Physical curl forces: dipole dynamics near optical vortices

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Abstract
The force on a particle with complex electric polarizability is known to be not derivable from a potential, so its curl is non-zero. This ‘curl force’ is studied in detail for motion near an anisotropic optical vortex of arbitrary strength. Fundamental questions are raised by the fact that although the curl force requires the polarizability to have a non-zero imaginary part, reflecting absorption or scattering (‘dissipation’) in the internal dipole dynamics, the particle motion that it generates is non-dissipative (volume-preserving in the position-velocity state space).

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1. Introduction

Recently [1], we explored the classical dynamics of a particle under the action of forces \( F(\mathbf{r}) \) that are not derivable from a potential. The condition for such forces is

\[
\nabla \times F(\mathbf{r}) \neq 0,
\]

so we introduced the terminology ‘curl forces’ to denote them. The fundamental significance of curl forces lies in the fact that they are Newtonian but not Hamiltonian or Lagrangian, that is, they are non-conservative—while also being non-dissipative because they preserve volume in the ‘state space’ of position \( \mathbf{r} \) and velocity \( \mathbf{v} \). Moreover, Noether’s theorem does not apply, so the link between symmetries and conservation laws is broken.

In [1] we left open the question of whether there are physical examples of curl forces. This has been the subject of intense controversy in the engineering literature, with claims [2, 3] and counter-claims [4], and an extensive even-handed review [5]; the curl forces have also been termed ‘follower forces’, ‘non-conservative positional forces’, ‘circulatory forces’ or ‘pseudo-gyroscopic forces’. Examples that have been considered involve complicated systems such as whirling flexible shafts [6, 7], pendulums with rotational dissipation [8, 9], thrust from
flexible rockets [2], and flailing water-hoses [5], and seem always accompanied by additional forces, for example friction (in which case the stability is dramatically altered by the presence of curl forces [10–15]). What has been lacking is a clear physical example of single-particle motion under the action of a curl force and in the absence of other forces.

In fact, one such example has been known for several decades: motion of small polarizable particles in optical electromagnetic fields. This has been much studied [16–20], not only theoretically but also experimentally because curl forces often accompany the gradient forces employed in optical traps and tweezers [16]. Moreover, the non-conservative nature of these forces has been noted [16, 21–23]. Our purpose here is to explore the curl aspects in a little more detail.

In section 2 we describe the curl nature of the well-known force on an electric dipole with complex polarizability, in particular the connection with the two-form of geometric phase theory. Although the derivation has been reproduced many times and is now textbook material [24], some aspects of it are not straightforward and illustrate the more general problem of separating fast and slow dynamics [25–27], as we explain in the appendix. In section 3 we calculate the curl force in detail for arbitrarily polarized paraxial fields near a general anisotropic vortex of arbitrary order, and analyse the resulting particle motion. Some questions raised by these curl forces and their quantum counterparts are discussed in section 4.

To avoid possible confusion, we note that in the context of optical forces the term ‘curl force’ has been employed [21, 22] in a very different sense from ours: to denote a force that can be expressed as the curl of a vector, i.e. \( \nabla \cdot F = 0 \); this contrasts with (1.1), in which the curl of \( F \) is non-zero.

2. Electromagnetic curl forces

For a small particle with induced electric-dipole moment described by a complex polarizability

\[
\alpha = \alpha_1 + i\alpha_2,
\]

in a monochromatic electromagnetic field with frequency \( \omega = ck \) whose spatial dependence is represented by the complex electric vector \( \mathbf{E}(r) \), the force, time-averaged over the optical period \( 2\pi/\omega \), is known [16] to be

\[
F(r) = \frac{1}{2} \text{Re} \left[ \alpha^* \mathbf{E}^*(r) \cdot (\nabla) \mathbf{E}(r) \right].
\]

(2.2)

Here the meaning of \( \cdot (\nabla) \), written in terms of components \( j \), is

\[
A_j(r) \cdot (\nabla) B_j(r) = \sum_j A_j(r) \nabla B_j(r).
\]

(2.3)

For familiar derivations, see for example [19, 24], also the appendix where some subtle aspects are discussed. The oscillating induced dipole moment can be classical or quantum, and the absorption or scattering associated with \( \alpha_2 \) could represent (for example) the loss associated with radiation from the dipole. Physically important extensions have been studied, for example dipoles that are magnetic as well as electric [28, 29], forces from evanescent waves [30], and larger particles where Mie scattering effects give rise to oscillations [18, 23, 29, 31] beyond the point dipole approximation. But for present purposes the simplest situation represented by (2.2) suffices.

