Suppression of superoscillations by noise

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Abstract
Bandlimited functions can vary faster than their highest Fourier component. Such ‘superoscillations’ result from near-perfect destructive interference among the Fourier components and correspond to large values of the phase gradient (local wavenumber). Superoscillations that are strong and extend over a large interval occur where functions are exponentially small. The associated interference is vulnerable to noise, in particular random phases. Averaging over the phases, modelled as independent Gaussian variables with a specified rms value, enables the suppression of superoscillations to be described quantitatively; very weak phase noise suffices. Strong noise generates functions that are essentially random, and the remaining well-understood superoscillations are localised in small intervals. The theory is illustrated by computations with an explicit superoscillatory function.

Keywords: phases, dephasing, bandlimited, random, interference

1. Introduction
Superoscillations are variations in a bandlimited function faster than its fastest Fourier component. As is now well understood [1–4], this phenomenon is neither paradoxical nor a violation of the uncertainty relation between the variances of the function and its Fourier transform, because superoscillations occur where the function is much smaller than elsewhere.

Such small values arise from near-perfect coherent destructive interference among the function’s Fourier components. For large superoscillations, persisting over an extended region (‘strong superoscillations’) this requires a delicate conspiracy of phases. Such interference is vulnerable to noise, especially in the form of random phases in the Fourier components. My purpose here is to outline a quantitative theory of the suppression of strong superoscillations by dephasing, explaining why very weak phase noise suffices.
In the absence of noise, the functions to be considered are of the form

\[ f(x) = \sum_{n=0}^{N} c_n \exp(ik_n x), \quad |k_n| \leq 1, \quad c_n \text{ real} , \tag{1.1} \]

in which the inessential restriction to \( c_n \text{ real} \) simplifies some of the formulas. We will consider functions with strong superoscillations near \( x = 0 \), which arise because of cancellations among the components, so the \( c_n \) will have different signs for different \( n \). It is convenient to scale \( f(x) \) to be unity at the origin, i.e.

\[ f(0) = \sum_{n=0}^{N} c_n = 1. \tag{1.2} \]

Superoscillations near \( x = 0 \) mean that \( |f(x)| \) gets rapidly large away from the origin, so the individual \( |c_n| \) must greatly exceed unity.

A convenient and much-studied [5–7] measure of superoscillations is the local phase gradient, namely the local wavenumber or, in quantum mechanics (after multiplication by \( \hbar \)), the real part of the weak value of momentum with the preselected state \( f(x) \) and position \( x \) postselected [7]:

\[ k(x) = \partial_x \arg f(x) = \text{Im} \frac{\partial_x f(x)}{f(x)}. \tag{1.3} \]

For functions of the type (1.1), superoscillations occur where \( |k(x)| > 1 \).
To illustrate the general theory, we will use the familiar [1, 8] superoscillatory function
\[
\begin{bmatrix}
\cos \frac{x}{N} + ia \sin \frac{x}{N}
\end{bmatrix}^N, \quad (a \gg 1, N \gg 1),
\]
which is of the form (1.1) with
\[
k_n = 1 - \frac{2n}{N}, \quad c_n = \frac{N!}{2^N(a + 1)^N} \frac{(-1)^n}{(N - n)!n!} \left( \frac{a - 1}{a + 1} \right)^n.
\]
The factor \((-1)^n\) is responsible for the destructive interference leading to superoscillations. It is convenient to choose \(N\) even; then \(f(x)\) is periodic with period \(\pi N\). The local wavenumber is, from (1.3)
\[
k(x) = \frac{a}{\cos^2 \left( \frac{x}{N} \right) + a^2 \sin^2 \left( \frac{x}{N} \right)}.
\]

The strength of the superoscillations is described by the parameter \(a\), because \(k(0) = a \gg 1\). The size of the region over which superoscillations occur is described by the parameter \(N\): \(k(x) > 1\) if \(|x| < N \arccot \sqrt{a}\), so the fraction of the \(x\) axis where \(f(x)\) is superoscillatory is \((2/\pi) \arccot \sqrt{a} \sim 2/(\pi \sqrt{a})\). What we are calling strong superoscillations correspond to \(a\) and \(N\) both large.

