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Weak value distributions for spin 1/2

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Abstract

The simplest weak measurement is of a component of spin 1/2. For this observable, the probability distributions of the real and imaginary parts of the weak value, and their joint probability distribution, are calculated exactly for pre- and postselected states uniformly distributed over the surface of the Poincaré-Bloch sphere. The superweak probability, that the real part of the weak value lies outside the spectral range, is 1/3. This case, with just two eigenvalues, complements our previous calculation (Berry and Shukla 2010 *J. Phys. A: Math. Theor.* **43** 354024) of the universal form of the weak value probability distribution for an operator with many eigenvalues.

1. Introduction

A weak measurement [1, 2] of a quantum observable $\hat{A}$, involving a preselected state $|\psi_0\rangle$ and a postselected state $|\psi_1\rangle$ leads to a weak value

$$A_{\text{weak}} = \frac{\langle\psi_1|\,\hat{A}\,|\psi_0\rangle}{\langle\psi_1\mid\psi_0\rangle} = A + \mathrm{i}A'. \tag{1.1}$$

The real and imaginary parts can be interpreted, as is now well understood [1, 3, 4], in terms of the shift (A) and momentum (A') of a pointer recording the measurement. An important feature of a weak measurement is that in contrast to the more familiar measurement, given by the expectation value $\langle\psi|\,\hat{A}\,|\psi\rangle$, the real part of the weak value A can lie far outside the spectrum of $\hat{A}$: it can be superweak [5], because the denominator in (1.1) is small when the pre- and postselected states are nearly orthogonal.

Recently [5], the typicality of superweakness was estimated, by calculating, for observables with $N \gg 1$ eigenvalues, the probability distribution of A over an ensemble of pre- and postselected states, and hence the probability that A lies outside the spectrum of $\hat{A}$. The result was a surprising universality: the distribution of A is largely independent of the ensemble of the states, with scaling governed by a single number characterising the distribution of eigenvalues. Moreover, superweakness for $N \gg 1$ was revealed as a surprisingly common phenomenon, whose probability could be as large as $1 - 1\sqrt{2} = 0.293$. Numerics indicated that the universal large-N distribution was a good approximation even down to $N = 5$. The study [5] generalized the earlier result [6] on the statistics of monochromatic superoscillations, that is waves in two dimensions that oscillate faster than the wavenumber of the consituent plane waves: the superoscillation probability was 1/3 (later generalised [7] to waves in arbitrary dimension).

Our purpose here is to complement these earlier studies by calculating the weak value distribution for the simplest case, i.e. $N = 2$. Without loss of generality, we can choose the observable for this 2-state system proportional to the z component of spin, namely

$$\hat{A} = \frac{2}{\hbar}\hat{S}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.2}$$

with eigenvalues +1 and −1. The states are represented by their directions on the Poincaré-Bloch sphere; in polar coordinates,

$$|\psi_0\rangle = \begin{pmatrix} \exp\left(-\frac{1}{2}\mathrm{i}\phi_0\right)\cos\frac{1}{2}\theta_0 \\ \exp\left(\frac{1}{2}\mathrm{i}\phi_0\right)\sin\frac{1}{2}\theta_0 \end{pmatrix}, \qquad |\psi_1\rangle = \begin{pmatrix} \exp\left(-\frac{1}{2}\mathrm{i}\phi_1\right)\cos\frac{1}{2}\theta_1 \\ \exp\left(\frac{1}{2}\mathrm{i}\phi_1\right)\sin\frac{1}{2}\theta_1 \end{pmatrix}. \tag{1.3}$$

The natural ensemble for these pre- and postselected states consists of independent distributions of these two directions on the sphere, uniform over the area of the sphere, that is with measure $\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$.

The weak value is calculated in section 2 as a function of the directions of the pre- and postselected states. The joint probability distribution $P_{\mathrm{joint}}\left(A, A'\right)$ of the real and imaginary parts of the weak value is calculated in section 3, and from this, in section 4, are calculated the separate distributions $P_{\mathrm{Re}}(A)$ and $P_{\mathrm{Im}}\left(A'\right)$. Superweak values correspond to $|A| > 1$, and from $P_{\mathrm{Re}}(A)$ we show that the probability for A to be found in this interval is 1/3. In a celebrated paper [8], it was shown that in a weak measurement the spin component of a spin 1/2 particle could exceed $100\hbar$; our formula for $P_{\mathrm{Re}}(A)$ enables the probability of this extraordinary occurrence to be calculated as 1/120 000.