After separating the real and imaginary parts of the polarizability, (2.2) becomes

\[
F(r) = \frac{1}{2} \alpha_1 \nabla (\mathbf{E}^*(r) \cdot \mathbf{E}(r)) + \frac{1}{2} \alpha_2 \text{Im} [\mathbf{E}^*(r) \cdot (\nabla) \mathbf{E}(r)].
\]

(2.4)
The first term is the gradient force responsible for the operation of optical particle manipulation (traps and tweezers), and of course has zero curl. The second term, often referred to as the ‘scattering force’ is the curl force we are concerned with. Its curl is

$$\nabla \times P(r) = \frac{1}{2} \alpha_2 \Im[\nabla E^*(r) \cdot \nabla E(r)],$$

(2.5)
in which the \(\cdot\times\) notation means

$$A(r) \cdot \times B(r) = \sum_j \nabla A_j(r) \times \nabla B_j(r).$$

(2.6)

(Some experiments [32–35], though expressed in terms of orbital angular momentum, can also be interpreted in terms of the action of the scattering/curl force.)

In a previous treatment [20], the expression occurring in (2.5) was identified as the vorticity of the Poynting vector. It has the same mathematical structure of the two-form whose flux through a closed loop generates the geometric phase for quantum state vectors transported around it [36]. The interpretation in the present context is similar: it measures the non-integrability of the curl force, that is, the path-dependence of the work done by \(F\) on the particle as it moves.

The curl force given by the second term in (2.4) is proportional to the orbital part \(P_{\text{orb}}\) of the Poynting vector \(P\) for the electromagnetic field \(E, H\):

$$P(r) = \Re[E^*(r) \times H(r)] = P_{\text{orb}}(r) + P_{\text{sp}}(r)$$

$$= \frac{1}{2\omega_0 \mu_0} (\Im[E^*(r) \cdot (\nabla \times E(r))] + \nabla \times \Im[E^*(r) \times E(r)]).$$

(2.7)

The second term \(P_{\text{sp}}(r)\) is the spin contribution [17, 20]. As has been emphasized [37], this spin vector plays no direct role in the force on the dipole in the electric-dipole model considered here (at least in lowest order in particle size: it can contribute in higher order, as can a magnetic dipole interaction [29, 38]). But of course the spin appears when the force is written in terms of \(P\), in which case (2.4) is

$$F(r) = \frac{1}{2} \alpha_1 \nabla (E^*(r) \cdot E(r)) + \frac{1}{2} \alpha_2 (\mu_0 \omega_0 P(r) - \nabla \times \Im[E^*(r) \times E(r)]).$$

(2.8)

This three-term representation is equally familiar as its equivalent (2.4) [22], and the last term—the spin force—is often referred to as the ‘curl force’ in the terminology, different from ours, mentioned at the end of the introduction. (In fact the curl force in (2.4) or (2.8) (proportional to \(\alpha_2\)) is a curl force in both senses, because as a consequence of Maxwell’s equations it is divergenceless; in (2.8) this follows from \(\nabla \cdot P(r) = 0\).

We note an interesting interpretation [37] of the terms in (2.4), in terms of the ‘weak measurement’ scheme [39, 40]. The second (curl force) term is the real part of the weak value of the orbital momentum with position \(r\) postselected, and the first (gradient) term is the imaginary part [41].

3. Paraxial fields near optical vortices

Now we consider the important case of a paraxial optical field, in a region where the local polarization is described by the complex unit vector \(e_p\) in the \(x, y\) plane, and the strength by the complex scalar wave \(\psi(x, y)\):

$$E(r) = \exp(ikz)\psi(x, y)e_p.$$  

(3.1)

Of course this transverse field is inevitably accompanied by a longitudinal (\(z\)) component (unless \(\psi(x, y)\) is a plane wave) but this is of order \(1/k\), and so paraxially negligible. The force (2.4) now becomes

$$F(r) = \frac{1}{4} \alpha_1 \nabla |\psi(r)|^2 + \frac{1}{2} \alpha_2 (\mu_0 \omega_0 |\psi^*(r)\nabla \psi(r)| + k |\psi(r)|^2 e_z).$$

(3.2)
We are interested only in the transverse force, represented by the first two terms, not the third term which is the longitudinal contribution along $e_z$. The curl force is the second term, proportional to $\text{Im} \{ \psi^* \nabla \psi \}$, which is the paraxial orbital current [17, 20].