Figure 1 illustrates these superoscillations. In figure 1(a), the clustering of zeros of \(\text{Re}\ f(x)\) near \(x = 0\) corresponds to the superoscillations, and the logarithmic scale is chosen to accommodate the very large increase away from \(x = 0\)—in this case rising to approximately \(10^8\). Figure 1(b) shows the strength \(k(0) = 6\) of the superoscillations, and their persistence over the range \(|x| < 3.88\) (shaded).

To incorporate the effect of noise, we note that since the superoscillations are associated with phase coherence (almost-perfect destructive interference), the natural choice for exploring the suppression of superoscillations is to introduce noise into the phases of the Fourier coefficients. Therefore we modify (1.1) as follows:
\[
f_\varepsilon(x) = \sum_{n=0}^{N} c_n \exp(i(k_n x + \varepsilon_n)), \quad \langle \varepsilon_n \rangle = 0, \quad \langle \varepsilon_n^2 \rangle = \varepsilon^2 \nonumber
\]
\[
k_\varepsilon(x) = \partial_x \arg f_\varepsilon(x) = \frac{\text{Im} \frac{\partial f_\varepsilon(x)}{f_\varepsilon(x)}}{f_\varepsilon(x)}.
\]
in which the $N + 1$ random phases $\varepsilon_n$ are independent and Gauss-distributed. Figure 2, which should be compared with figure 1(a), illustrates how weak random phases suppress the superoscillations near $x = 0$.

In section 2 we will calculate the mean, i.e. noise-averaged, intensity $I(x)$, and in section 3 we will calculate the mean wave number $K(x)$. These quantities are defined by

\[ I(x) \equiv \langle |f_n(x)|^2 \rangle \quad (a) \]
\[ K(x) \equiv \langle k_n(x) \rangle \quad (b) \]

in which the averages $\langle \cdots \rangle$ are over all the random phases $\varepsilon_n$. These are local measures of superoscillation. Global measures of superoscillation yield $I(x)$ involve integrals over $x$ and can be optimised [9]; and a statistical analysis of the quality of the yield when it is not optimised has been carried out, in a study [10] complementary to that presented here.

In section 4 we contrast strong extended superoscillations suppressed by weak noise with the localised superoscillations that remain when the phase noise is strong and the Fourier components generate effectively random functions.

2. Intensity dephasing

In (1.8) the intensity involves a double Fourier sum, whose separation into diagonal and off-diagonal contributions gives

\[ I(x) = \sum_{n=0}^{N} |c_n|^2 + 2 \text{Re} \sum_{n=0}^{N} \sum_{m=0}^{n-1} c_n c_m \exp(i(k_n - k_m)x) \exp(i(\varepsilon_n - \varepsilon_m)). \]  
(2.1)

To evaluate the average, we use the independence of $\varepsilon_n$ and $\varepsilon_m$, and the standard Gauss average

\[ \langle \exp(i\varepsilon_n) \rangle = \exp\left(-\frac{1}{2} \varepsilon^2\right). \]  
(2.2)

It is convenient to define the quantity

\[ S_2 \equiv \sum_{n=0}^{N} |c_n|^2, \]  
(2.3)

which is large because the $|c_n|$ are large for the superoscillatory functions $f(x)$ that we are considering.

Thus the mean intensity is

\[ I(x) = S_2 + \exp(-\varepsilon^2)(|f(x)|^2 - S_2) \]
\[ = (1 - \exp(-\varepsilon^2))S_2 + \exp(-\varepsilon^2)|f(x)|^2. \]  
(2.4)