2. Calculation of weak values

A straightforward calculation from (1.1)–(1.3) gives the weak values in terms of the directions of the pre-and postselected states as

$$\begin{aligned} A &= \frac{\cos\theta_0 + \cos\theta_1}{1 + \cos\theta_0\cos\theta_1 + \sin\theta_0\sin\theta_1\cos\phi}, \\ A' &= \frac{\sin\theta_0\sin\theta_1\sin\phi}{1 + \cos\theta_0\cos\theta_1 + \sin\theta_0\sin\theta_1\cos\phi}, \end{aligned} \tag{2.1}$$

where $\phi = \phi_1 - \phi_0$ (reflecting the azimuthal symmetry with respect to the observable). The large superweak values are associated with the singularities at $\theta_1 = \pi - \theta_0$, $\phi = \pi$ where the denominators vanish, corresponding to orthogonality of the pre- and postselected states.

Figure 1 illustrates the geometry of A and A' in the natural space

$$c_0 = \cos\theta_0, \qquad c_1 = \cos\theta_1, \qquad \phi \tag{2.2}$$

in whose volume the distribution of states is uniform.

For a technical reason that will become clear, it is convenient to immediately transform from polar coordinates θ, ϕ on the sphere to stereographic coordinates ρ, ϕ on the plane; the radial coordinate is

$$\rho = \tan\tfrac{1}{2}\theta. \tag{2.3}$$

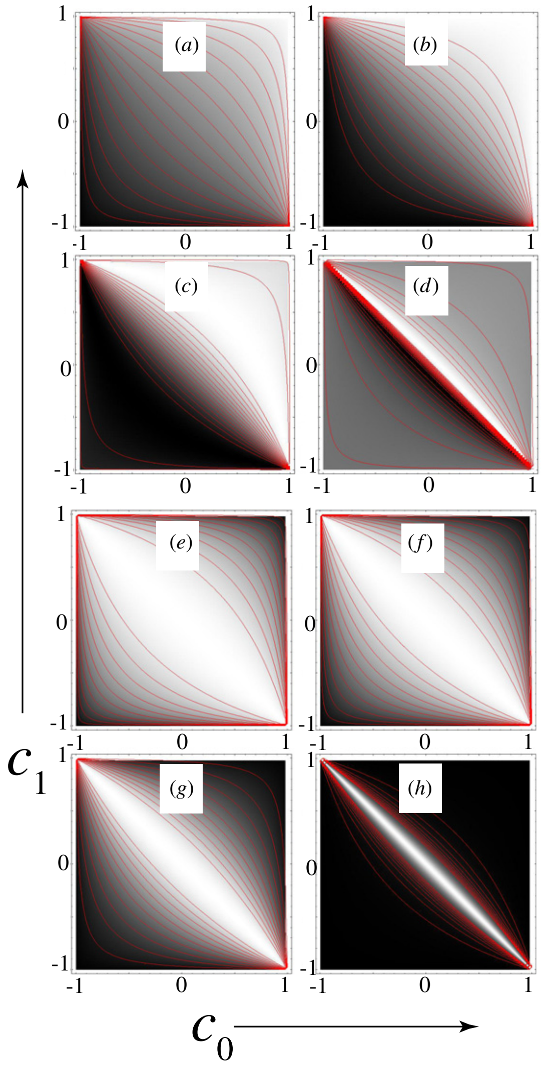

Figure 1. Real part A (*a*)–(*d*) and imaginary part A' (*e*)–(*h*) of weak value for spin $1/2$ measurements as function of $c_0 = \cos\theta_0$ and $c_1 = \cos\theta_1$, for (*a*), (*e*): $\phi = \pi/8$, (*b*), (*f*): $\phi = \pi/2$, (*c*), (*g*): $\phi = 3\pi/4$, (*d*), (*f*): $\phi = 31\pi/32$, as density-shaded contour plots (larger values lighter). The singularities at $c_1 = -c_0$, $\phi = \pi$ correspond to orthogonality of the pre- and postselected states.