Near any vortex, $\psi$ can be represented, up to an irrelevant constant multiplier, by

$$\psi (r) = (x + i y)^m,$$  \hspace{1cm} (3.3)

in which $m$ is a positive integer and $b$ is a complex constant representing the anisotropy of the vortex, whose strength (signed number of $2\pi$ phase rotations around it) is $m \text{sign} (\text{Re} b)$ [42]. The curl part of the force (3.2) is

$$\mathbf{F}_{\text{curl}} (r) = \frac{1}{2} \alpha_2 m (x + i y)^{2m-2} (\text{Re} b) \mathbf{e}_\theta ,$$  \hspace{1cm} (3.4)

so the force is directed azimuthally [42], even for anisotropic vortices where $b \neq \pm 1$. The curl is

$$\nabla \times \mathbf{F}_{\text{curl}} (r) = \alpha_2 m^2 \text{Re} b (x + i y)^{2m-1} \mathbf{e}_z .$$  \hspace{1cm} (3.5)

To examine the dynamics, we consider just the isotropic case $b = 1$. Then the acceleration of the dipole, after scaling time to eliminate irrelevant constants, is

$$\ddot{r} = r^{2m-1} \mathbf{e}_\theta , \quad \ddot{\theta} = r^{\sigma} , \quad r^{\sigma} \ddot{r} + 2 r \ddot{\theta} = r^{2m} .$$  \hspace{1cm} (3.6)

Here we see the non-applicability of Noether’s theorem in its purest form: the force is rotationally symmetric, but the tangential torque causes the angular momentum to change monotonically. In [1] the curl dynamics (3.6) was studied in detail, and the analytic determination of the spiral particle trajectories reduced to the study of the Emden–Fowler equation [43, 44].

The linear case, corresponding to the simplest vortex with strength $m = 1$, is integrable, with orbits parameterised by constants $A$ and $B$:

$$x(t) + i y(t) = r(t) \exp (i \theta (t)) = A \exp \left( \frac{t + i t}{\sqrt{2}} \right) + B \exp \left( - \frac{t + i t}{\sqrt{2}} \right) .$$  \hspace{1cm} (3.7)

For each term, the angular velocity is constant, and the distance from the vortex increases or decreases exponentially. For $m > 1$, the dynamics is nonlinear and seems nonintegrable. The following particular exact solutions are known, representing the dipole spiralling in from infinity for $t > 0$, with the angular velocity decreasing linearly and the distance diminishing as a power-law:

$$\theta (t) = - \frac{\sqrt{m}}{m - 1} \log t , \quad r (t) = \left( \frac{(m + 1) \sqrt{m}}{(m - 1)^2} \right)^{1/(2m-2)} \frac{1}{t^{1/(m-1)} .}$$  \hspace{1cm} (3.8)

Figure 1 illustrates these ‘pure curl force’ orbits.

In practice, the polarizability will usually not be purely imaginary, so the (3.2) will include the gradient force as well as the curl force, giving an additional radial contribution to the spiralling near a vortex [22]. Including the gradient force for the symmetric unit-strength vortex (3.3) with $b = 1, m = 1$, the dynamics is again linear and the solution, generalizing (3.7), can be written, after scaling $r$ and $t$, up to a rotation, as

$$x(t) + i y(t) = \cos \beta \exp \left( t \exp \left( \frac{1}{2} i \mu \right) \right) + \sin \beta \exp \left( i \sigma \right) \exp \left( - t \exp \left( \frac{1}{2} i \mu \right) \right) ,$$

where $\tan \mu = \alpha_2 / \alpha_1$.  \hspace{1cm} (3.9)

The gradient force is repulsive if $\alpha_1 > 0$, attractive if $\alpha_1 < 0$. If the two terms in (3.9) have different magnitudes (i.e. if $\beta \neq \pi / 4$), the particle spirals in, turns smoothly, and then spirals out at a different rate, as illustrated in figure 2.
4. Discussion

