Curvature of wave streamlines

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Abstract

Wave streamlines are integral curves of the local wavevector (phase gradient). An exact formula is derived, giving the curvature of streamlines as the component transverse to the local wavevector of the gradient of the logarithm of the local wavenumber. The formula is applied to quantum particles moving in a potential and classical light in the presence of a refractive-index gradient. Three limiting regimes are encompassed. The first is geometrical, in which the bending of streamlines arises solely from the classical force or optical index gradient. The second and third limits concern singularities in the pattern of wave streamlines, of two types: optical vortices, near which the streamlines are asymptotically circular, and phase saddles, near which the streamlines are asymptotically hyperbolic.

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1. Introduction

In classical mechanics, particle trajectories bend under the action of a transverse force, for example gravity. In geometrical optics, light rays bend when there is a transverse gradient of refractive index, as in the mirage [1–3]. In wave physics (quantum or optical), the analogous trajectories are the streamlines: lines of energy flow or current, given by integral curves of the local wavevector [4, 5]. These can bend even in free space where there are no forces or index gradients. Extreme bending occurs near singularities in the pattern of streamlines; these are wave vortices (a.k.a., phase singularities, nodes or wave dislocations), where the streamlines are circles which can be arbitrarily small [6, 7], and phase saddles, where the streamlines are hyperbolae [8]. Many calculations, in two or three dimensions (e.g. [9]) show these essentially wave features.

My purpose here is to derive a surprisingly simple exact formula for the curvature of streamlines in wave physics, in the presence of forces or index gradients, encompassing the classical/geometrical and phase singularity limits. The formula can be expressed in a way that nicely illustrates the optical-mechanical analogy.

The theory involves complex scalar wavefunctions representing monochromatic waves, depending on position \( \mathbf{r} \) and expressed in terms of modulus and phase (both real):

\[
\psi (\mathbf{r}) = \rho (\mathbf{r}) \exp (i \chi (\mathbf{r})).
\]  

(1.1)
The local wavevector, everywhere tangent to the streamlines, is the phase gradient
\[ k(r) = \nabla \chi(r) = \text{Im} \nabla \log \psi(r), \]  
with length \( k(r) = |k(r)| \). The restriction to monochromatic waves is imposed because for polychromatic waves the streamlines move and do not correspond to particle or optical trajectories. The restriction to scalar waves means that optical polarization and quantum spin effects will be neglected. These restrictions are discussed further in section 4.

2. Curvature formula

For any family of curves in the plane or in space, described by a field of unit tangent vectors \( T(r) \), the curvature vector \( C(r) \), whose magnitude is the scalar curvature of the curve passing through \( r \), is the rate of change of \( T \) along the path [10], which is perpendicular to \( T \):  
\[ C(r) = (T(r) \cdot \nabla)T(r) = \nabla_\perp T(r), \]  
in which, here and hereafter, the subscript \( \perp \) denotes the component perpendicular to \( T(r) \). For waves \( \psi(r) \), \( T(r) \) is parallel to the local wavevector \( k(r) \), so  
\[ C(r) = \left( \frac{k(r)}{k(r)}, \frac{k(r)}{k(r)} \right) = \frac{1}{k(r)^2} \left( k(r) \cdot \nabla \right) k(r). \]  
The wavevector field (1.2) has the special feature, implied by its origin as a phase gradient, that its curl is zero. Therefore, from an elementary vector identity,  
\[ (k(r) \cdot \nabla)k(r) = \frac{1}{2} \nabla(k(r) \cdot k(r)) = \frac{1}{2} \nabla |k(r)|^2. \]  
Thus we obtain the curvature formula
\[ C(r) = \frac{1}{2k(r)^2} \nabla_\perp |k(r)|^2 = \nabla_\perp \log k(r) \]
\[ = \nabla \log k(r) - \frac{1}{k(r)^2} \left( k(r) \cdot \nabla \log k(r) \right) k(r) \]
\[ = \frac{1}{k(r)^2} \left( k(r) \times \left( \nabla \log k(r) \times k(r) \right) \right). \]  

