Yakir Aharonov is one of the leading figures in the foundations of quantum physics. His contributions range from the celebrated Aharonov-Bohm effect (1959), to the more recent theory of weak measurements (whose experimental confirmations were recently ranked as the two most important results of physics in 2011). This volume will contain 27 original articles, contributed by the most important names in quantum physics, in honor of Aharonov’s 80-th birthday.

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Chapter 21
Superoscillations, Endfire and Supergain

M.V. Berry

Abstract Superoscillatory functions vary faster than their fastest Fourier component. Here they are employed to give an alternative description and explicit recipe for creating endfire arrays with supergain, that is antennas with radiation patterns concentrated in an arbitrarily narrow angular range and of arbitrary form. Two examples are radiation patterns described by sinc and Gaussian functions.

[Editor’s note: for a video of the talk given by Prof. Berry (titled ‘Weak Value Probabilities’) at the Aharonov-80 conference in 2012 at Chapman University, see quantum.chapman.edu/talk-6.]

Dedicated to Yakir Aharonov on his 80th birthday: still quick, still deep, still subtle.

21.1 Introduction

The three elements of the title are connected. My aim here is to show how, in the hope of giving further insight into them.

A central ingredient of Aharonov’s weak measurement scheme [1–3] is the concept of superoscillations. This is a property of band-limited functions: they can oscillate arbitrarily faster than their fastest Fourier component, over arbitrarily long intervals [4–6]. The price paid for this apparently paradoxical behaviour is that the functions are exponentially larger outside the superoscillatory region than in it.

An endfire array is an antenna in the form of a set of radiating sources arranged on a line (Fig. 21.1), the object of interest being the radiation pattern, that is, the angular distribution of wave intensity in the far field. It has been known at least since the 1940s [7–10] that if the array contains sufficiently many sources their strengths and phases can be so arranged that the radiation pattern is confined to an arbitrarily narrow region near the forward direction, even if the line containing all the sources...
is arbitrarily smaller than the wavelength of the radiation. This phenomenon, apparently contradicting diffraction-theory folklore about resolving power, is supergain (also called superdirectivity). The concept, originating in radar physics, has been applied in optics [11–17].

Section 21.2 gives a general theory of supergain in terms of superoscillations, leading to a simple recipe for creating radiation patterns of any form, and connects this with the traditional explanation involving complex-variable theory. Section 21.3 gives two explicit examples: radiation patterns in the form of sinc functions and gaussians.

21.2 General Theory

Consider \(N + 1\) sources (\(N\) even) arranged uniformly on a line of length \(L\) (Fig. 21.1), so their spacing is \(l = L/N\). The sources are point radiators of monochromatic scalar waves with wavelength \(\lambda\) and wavenumber \(k = 2\pi/\lambda\). With excitation amplitudes \(A_m\), the far-field angular amplitude is

\[
\psi(\theta) = \frac{1}{2N} \sum_{m=-N/2}^{N/2} A_m \exp(-imk l \cos \theta + imk l)
\]

\[= \frac{1}{2N} \sum_{m=-N/2}^{N/2} A_m \exp\left(2imk l \sin^2 \frac{1}{2} \theta\right). \quad (21.1)\]

The phases \(imk l\) have been included so that if all the \(A_m\) have the same phase the sources add coherently in the forward direction \(\theta = 0\).

In this simplest case—all the \(A_m\) equal—the angular distribution is

\[
A_m = \frac{1}{N + 1}, \quad \psi(\theta) = \frac{\sin((N+1)kL \sin^2 \frac{1}{2} \theta)}{(N+1) \sin(\frac{1}{N}kL \sin^2 \frac{1}{2} \theta)} \approx \frac{\sin(kL \sin^2 \frac{1}{2} \theta)}{kL \sin^2 \frac{1}{2} \theta}. \quad (21.2)
\]

The zero-amplitude directions (‘cones of silence’) are

\[
\theta_n = 2 \sin^{-1} \left(\sqrt{\frac{\pi Nn}{(N+1)kL}}\right), \quad 1 \leq |n| \leq \frac{kL}{\pi} = \frac{2L}{\lambda}. \quad (21.3)
\]
In particular, for large $N$, representing a line source, and $kL \gg 1$, the first zero is

$$\theta_1 \approx 2 \sqrt{\frac{\pi}{kL}}, \quad (21.4)$$

as expected on the basis of the Rayleigh resolution criterion. As is well known, and as Fig. 21.2 illustrates, a narrow radiation pattern requires $kL \gg 1$.