(This figure is in colour only in the electronic version)

Then an elementary calculation from (1.1) gives the weak value for each pair of pre- and postselected states as

$$
\begin{aligned}
A &= \frac{1-\rho_0^2\rho_1^2}{1+\rho_0^2\rho_1^2+2\rho_0\rho_1\cos\phi} \equiv \frac{Y}{X} \\
A' &= \frac{2\rho_0\rho_1\sin\phi}{1+\rho_0^2\rho_1^2+2\rho_0\rho_1\cos\phi} \equiv \frac{Z}{X}.
\end{aligned}
\tag{2.4}
$$

3. Joint probability distribution of real and imaginary weak values

From the symmetry of the observable $\hat{A}$ in (1.1), of the weak value (2.1) under exchange of $|\psi_0\rangle$ and $|\psi_1\rangle$, and the uniform distributions of $|\psi_0\rangle$ and $|\psi_1\rangle$ on the sphere, it follows that the joint distribution $P_{\text{joint}}(A, A')$ depends only on the absolute values $|A|$ and $|A'|$, so we only need perform the calculations for $A \geqslant 0$ and $A' \geqslant 0$. This will be assumed in what follows, though we will not always indicate the absolute values.

The desired probability distributions are

$$\begin{aligned} &P_{\text{Re}}(A) = \int_{-\infty}^{\infty} \mathrm{d}A' P_{\text{joint}}(A, A'), \qquad P_{\text{Im}}(A') = \int_{-\infty}^{\infty} \mathrm{d}A P_{\text{joint}}(A, A') \\ &P_{\text{joint}}(A, A') = \left\langle \delta\left(A - \frac{Y}{X}\right) \delta\left(A' - \frac{Z}{X}\right) \right\rangle = \frac{Y^2}{A^2} \langle \delta(AX - Y)\delta(A'X - Z) \rangle, \end{aligned} \tag{3.1}$$

where the angle brackets represent ensemble averages. Now we note that the radial dependencies in the weak values (2.4) only involve the combination $\rho_0\rho_1$. This leads to a simplification: for any function F, the average, incorporating uniform distribution on the sphere of states, is

$$\begin{aligned} \langle F(\rho_0\rho_1, \phi) \rangle &= \frac{1}{8\pi} \int_0^{\pi} \mathrm{d}\theta_0 \sin\theta_0 \int_0^{\pi} \mathrm{d}\theta_1 \sin\theta_1 \int_0^{2\pi} \mathrm{d}\phi F(\rho_0\rho_1, \phi) \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{\mathrm{d}\rho_0 \rho_0}{\left(1 + \rho_0^2\right)^2} \int_0^{\infty} \frac{\mathrm{d}\rho_1 \rho_1}{\left(1 + \rho_1^2\right)^2} \int_0^{2\pi} \mathrm{d}\phi F(\rho_0\rho_1, \phi) \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{\mathrm{d}\rho_0 \rho_0^3}{\left(1 + \rho_0^2\right)^2} \int_0^{\infty} \frac{\mathrm{d}v v}{\left(\rho_0^2 + v^2\right)^2} \int_0^{2\pi} \mathrm{d}\phi F(v, \phi) \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{\mathrm{d}v v}{(1 - v^2)^2} \left(\frac{1 + v^2}{1 - v^2} \log\frac{1}{v} - 1 \right) \int_0^{2\pi} \mathrm{d}\phi F(v, \phi). \end{aligned} \tag{3.2}$$

The third equality follows after substituting $\rho_0\rho_1 = v$, and the fourth from evaluating the integral over ρ_0.