This formula applies generally, to complex scalar waves (1.1) of any type, irrespective of the equations (linear or nonlinear) that they satisfy. We will apply it to nonrelativistic quantum particles of mass \( m \) with energy \( E \), in a potential \( U(r) \), for which \( \psi \) satisfies the time-dependent Schrödinger equation, and light with vacuum wavenumber \( k_0 \) in a varying refractive index \( n(r) \), for which \( \psi \) satisfies the Helmholtz equation. Both equations are of the form
\[ \nabla^2 \psi(r) + A(r) \psi(r) = 0, \]  
with
\[ A(r) = \begin{cases} \frac{2m}{\hbar^2} (E - U(r)) & \text{(quantum)}, \\ \frac{1}{k_0^2 n^2(r)} & \text{(optical)} \end{cases} \]  
incorporating the familiar optical-mechanical analogy. In optics, the vacuum wavenumber \( k_0 \) must be distinguished from the length \( k(r) \) of the local wavevector. In particular, \( k(r) \) can greatly exceed \( k_0 \) and approaches infinity near vortices—an example of the now well-understood phenomenon of ‘superoscillation’ [11–15].

Expressed in terms of the modulus and wavevector in (1.1) and (1.2), equation (2.5) becomes the pair
\[ \frac{\nabla^2 \rho(r)}{\rho(r)} = k(r)^2 - A(r) \]
\[ \nabla \cdot k(r) = -2k(r) \cdot \nabla \log \rho(r). \]  

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Using the first of these equations, the curvature vector (2.4) can be written
\[
C(r) = \frac{1}{2k^2(r)} \nabla_\perp \left( A(r) + \frac{\nabla^2 \rho(r)}{\rho(r)} \right).
\] (2.8)

For quantum waves, this has an interesting interpretation. According to the Madelung–Bohm–de Broglie (MBdB) picture [16, 17], we can think of \( \psi \) as representing a swarm of particles whose local velocity \( v(r) = \text{momentum/mass} \) where momentum = \( h \times \text{local wavevector} \), that is
\[
v(r) = \frac{h}{m} k(r).
\] (2.9)
(The particles are moving but for monochromatic waves the velocity field is stationary so there is no time dependence.) Then it is a standard result of MBdB theory that the particle acceleration is given by the Newton-type equation
\[
\partial_t v = -\frac{\nabla(U + Q)}{m},
\] (2.10)
in which the external potential \( U(r) \) is supplemented by a quantum potential \( Q(r) \) [16, 17], depending on the wavefunction modulus:
\[
Q(r) = -\frac{\hbar^2}{2m} \frac{\nabla^2 \rho(r)}{\rho(r)}.
\] (2.11)
Thus (2.8) can be written in a form in which the classical and quantum forces are clearly separated:
\[
C(r) = -\frac{1}{mv^2(r)} \nabla_\perp (U(r) + Q(r)).
\] (2.12)
The formulas (2.4), (2.8) and (2.12) are the central results of this paper.

3. Limiting cases

Now we look at the limiting cases. In the geometrical-optics regime, for places \( r \) reached by one ray (i.e. no interference), we can write the wavefunction using the simplest eikonal or WKB approximation [18] as
\[
\psi(r) \sim \exp \left[ ik_0 \int^r n(r') \mathbf{T}(r') \cdot dr' \right],
\] (3.1)
in which \( \mathbf{T}(r) \) is the tangent to the ray at \( r \). Thus, from (1.2), the local wavevector is
\[
k(r) \approx k_0 n(r) T(r), \quad \text{i.e. } k(r) = k_0 n(r),
\] (3.2)
and the curvature (2.4) becomes
\[
C(r) \approx \nabla_\perp \log n.
\] (3.3)
This can be obtained alternatively from (2.8), in which the term \( A(r) \), involving \( k_0^2 \), dominates in the geometrical-optics limit.

Equation (3.3) is the standard geometrical-optics result [18]: the ray bends in the direction of increasing refractive index, and its curvature is the logarithmic derivative of the refractive index normal to the ray. A well-known application is to the mirage [1–3], in which light bends upwards or downwards when the refractive index of air increases above a hot road or decreases above a cold sea. The analogous formula for the bending of a classical trajectory is
\[
C(r) \approx \nabla_\perp \log(E - U(r)).
\] (3.4)
The second limiting case is close to a wave vortex line, where, in the plane perpendicular to the line, with coordinates $x, y$, the wavefunction vanishes linearly:

$$\psi(r) \approx ax + by = r(a \cos \phi + b \sin \phi),$$  \tag{3.5}

in which $a$ and $b$ are complex coefficients. The index of the vortex ($+1$ or $-1$) is the sign of $\text{Im}(a^*b)$ \cite{6}. The curvature can be calculated directly from (2.4), or alternatively from (2.8), in which it is now the ‘quantum force’ term that dominates. The calculation is easy in polar coordinates: the quantum force term is