However, by suitable choice of the strengths and phases of the amplitudes $A_m$ it is possible to create radiation patterns as narrow as desired and indeed of arbitrary shape, even for $kL \ll 1$, i.e. $L \ll \lambda$—that is, to get supergain. To achieve supergain using superoscillatory functions, we first note that $\theta$ is not the natural variable, because $\psi(\theta)$ in (21.1), is not band-limited. This follows from the Bessel-Fourier relation

$$\exp(it \cos \theta) = \sum_{n=-\infty}^{\infty} i^n J_n(t) \exp(in\theta), \quad (21.5)$$

indicating that each term in (21.1) contains infinitely many angular Fourier components (i.e. angular momenta). However, defining the new variable $x$ by

$$x \equiv kL \sin^2 \frac{1}{2} \theta \quad (21.6)$$

gives the function

$$f(x) = \psi(\theta) = \sum_{m=-\frac{1}{2}N}^{\frac{1}{2}N} A_m \exp\left(\frac{2imx}{N}\right), \quad (21.7)$$

which is band-limited, with largest Fourier components proportional to $\exp(\pm ix)$.

It is possible to create such functions with arbitrarily fast oscillations for small $|x|$, e.g.

$$f_{\text{super}}(x, a) \approx \exp(iax) \quad (a > 1, |x| \ll 1, N \gg 1). \quad (21.8)$$
The most familiar and much-studied [13] superoscillatory function, to be used extensively in the following, is

\[
  f_{\text{super}}(x) = \left( \cos \frac{x}{N} + ia \sin \frac{x}{N} \right)^N. \quad (21.9)
\]

This is periodic in \(x\) with period \(N\). The small-\(x\) approximation \(\exp(iax)\) oscillates rapidly but does not lead to a concentrated radiation pattern.

However, the simple procedure of integrating over the scale variable \(a\), with a suitable weight function, enables any radiation pattern to be created, including the narrow ones corresponding to supergain. (This is a simplified variant, suggested to me by Professor Sandu Popescu, of a procedure previously used in an acoustic example in Sect. 4 of [4].) Let \(g(a)\) be the Fourier transform of the desired angular amplitude, and construct the function

\[
  f(x) = \int_{-\infty}^{\infty} da f_{\text{super}}(x, a) g(a). \quad (21.10)
\]

Then, for \(|x| \ll 1\), \(f(x)\) has, from (21.8), the desired form

\[
  f(x) \approx \int_{-\infty}^{\infty} da \exp(iax) g(a). \quad (21.11)
\]

If \(g(a)\) is even, as it will be in the examples in Sect. 21.3, \(f(x)\) is a real function. The radiation pattern thus constructed is

\[
  \psi(\theta) = f \left( kL \sin^2 \frac{1}{2} \theta \right). \quad (21.12)
\]

As \(\theta\) increases, this samples \(f(x)\) from \(x = 0\), corresponding to \(\theta = 0\), to \(x_{\text{max}} = kL\), corresponding to the backward direction \(\theta = \pi\). It is necessary to take \(N\) large enough to ensure that \(x_{\text{max}}\) lies in the superoscillatory interval where (21.8) applies, and this will be illustrated in Sect. 21.3. For \(x > x_{\text{max}}\) (‘beyond backwards’) the values of \(\theta\) given by (21.6) would be complex.