To calculate $P_{\text{joint}}\left(A, A'\right)$, the two integrals will be eliminated by the two δ-functions in (3.1). For the ϕ integration, after using $\int \mathrm{d}x \delta\left(f\left(x\right)\right) \delta\left(g\left(x\right)\right) = \sum_i \left|f\left(x_i\right)\right|^{-1} \delta\left(g_i\left(x\right)\right)$, where x_i are the zeros of $f(x)$ in the integration range, we get

$$\begin{aligned} \int_0^{2\pi} \mathrm{d}\phi F(v, \phi) &= \frac{(1 - v^2)}{A^2} \int_0^{2\pi} \mathrm{d}\phi \delta((A + 1)v^2 + 2Av\cos\phi + A - 1) \\ &\quad \times \delta(A'(v^2 + 2v\cos\phi + 1) - 2v\sin\phi) \\ &= \frac{(1 - v^2)}{2A^3|\sin\phi_{\text{c}}|} [\delta(A'(v^2 + 2v\cos\phi_{\text{c}} + 1) - 2v\sin\phi_{\text{c}}) \\ &\quad + \delta(A'(v^2 + 2v\cos\phi_{\text{c}} + 1) + 2v\sin\phi_{\text{c}})]. \end{aligned} \tag{3.3}$$

The second equality results from the δ-function containing A, and involves

$$\cos\phi_{\text{c}} = \frac{1 - A - (A + 1)v^2}{2Av}, \qquad \sin\phi_{\text{c}} = \frac{A + 1}{2Av} \sqrt{(1 - v^2)\left(v^2 - \left(\frac{A - 1}{A + 1}\right)^2\right)}, \tag{3.4}$$

in which the square root is positive and there are two terms because for each value of $\cos\phi_c$ there are two values of $\sin\phi_c$.

After noting that the v integration depends only on $v^2 = u$, the joint probability distribution becomes

$$P_{\text{joint}}(A, A') = \frac{1}{\pi A(A+1)} \int_{\left(\frac{A-1}{A+1}\right)^2}^{1} \frac{\mathrm{d}u}{1-u} \frac{\left(\frac{1}{2}(1+u)\log\frac{1}{u} - (1-u)\right)}{\sqrt{(1-u)\left(u - \left(\frac{A-1}{A+1}\right)^2\right)}}$$
$$\times\delta\left(A'(1-u) - \sqrt{(1-u)\left(u - \left(\frac{A-1}{A+1}\right)^2\right)}\right), \tag{3.5}$$

in which the restriction of the limits of the integral arise from the condition $|\sin\phi_c| \leqslant 1$. The argument of the remaining δ-function vanishes for $u = u_{c1}$ and $u = u_{c2}$, where

$$u_{c1} = 1 - \frac{4A}{(1+A^2)(1+A'^2)}, \qquad u_{c2} = 1. \tag{3.6}$$

The value u_{c2} does not contribute, because the prefactor in (3.5) vanishes for $u = 1$, leading to the final result for the joint distribution: reinstating the absolute value,

$$P_{\text{joint}}(A, A') = \frac{(1+|A|)}{2\pi A^2}\left(\frac{(1+u_{c1})}{2(1-u_{c1})}\log\frac{1}{u_{c1}} - 1\right). \tag{3.7}$$

Figure 2 shows the distribution. It is clear that A and A' are strongly correlated. At the eigenvalues $A = \pm 1$, $A' = 0$, P_{joint} has a logarithmic singularity, whose form is

$$P_{\text{joint}}(1+\varepsilon, 0) \approx \frac{1}{\pi}\log\left(\frac{2}{\mathrm{e}|\varepsilon|}\right), \qquad P_{\text{joint}}(1, \varepsilon) \approx \frac{1}{\pi}\log\left(\frac{1}{\mathrm{e}\varepsilon}\right), \qquad \varepsilon \ll 1. \tag{3.8}$$

Away fom the eigenvalues, P_{joint} decays rapidly.

4. Real and imaginary weak value distributions

For the real part of the weak value, (3.1), (3.6) and (3.7) give

$$P_{\text{Re}}(A) = 2\int_0^\infty \mathrm{d}A' P_{\text{joint}}(A, A') = \frac{1}{3}\left(\Theta(1-|A|) + \frac{1}{|A^3|}\Theta(|A|-1)\right), \tag{4.1}$$

in which Θ denotes the unit step. (Actually, we found it simpler to obtain this result by integrating over A' first and evaluating the u integral by a contour deformation around a branch cut, thereby eliminating the logarithm in (3.2).)