Because $S_2 \gg 1$, very small noise $\varepsilon$, in fact $\varepsilon > \varepsilon_c$ where

\[ \varepsilon_c = \frac{1}{\sqrt{S_2}}, \]  
(2.5)

suffices to destroy the coherent destructive interference at $x = 0$; the intensity rapidly increases from its noise-free value $I(0) = 1$. With further increase of the noise, to $\varepsilon \gg 1$, the intensity rises to its fully dephased value $I(0) = S_2$. The limiting behaviours are

\[ I(x) \approx \begin{cases} |f(x)|^2 + \varepsilon^2 S_2 & (\varepsilon \ll \varepsilon_c) \\ \frac{S_2}{\varepsilon} & (\varepsilon \gg 1) \end{cases}. \]  
(2.6)
For the function (1.4), the sum $S_2$ can be evaluated as polynomials of degree $a^{2N}$, expressed in terms of hypergeometric functions:

$$S_2 = \left( \frac{1}{2} (1 + a) \right)^{2N} 2F_1 \left(-N, -N, 1, \left( \frac{a - 1}{a + 1} \right)^2 \right) \approx \left( \frac{1}{2} a \right)^{2N} \frac{(2N)!}{(N!)^2}. \quad (2.7)$$

As an illustration of its large value

$$a = 6, N = 10 \Rightarrow S_2 = \frac{171399067177014648449}{262144} = 6.53... \times 10^{14}. \quad (2.8)$$

Figure 3. Dots: mean intensity at the origin $I(0)$ for $a = 6, N = 10$, averaged over 100 sets of $N + 1 = 11$ random phases, as a function of rms noise $\varepsilon$; red curve: theoretical formula (2.4). (a) For range $0 < \varepsilon < \frac{\varepsilon_c}{5}$; (b) for magnified range $0 < \varepsilon < 5\varepsilon_c$. For this case, $\varepsilon_c = 3.9 \times 10^{-8}$.

Figure 4. $\log_{10}I(x)$ of mean intensity $I(x)$ for $a = 6, N = 10$ and the indicated values $\varepsilon$ of rms random phases. Black dashed curves: averaged over 100 sets of $N + 1 = 11$ random phases; red curves: theoretical formula (2.4).
Figure 3 illustrates how the average intensity $I(0)$ increases with the noise $\varepsilon$; the fluctuations about the mean are sensitive to the value of $\varepsilon$, increasingly so as $\varepsilon$ increases. Figure 4 illustrates the increase of $I(x)$ away from the superoscillatory region, for different values of $\varepsilon$. The fluctuations are invisible on the scale of the picture, because for fixed $\varepsilon$ the intensity is a smooth function of $x$.

3. Wavenumber dephasing

A feature that the mean wavenumber $K(x)$ in (1.8b) shares with all statistical calculations involving weak values [5, 6, 11] is the need to accommodate the denominator, which (see (1.7)) is the random-phased function $f_\varepsilon(x)$. First we study the wavenumber at the origin, where the superoscillations are strongest and therefore most sensitive to noise. To make the denominator manageable, we represent it as a formal Laplace transform. Thus the mean wavenumber is

$$K(0) = \left\langle \partial_x f_\varepsilon(0) \int_0^\infty ds \exp(-sf_\varepsilon(0)) \right\rangle. \quad (3.1)$$

The suppressions we study in this section involve $\varepsilon_n \ll 1$, so it suffices to retain the terms linear in $\varepsilon_n$ in the random phase factors:

$$f_\varepsilon(0) = 1 + i \sum_{n=0}^N c_n \varepsilon_n + O(\varepsilon_n^2). \quad (3.2)$$

It might seem that consistency requires the analogous expansion in $\partial_x f_\varepsilon(0)$. But this is not the case, because fluctuations in the numerator have a much smaller effect than in the denominator; a more elaborate analysis confirms this. Thus, after noting that in the Laplace transform the different and statistically independent $\varepsilon_n$ occur in separate factors, the average becomes

$$K(0) \approx k(0) \int_0^\infty ds \exp(-s) \prod_{n=0}^N \left\langle \exp(-s\varepsilon_n c_n) \right\rangle$$

$$= k(0) \int_0^\infty ds \exp\left(-s - \frac{1}{2} \varepsilon^2 S_2 \right). \quad (3.3)$$

Figure 5. Dots: mean local wavenumber at the origin $K(0)$ for $a = 8, N = 10$, averaged over 500 sets of $N + 1 = 11$ random phases, as a function of rms noise $\varepsilon$; red curve: theoretical formula (3.4).
Evaluating the integral gives

\[
K(0) = k(0) \sqrt{\frac{\pi}{2\xi^2 S_2}} \exp\left(\frac{1}{2\xi^2 S_2}\right) \text{erfc}\left(\frac{1}{\xi} \sqrt{\frac{1}{2S_2}}\right)
\]  

(3.4)

Figure 5 illustrates how this describes the suppression of the superoscillations with increasing noise, in terms of the natural variable \(\xi\sqrt{S_2} = \xi/\xi_c\).