The formula (21.12) represents the far field. We will see that the coefficients \(A_m\), whose near-cancellation is responsible for supergain, take enormously large values (see Fig. 21.5 later). This implies that the wave intensity in the near field is correspondingly large, a phenomenon responsible for the well-known inefficiency of supergain antennas [8–10].

The conventional way of understanding supergain [7, 11] is via the complex variable

\[
  z = \exp \left( \frac{2ix}{N} \right) = \exp \left( 2ikL \sin^2 \frac{1}{2} \theta \right). \quad (21.13)
\]
and the corresponding function

\[ F(z) = \psi(\theta) = f(x) = \sum_{m=-\frac{1}{2}}^{\frac{1}{2}N} A_m z^m. \]  

(21.14)

Physical directions \(-\pi \leq \theta \leq \pi\) (i.e. real \(\theta\), so \(\sin^2(\theta/2) \leq 1\)) correspond to \(z\) on the unit circle and \(|\arg z| \leq 2kl\); for fixed \(L\), this region shrinks towards the forward direction \(z = 1\) as \(N\) increases. The zeros of \(F(z)\) correspond to the cones of silence. In the simplest case discussed earlier, in which all \(A_m\) are in phase, we have

\[ A_m = \frac{1}{N + 1}, \quad F(z) = \frac{(z^{N+1} - 1)}{(z - 1)}, \]  

(21.15)

whose zeros, uniformly distributed round the circle, are the \((N + 1)\)th roots of unity with \(z = 1\) excluded. Supergain [11] corresponds to choosing the \(A_m\) so that the zeros, whose density increases with \(N\), get concentrated into the physical region near \(\arg z \ll 1\), that is, near the forward direction.

### 21.3 Examples: Sinc and Gaussian Radiation Patterns

#### 21.3.1 Sinc Radiation Pattern

The simplest implementation of (21.10) is to make \(g(a)\) constant on a finite interval and zero outside, that is, the Fourier transform of the sinc function. Then if the width of the interval is \(2/w\), the function \(f(x)\) given by (21.10) is

\[ f(x, w) = 2w \int_{1/w}^{1/w} da f_{\text{super}}(x, a) = \frac{w}{(N + 1) \sin \frac{x}{N}} \text{Im} \left[ \left( \cos \left( \frac{x}{N} + \frac{i}{w} \sin \frac{x}{N} \right) \right)^{N+1} \right], \]  

(21.16)

and the radiation pattern is

\[ \psi(\theta, w) = \frac{w}{(N + 1) \sin(kl \sin^2 \frac{1}{2} \theta)} \times \text{Im} \left[ \left( \cos \left( kl \sin^2 \frac{1}{2} \theta \right) + \frac{i}{w} \sin \left( kl \sin^2 \frac{1}{2} \theta \right) \right)^{N+1} \right]. \]  

(21.17)

When \(w = 1\), this reproduces the radiation pattern of a line of coherently radiating sources (Eq. (21.2)). Otherwise, the analogous relation to (21.8) for small \(x\) and
Fig. 21.3 Radiation patterns with supergain: polar plots of the sinc-based pattern $\psi^2(\theta)$ given by (21.17) for $kL = \pi/2$ (that is, an endfire array of length $L = \lambda/4$), and $w = 1/40$, for the indicated values of $N$. The dots indicate the origin $\psi = 0$, and the dashed lines indicate the forward direction $\theta = 0$.

Fig. 21.4 Modulus of sinc-based function $|f(x, 1/40)|$ (Eq. (21.16)) for $N = 100$. (a) includes a complete period $|x| \leq 50\pi$, and the sinc function near the origin is invisible; (b) includes the much smaller range $|x| \leq 0.2\pi$, revealing the sinc.

For large $N$, namely

$$f(x, w) \approx \frac{\sin(x/w)}{(x/w)} \quad (x \ll 1, N \gg 1), \quad (21.18)$$

gives the approximate pattern

$$\psi(\theta, w) \approx \frac{w \sin(kLw \sin^2 \frac{1}{2} \theta)}{kL \sin^2 \frac{1}{2} \theta}, \quad (21.19)$$

and hence supergain if $w < 1$. Figure 21.3 shows how supergain emerges as $N$ increases, in the form of a narrow sinc radiation pattern for a short endfire array with length $L = \lambda/4$, and the small value $w = 1/40$.