$$\frac{1}{2} \nabla^2 \rho(r) = -\frac{\text{Im}^2(a^*b)}{\rho(r)^4} r,$$ \tag{3.6}

and from (1.2) the local wavenumber is

$$k(r)^2 = \frac{\text{Im}^2(a^*b)}{\rho(r)^4} r^2.$$ \tag{3.7}

It follows that in the curvature all dependence on $a$ and $b$ cancels, giving the result

$$C(r) \approx -\frac{r}{r^2}, \quad r = |x, y|.$$ \tag{3.8}

This reproduces the known features of the flow lines near a vortex \cite{6}: independent of the anisotropy of the phase which is encoded in $a$ and $b$, these are circles, which therefore have curvature $1/r$.

In three dimensions, $k(r)$ has a component parallel to the vortex line, so, as is well known \cite{7}, the streamlines are helices. However, close to the vortex the singular part (3.8) of the curvature dominates.

The third limiting case is close to a phase saddle. This corresponds to $\nabla \chi(r) = k(r) = 0$, where the second of equations (2.7) gives

$$\nabla \cdot k(r) = \nabla^2 \chi(r) = 0.$$ \tag{3.9}

This reproduces the known result \cite{8} that for waves satisfying (3) there are no phase extrema (because these would require $|\nabla^2 \chi| > 0$. Thus in two dimensions the local form near a phase saddle is

$$\chi(r) \approx \frac{1}{2}(ax^2 + 2bxy - ay^2),$$ \tag{3.10}

(in which $a$ and $b$ are now real constants) and the local wavevector is

$$k(r) \approx (ax + by)e_x + (bx - ay)e_y.$$ \tag{3.11}

The corresponding streamlines, everywhere parallel to $k(r)$, are the hyperbolas

$$b(y^2 - x^2) + 2axy = \text{constant}.\tag{3.12}$$

For the curvature near a saddle, (2.4) leads, leading after some elementary calculations, to the formula

$$C(r) \approx \frac{b(y^2 - x^2) + 2axy}{(a^2 + b^2)(x^2 + y^2)} ((-bx + ay)e_x + (ax + by)e_y).$$ \tag{3.13}

The corresponding scalar curvature is

$$|C(r)| = \frac{|b(y^2 - x^2) + 2axy|}{\sqrt{a^2 + b^2}(x^2 + y^2)^{3/2}},$$ \tag{3.14}

which diverges as $1/r$ at the saddle, analogous to (3.8) near a vortex. Generalization to phase saddles in three dimensions is straightforward.
4. Concluding remarks

Equation (2.4) is the general formula for the streamline curvature of waves of any type, in the plane or in space; the curvature is the transverse gradient of the logarithm of the local wavenumber. It is hard to imagine a simpler expression. For quantum and optical waves, (2.8) and (2.12) give the curvature in a form where the classical force or optical refractive-index gradient appears separately from purely wave effects associated with the modulus \( \rho(r) \).

Only monochromatic waves represented by complex scalar functions of position have been considered. For polychromatic light, or quantum waves represented by superpositions of energy states, the wavefunctions \( \psi(r, t) \) are time-dependent, and the trajectories, curves in spacetime followed by ‘particles’ envisaged as having momentum tangent to the instantaneous wavevector \( k(r, t) \) (equation (1.2)), are distinct from the streamlines. The latter are the instantaneous integral curves of \( k(r, t) \); over time, they move and cover 2-surfaces in spacetime, and their significance is moot.

In optics, the scalar wave picture of light is an approximation to the full electromagnetic vector wavefield. It might seem natural that the generalization of the streamlines considered here would be the integral curves of the Poynting vector

\[
S(r) = \text{Re}E^*(r) \times H(r).
\]  

But this is not obvious. When the electric and magnetic fields are related by Maxwell’s equations, \( S(r) \) separates naturally into a part representing the orbital current and a part representing the spin current [4, 19]. Moreover, for nonparaxial fields the separation is different when expressed in terms of \( E \) alone or \( H \) alone (though there are reasons to define the spin and orbital currents as the mean of the electric and magnetic versions (‘electric-magnetic democracy’ [4, 20])). The corresponding vectors—the full \( S \), the orbital current alone, or the separate electric or magnetic versions—are different for general polarized light fields, and their integral curves generate different trajectories, with different curvatures. It is not clear whether any one of these trajectories enjoys a privileged status, or whether different trajectories are relevant to different physical situations.

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