A similar calculation gives the mean wavenumber for \(x > 0\). A subtlety is that in the counterpart of (3.3) the coefficient of \(S_2^2\) in the exponent of the integral can sometimes be complex with positive real part. Interpreting this by rotating the integration contour leads to

\[
K(x) = \text{Im} \left[ \frac{\partial f(x)}{f(x)} \sqrt{\frac{\pi}{2S_2(x)}} \exp\left(\frac{1}{2\xi^2 S_2(x)}\right) \text{erfc}\left(\frac{1}{\xi} \sqrt{\frac{1}{2S_2(x)}}\right) \right]
\]

(3.5)

in which \(S_2(x)\) is the following complex function, generalising \(S_2\) in (2.3):

\[
S_2(x) \equiv \sum_{n=0}^{N} c_n^2 \exp(2ik_n x) / f(x)^2.
\]  

(3.6)

We can anticipate interesting behaviour for \(K(x)\) by arguing that the noiseless wavenumber \(k(x)\) decreases as \(x\) increases, and so should be less sensitive to dephasing. As figure 6 illustrates, the formula (3.5) supports this expectation, with the denominator \(1/f(x)^2\) in (3.6) generating negative spikes in the average \(K(x)\).

4. Concluding remarks

It is clear that the foregoing theory explains the extreme sensitivity of superoscillations to noise. They are easily suppressed by random phases with very small rms strength \(\xi \sim \xi_c\). As discussed elsewhere, the sensitivity threatens the practicality of optical superresolution techniques based on superoscillations [12].

To avoid confusion, it should be noted that suppression is not the same as elimination. When \(\xi > 2\pi\), the initially strongly superoscillatory function \(f(x)\) is completely dephased and becomes effectively a random wave with coefficients \(c_n\). But random waves also typically contain superoscillations, as initially understood in \(D = 2\) dimensions [5] and then for general
including $D = 1$ [6]. The difference is that these remaining superoscillations are localized: by contrast, the strong superoscillations studied in this paper extend over longer intervals (e.g. quantified by $N$ for the function (1.4)).

The random-wave superoscillations are also rarer for random waves of the type (1.7) than for the standard superoscillatory function (1.4). To see this, note first that a simple modification of the argument in [6] gives the probability that a randomly chosen $x$ is superoscillatory (i.e. $|k(x)| > 1$) as

$$P_{\text{super}} = 1 - \frac{1}{\sqrt{1 + \langle k^2 \rangle}} = \frac{\sum_{n=0}^{N} c_n^2 k_n^2}{2}.$$  \hspace{1cm}  (4.1)

The coefficients $c_n^2$ in (1.5) are approximately Gauss-distributed [8], and asymptotics for $N \gg 1$ and $a \gg 1$ gives

$$\langle k^2 \rangle \approx \frac{1}{a^2} + \frac{1}{2N^2} \rightarrow \frac{1}{a^2} \text{ as } N \rightarrow \infty,$$  \hspace{1cm}  (4.2)

so

$$P_{\text{super}} \approx \frac{1}{2a^2}.$$  \hspace{1cm}  (4.3)

This is much smaller than the value for the deterministic function (1.4), namely

$$P_{\text{super(equation(1.4))}} \sim \frac{2}{\pi \sqrt{d}}$$  \hspace{1cm}  (4.4)

(see the remarks after (1.6)).
Figure 7, which is the counterpart of figure 1 with strong dephasing, illustrates two of the remaining superoscillations (shaded) in a sample random wave (1.7).

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