When $N$ is too small, the radiation is concentrated near the backward direction (e.g. for $N = 200$ in Fig. 21.3), because the limiting value $x_{\text{max}} = kL$ sampled in $f(x, w)$ (Eq. (21.16)) is outside the region near $x = 0$ where the desired sinc function occurs, and instead lies in the exponentially large region beyond. Figure 21.4 illustrates the enormous disparity of values between these regions. For a quantitative understanding, we approximate $\psi(\theta)$ in (21.17) near $\theta = \pi$ and for $w < kL$:

$$\psi(\pi - \mu) \approx \left(\frac{w}{kL}\right) \sin\left(\frac{kL}{w} \left(1 - \frac{1}{4} \mu^2 \right)\right) \times \exp\left(\frac{1}{2N} \left(\frac{kL}{w}\right)^2 \left(1 - w^2\right) \left(1 - \frac{1}{2} \mu^2 \right)\right) \quad (\mu \ll 1). \quad (21.20)$$
This is exponentially large unless $N$ exceeds a critical value $N_c$, which can be defined by requiring $\psi^2$, averaged over the oscillations near $\theta = \pi$, to equal the value $\psi(\theta = 0) = 1$. For $N > N_c$, the value near $\theta = \pi$ is exponentially small, corresponding to supergain. From (21.20),

$$N_c = \left( \frac{kL}{w} \right)^2 \frac{(1 - w^2)}{2 \log \frac{\sqrt{2}kL}{w}}. \quad (21.21)$$

For the parameters of Fig. 21.3, $N_c = 440$, in fair agreement with the crossover to supergain.

The zeros of the radiation pattern are the directions

$$\theta_n = 2 \sin^{-1} \left( \frac{1}{N} \tan^{-1} \left( w \tan \frac{n\pi}{N + 1} \right) \right) \quad (1 \leq |n| \leq \frac{1}{2}N). \quad (21.22)$$

When $w = 1$ these cones of silence coincide with those of the equally phased array in (21.3). For small $w$ they cluster closer to $\theta = 0$:

$$\theta_n \approx 2 \sqrt{\frac{wn\pi}{kL}} \quad (w \ll 1, N \gg 1, n \ll N). \quad (21.23)$$

In terms of the variable $z$ of conventional supergain theory, the zeros cluster closer to $z = 1$ on the unit circle.

The remarkable superdirectivity accomplished by superoscillation is a consequence of near-destructive interference between successive sources. For the pattern (21.17), the excitation amplitudes in (21.16) are

$$A_m(w) = \frac{wN!}{2^{N+1}(\frac{1}{2}N + m)!((\frac{1}{2}N - m)! \int_{-1/w}^{1/w} da (1 - a^2)^{\frac{1}{2}N} \left( \frac{1 + a}{1 - a} \right)^m}$$

$$= \frac{wN!}{2^{\frac{1}{2}N - m + 1}(\frac{1}{2}N - m)!} \times \sum_{k=0}^{\frac{1}{2}N + m} \frac{(-1)^{\frac{1}{2}N - m}(\frac{1}{w} - 1)^{\frac{1}{2}N - m + k + 1} + (-1)^k(\frac{1}{w} + 1)^{\frac{1}{2}N - m - k + 1}}{2^k k!(\frac{1}{2}N + m - k)!((\frac{1}{2}N - m + k + 1))}. \quad (21.24)$$

(The sum can be expressed in terms of incomplete beta functions.) It is clear that $A_{-m}(w) = A_m(w)$, so we need study only $m \geq 0$. The simplest case is $w = 1$, for which $A_m(1) = 1/(N + 1)$, corresponding to all sources in phase. For general $w$, the coefficients for the first few values of $N$ are shown in Table 21.1.