What is the origin of this optical curl force? As described explicitly in the appendix, the external forces on the separate charges in the dipole are Hamiltonian: Coulomb plus Lorentz. Therefore the curl force reflects the influence of the internal dynamics on the motion of the dipole. Since the curl force is proportional to the imaginary part $\alpha_2$ of the polarizability, it arises only if the internal dynamics is lossy, that is, if there is absorption or scattering (‘dissipation’). To maintain the oscillations of the dipole, the energy lost must be supplied by the electromagnetic field. Nevertheless, the motion of the dipole, governed by the curl force, is not dissipative. In a sense, this phenomenon is the reverse of macroscopic irreversibility, in which effective dynamics involving friction arises from microscopic dynamics that is purely Hamiltonian.
There are many different kinds of physical vortex fields, for example in classical fluids and in superconductors and superfluids. It would be interesting to explore the possibility that such vortices can generate forces of the curl type, driving the motion of particles with internal dynamics, as in the optical case considered here.

Curl forces raise interesting questions. Can the dynamics of an object with internal dynamics, modelled as a point particle, be determined by a curl force if the internal dynamics is not lossy? More generally, for a composite system where all forces, external and internal, are Hamiltonian, can the effective dynamics of part of the system, emergent in a well-defined asymptotic limiting regime, be non-Hamiltonian and also non-dissipative?

More fundamental is the question of the quantum-mechanical counterpart of dynamics under curl forces. In the absence of a Hamiltonian or Lagrangian description, the usual routes to quantization are blocked. A clue is provided by the fact that classical curl forces require the moving particle to possess internal structure. This suggests that analogous quantum effects might be associated with the particle’s internal structure. And indeed, the possibility has been envisaged [20, 45] that the optical vortex forces considered here are classical averages, which, when deconstructed quantum-mechanically, correspond to large momentum transfers to the particle. These correspond to ‘superkicks’, proportional to the local ‘weak value’ [40, 45] of the optical momentum with position postselected. A detailed study [46] in which the internal structure of the particle is modelled quantum-mechanically as a two-level atom, confirms this predicted large momentum transfer.

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Appendix. Dipole force approximation

We follow standard derivations (e.g. [19, 20, 24]) of the force (2.2), while being more explicit about the approximations involved. Referring to figure A1, we model the dipole classically, as a pair of charges $q$ and $-q$, each with mass $\frac{1}{2}m$, situated at $r_1$ and $r_2$ and with nonrelativistic velocities $v_1$ and $v_2$. The separation (vector from 1 to 2) is $d$, and we are interested in the dynamics of the centre of mass $r = \frac{1}{2}(r_1 + r_2)$. We write the full time-dependent

\[
\begin{align*}
\dot{r}_1 &= v_1, \\
\dot{r}_2 &= v_2, \\
\dot{d} &= \frac{2}{m}(qv_2 - qv_1),
\end{align*}
\]

Figure A1. Geometry of model dipole.
electromagnetic field as
\[
E_{\text{tot}}(r, t) = \text{Re}[E(r) \exp(-i\omega t)],
\]
\[
B_{\text{tot}}(r, t) = \text{Re}[B(r) \exp(-i\omega t)].
\]
(A.1)

This acts on the charges via the Coulomb and Lorentz forces, which of course are Hamiltonian. But there are also internal forces between the particles, which we represent as connected by a spring with natural frequency \(\omega_0\) and damping constant \(\Omega\). Without approximation, the total forces on the two particles are
\[
F_2 = \frac{i}{2}m\ddot{q} = q(E_{\text{tot}}(r_2, t) + v_2 \times B_{\text{tot}}(r_2, t)) - \frac{i}{2}m\omega_0^2 d - \frac{1}{4}m\omega \dot{d}
\]
\[
F_1 = \frac{i}{2}m\ddot{q} = -q(E_{\text{tot}}(r_1, t) + v_1 \times B_{\text{tot}}(r_1, t)) + \frac{i}{2}m\omega_0^2 d + \frac{1}{4}m\omega \dot{d}.
\]
(A.2)

To get the total force \(F_d\) on the dipole, acting on the total mass \(m\) at the centre of mass \(r\), we add the two forces. Of course the internal forces cancel (Newton’s third law), and we have, again exactly,
\[
F_d = m\ddot{q} = q(E_{\text{tot}}(r_2, t) - E_{\text{tot}}(r_1, t) + v_2 \times B_{\text{tot}}(r_2, t) - v_1 \times B_{\text{tot}}(r_1, t)).
\]
(A.3)

