersion can be detected by the change in shape of a pulse as it propagates (by "shape" we mean the time dependence of the wave function at a fixed position, not the spatial dependence at an instant—this would vary in an inhomogeneous nondispersive medium). The shape of the pulse used to detect dispersion may be quite arbitrary, except that it must not be sinusoidal, because *the sinusoid is the only wave that travels without change of shape.*

What is the basic physical reason for this remarkable property? We feel that the following elementary explanation could usefully be presented to undergraduates. Consider a dispersive medium (e.g., a water surface or a dielectric) which is in equilibrium when no wave is passing through it. The advancing wave acts on the matter in the medium and the resulting forced motion determines the subsequent propagation (the motion may either constitute the wave itself—mechanical waves, or else provide a source of "secondary wavelets"—electromagnetic waves). The wave will not disperse if the time dependence $x(t)$ of the forced motion is in some sense the same as the time dependence $f(t)$ of the exciting wave. To find the pulse shape $f(t)$ for which this holds, we realize that the medium in equilibrium is in a configuration of minimum potential energy, so that any small motion $x(t)$ must occur in a quadratic potential well. Thus the matter satisfies the equation for a *forced harmonic oscillator*, namely

$$d^2x/dt^2+\omega_0^2x=f(t), \qquad (1)$$

where ω_0 is a resonant frequency (more generally, x is one of the "Lagrangian normal coordinates" and ω_0 is the frequency of the corresponding normal mode). For propagation without dispersion, the simplest assumption is

$$x(t)=Af(t), \qquad (2)$$

where A is a constant. Inserting this into Eq. (1), we find that the only solutions are

$$f(t)=B\exp(\pm i\omega t), \qquad A=(\omega_0^2-\omega^2)^{-1}, \qquad (3)$$

where B and ω are arbitrary. But these "undispersed waves" are harmonic, which is what we wanted to show. [More generally, the assumed relation between $x(t)$ and $f(t)$ could involve a time delay, or $x(t)$ might keep in step with a derivative of $f(t)$, so that instead of Eq. (2) we would have

$$x(t)=A(d^n/dt^n)f(t-\tau).$$

A slightly less elementary mathematical argument shows that in this case also $f(t)$ must be harmonic, although A is no longer given by Eq. (3).]

The fundamental physical reason for the importance of sinusoidal waves therefore lies in the behavior of matter executing small motions near equilibrium. (It is also worth pointing out that nonsinusoidal linear waves in a dispersive medium usually satisfy not a simple "wave equation," but a complicated set of coupled equations; surface waves on water provide a good example.) We do not claim that our argument is original; it was certainly known to Lord Rayleigh, from whose *Theory of Sound* (Macmillan, London, 1877), Vol. 1, p. 17, we quote:

> ...it is a consequence of the general theory of vibration that the particular type, now suggested as corresponding to a simple tone, is the only one capable of preserving its integrity among the vicissitudes which it may have to undergo.

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