The case of interest for supergain is $N \gg 1, w < 1$. Then Stirling’s formula, and expansion of the integrands about their maxima, gives the asymptotics in terms of
Table 21.1 Coefficients $A_m(w)$ for $N = 2, 4, 6$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$N = 2$</th>
<th>$N = 4$</th>
<th>$N = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{6w^2}(-1 + 3w^2)$</td>
<td>$\frac{1}{40w^4}(3 - 10w^2 + 15w^4)$</td>
<td>$\frac{1}{112w^6}(-5 + 7w^2(3 - 5w^2 + 5w^4))$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{12w^2}(1 + 3w^2)$</td>
<td>$\frac{1}{20w^4}(-1 + 5w^4)$</td>
<td>$\frac{1}{448w^6}(15 + 7w^2(-3 - 5w^2 + 15w^4))$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{80w^4}(1 + 10w^2 + 5w^4)$</td>
<td>$\frac{1}{224w^6}(-3 + 7w^2(-3 + 5w^2 + 3w^4))$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{448w^6}(1 + 7w^2(3 + 5w^2 + w^4))$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 21.5 Points: moduli of excitation coefficients $A_m$ (Eq. (21.24)) for endfire array with $N + 1 = 101$ radiating sources, for (a) $w = 0.7$, (b) $w = 0.4$, (c) $w = 0.2$, (d) $w = 0.01$. The curves show the approximation (21.25)

two Gaussians:

\[
A_m(w) \approx \frac{(-1)^{\frac{1}{2}N+m}}{N^{\frac{3}{2}}w^N\sqrt{2\pi(1-w^2)}} \left[ \exp\left( -\frac{2(m - \frac{1}{2}N w)^2}{N(1-w^2)} \right) \right. \\
\left. + \exp\left( -\frac{2(m + \frac{1}{2}N w)^2}{N(1-w^2)} \right) \right]. \tag{21.25}
\]

Note the alternating signs, indicating the near-cancellation responsible for supergain. Figure 21.5 shows the moduli of the coefficients. The agreement with the asymptotic formula appears excellent. However, using (21.25) instead of the exact coefficients in (21.1) leads to numerical instability and no hint of supergain: the interference is far too delicate to be captured by this lowest level of asymptotics. (For $w > 1$, the asymptotics is different: all amplitudes are in phase and there is no supergain.)
21.3.2 Gaussian Radiation Pattern

The simplest example of a desired radiation pattern without zeros is the Gaussian, for which (21.10) is

\[
 f(x, w) = \frac{w}{\sqrt{2\pi}} \int_{-\infty}^{\infty} da f_{\text{super}}(x, a) \exp\left(-\frac{1}{2} a^2 w^2\right) 
 = \frac{\sin^N(x/N)}{(N+1)wN^{2N/2}} \left( \frac{w}{\sqrt{2}} \cot\left(\frac{x}{N}\right) H_{N+1}\left(\frac{w}{\sqrt{2}} \cot\left(\frac{x}{N}\right)\right) 
 - \frac{1}{2} H_{N+2}\left(\frac{w}{\sqrt{2}} \cot\left(\frac{x}{N}\right)\right) \right),
\]

where \( H \) denotes the Hermite polynomials. For small \( x \), this gives the desired Gaussian with width \( w \), and supergain for \( w < 1 \):

\[
 f(x, w) \approx \exp\left(-\frac{x^2}{2w^2}\right) \quad (x \ll 1).
\]

(This limiting form, obvious from (21.11), can also be obtained from (21.26) by a large \( N \) resummation of the power series for the Hermite polynomials.) As in the case of the sinc pattern, enormous magnification is required to see the Gaussian, as Fig. 21.6 illustrates.

Some of the corresponding radiation patterns, obtained by the substitution (21.12), are shown in Fig. 21.7. In this case, the crossover to supergain as \( N \) increases is much more sensitive than in the sinc case (cf. Fig. 21.3).
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References

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