Taking the dipole limit \(d \to 0\) and introducing the dipole moment \(p \equiv qd\), we find, to lowest order in \(d\),
\[
F_d = (p \cdot \nabla)E_{\text{tot}}(r, t) + \dot{p} \times B_{\text{tot}}(r, t) + v \times (p \cdot \nabla)B_{\text{tot}}(r, t).
\]
(A.4)

The first two terms will turn out to be of the same order of magnitude, but for the nonrelativistic motion we are considering the third term is much smaller. This is because, symbolically
\[
|\dot{p}| \sim \omega |p| = ck|p| \quad \text{and} \quad |v \times (p \cdot \nabla)| \leq v|p|/c,
\]
(A.5)

so the ratio of terms does not exceed \(v/c\). Thus
\[
F_d = (p \cdot \nabla)E_{\text{tot}}(r, t) + \dot{p} \times B_{\text{tot}}(r, t).
\]
(A.6)

This involves the internal dynamics of the dipole through the time-dependent moment \(p\), for which we take the difference of the equations (A.2), to get—again exactly—
\[
\ddot{p} + \omega_0^2 p + \Omega \dot{p} = \frac{2q^2}{m} (E_{\text{tot}}(r_2, t) + E_{\text{tot}}(r_1, t) + v_2 \times B_{\text{tot}}(r_2, t) + v_1 \times B_{\text{tot}}(r_1, t)).
\]
(A.7)

In the dipole limit \(d \to 0\) we get, to lowest order and again after neglecting a magnetic term which by inequalities analogous to (A.5) is nonrelativistically small, the dipole dynamics
\[
\ddot{p} + \omega_0^2 p + \Omega \dot{p} = \frac{4q^2}{m} E_{\text{tot}}(r, t).
\]
(A.8)

The familiar solution of this equation assumes \(p\) to have the same time-dependence \(\exp(-i\omega t)\) as the fields (A.1), and gives the induced dipole moment
\[
p = \text{Re}[\alpha E(r) \exp(-i\omega t)],
\]
(A.9)

with the complex polarizability
\[
\alpha \equiv \alpha_1 + i\alpha_2 = \frac{4q^2}{m(\omega_0^2 - \omega^2 - i\omega \Omega)}.
\]
(A.10)

But this is an approximation to the solution of (A.8), because the fields depend on \(r\), which depends on \(t\) because the dipole is moving; it is an adiabatic approximation, whose validity again requires the \(r\) motion to be nonrelativistic.
Substituting (A.9) into the force (A.6), and taking the time average of the force $F_d$ (over times longer than the optical period $1/\omega$—a procedure whose rigorous justification requires some care), we find the time-averaged force

$$F(r) = \frac{1}{2} \text{Re}[\alpha^*(\mathbf{E}^* (\mathbf{r}) \cdot \nabla) \mathbf{E}(\mathbf{r})] - \frac{1}{2} \text{Im}[\omega \alpha^* \mathbf{E}^* (\mathbf{r}) \times \mathbf{B}(\mathbf{r})]$$

where Maxwell’s equation has been used to eliminate $\mathbf{B}(\mathbf{r})$. Now use of a vector identity completes the derivation of the standard formula (2.2).

We emphasize a non-trivial feature of the adiabatic approximation (A.9): it is necessary in order to generate the force (A.11) as a function of dipole position $\mathbf{r}$ alone. Without the adiabatic approximation, the force (A.6) will depend on $t$ as well as $\mathbf{r}$ and the separation of internal and external dynamics would be much harder. The situation is analogous to the Born–Oppenheimer separation of nuclear and electronic dynamics in molecular quantum physics [47, 48], but with the important difference that the Born–Oppenheimer force is a pure gradient, in contrast to the optical force which has a curl contribution. As in the Born–Oppenheimer case, systematic corrections to the dipole dynamics beyond the adiabatic regime can be found but are complicated. We are currently studying this.

References

[29] Bekshaev A Y 2013 Sub-wavelength particles in an inhomogeneous light field: optical forces associated with the spin and orbital energy flows J. Opt. 15 044004
[43] Polyanin A D and Zaitsev V F 2003 Handbook of Exact Solutions for Ordinary Differential Equations (Boca Raton, FL: Chapman and Hall/CRC)