The distribution $P_{\text{Re}}(A)$ (figure 3) is uniform for $|A| < 1$, i.e. between the eigenvalues, and decays in the superweak region outside. The power-law decay is similar to those previously found [5–7] for statistics of quotients of random variables (here Y/X in (2.4)). The probability of finding a superweak value is

$$P_{\text{superweak}} = 2\int_1^\infty \mathrm{d}A\, P_{\text{Re}}(A) = \frac{1}{3}. \tag{4.2}$$

In [8], it was envisaged that a weak measurement of a spin component could yield a value exceeding $100\hbar$. The probability that this would occur with a random choice of pre- and postselected states can now be calculated:

$$P_{S_z > 100\hbar} = \frac{2}{3}\int_{200}^\infty \frac{\mathrm{d}A}{A^3} = \frac{1}{120\,000}. \tag{4.3}$$

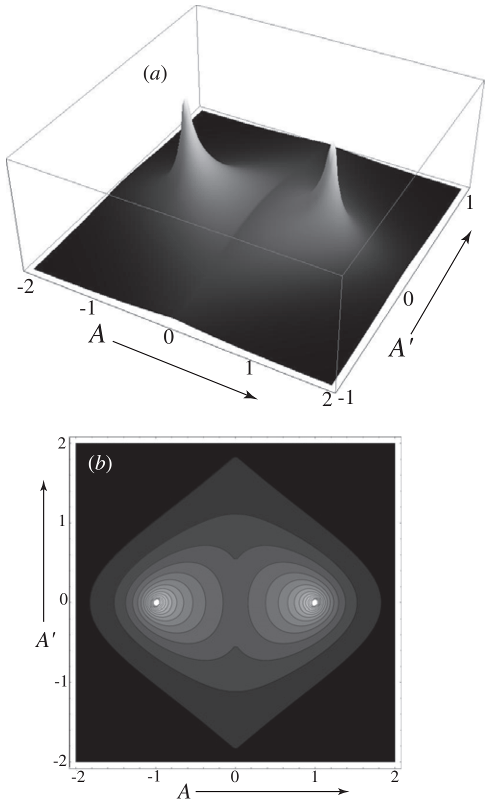

Figure 2. Joint probability distribution $P_{\rm joint}(A,\ A')$ of real and imaginary parts of $A_{\rm weak}$ (equation (3.7)): (*a*) 3D plot, as a surface; (*b*) contour plot.

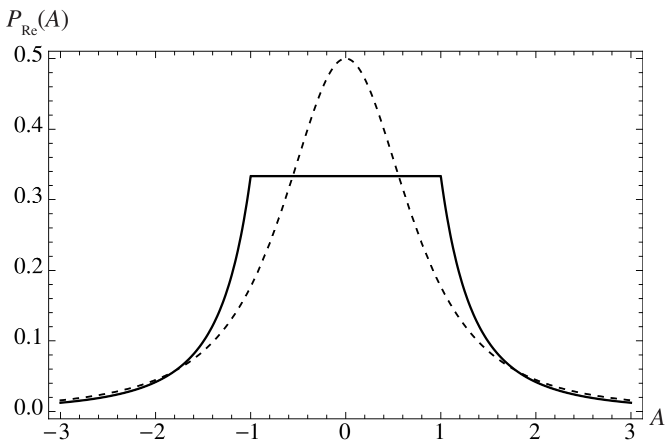

Figure 3. Probability distribution $P_{\rm Re}(A)$ for $A = {\rm Re}A_{\rm weak}$. Full curve: spin 1/2 (equation (4.1)); dashed curve: universal result for many states, from [5].

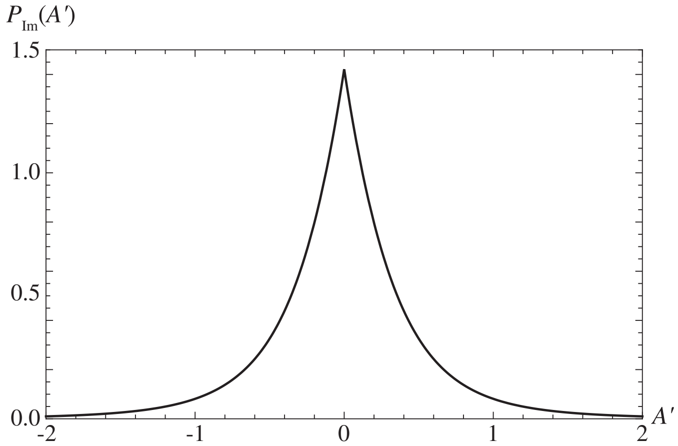

Figure 4. Probability distribution $P_{\mathrm{Im}}(A')$ for $A' = \mathrm{Im}A_{\mathrm{weak}}$ (equation (4.4)).

Similarly, the distribution of the imaginary part is

$$P_{\mathrm{Im}}(A') = \frac{1}{\pi(1+A'^2)} \times \left[2 - 3A'^2 - 6|A'|(1+A'^2)\tan^{-1}\frac{1}{|A'|} + (1+4A'^2+3A'^4)\left(\tan^{-1}\frac{1}{|A'|}\right)^2\right]. \tag{4.4}$$

As illustrated in figure 4 (and not obvious from the formula), this is a rapidly decaying function, with asymptotic behaviour

$$P_{\mathrm{Im}}(A') \approx \begin{cases} \dfrac{\pi}{4} + \dfrac{2}{\pi} - 4|A' & (|A'| \ll 1) \\ \dfrac{2}{3\pi|A'|^4} & (|A'| \gg 1). \end{cases} \tag{4.5}$$

5. Concluding remarks

The weak value probability distributions (4.1) and (4.2) for this simplest case of just $N = 2$ eigenvalues differ in two respects from the previously found distribution [5] that emerges as N increases and that is universal (as a consequence of the central limit theorem for the eigenvalue sums implicit in (1.1)). The first difference concerns $P_{\mathrm{Re}}(A)$. The universal distribution $P_{\mathrm{Re}}(A)$ is a smooth function, in which the only indication of the extent of the spectrum of the observable $\hat{A}$ is a scaling variable quantifying the way in which the N eigenvalues are distributed within the spectral range. By contrast, for $N = 2$ there is a discontinuity of slope at the eigenvalues $A = \pm 1$.

The second difference concerns $P_{\mathrm{Im}}(A')$. For large N, this is the same as $P_{\mathrm{Re}}(A)$ [5], but for $N = 2$ the forms of $P_{\mathrm{Im}}(A')$ and $P_{\mathrm{Re}}(A)$ are very different.

Nevertheless, the distributions for $N = 2$ and for large N decay in the same way for large $|A|$: as $1/|A|^3$. Moreover, the superweak probabilities are not very different: for large N, $P_{\mathrm{superweak}}$ can be as large as $1 - 1/\sqrt{2} = 0.293\ldots$, whereas for $N = 2$, $P_{\mathrm{superweak}} = 1/3$ – intriguingly, the same as the superoscillation probability [6] for gaussian random monochromatic waves in two dimensions. These similarities are compatible with our previous observation [5] that the $N>>1$ distribution fits those computed numerically even down to $N = 5$.

Finally, we emphasize that the distribution of superweak values is very different from that of the expectation values in a conventional measurement. For the observable (1.2), the expectation value (which of course is real) is

$$A_{\mathrm{exp}} = \langle\psi|\hat{A}|\psi\rangle = \cos\theta, \tag{5.1}$$

whose probabilty distribution is

$$P_{\mathrm{exp}}(A_{\mathrm{exp}}) = \tfrac{1}{2}\int_0^\pi \mathrm{d}\theta \sin\theta\, \delta(A_{\mathrm{exp}} - \cos\theta) = \tfrac{1}{2}\Theta(1 - |A_{\mathrm{exp}}|). \tag{5.2}$$

This is restricted to the interval $|A|\leqslant 1$ and uniform within it.

Acknowledgments

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