Stable and unstable Airy-related caustics and beams

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Abstract

Optical beams with an underlying caustic structure are stable under perturbation if the caustics belong to the catastrophe-theory classification; otherwise they are unstable. The original Airy beam in two spatial dimensions, with its curved caustic, is stable in this sense. But the separable Airy-product beam in three dimensions is unstable: under separability-breaking perturbation, it unfolds into the hyperbolic umbilic diffraction catastrophe, which is stable. By including initial phase factors, a variety of new exact solutions of the paraxial wave equation can be generated, corresponding to Pearcey and higher-catastrophe beams with stable caustics, and with the associated diffraction catastrophes appearing in their canonical forms or as deformations of these.

Keywords caustics, accelerated beams, catastrophes, paraxial

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1. Introduction and preliminaries

After it was understood that quantum waves described by Airy functions can accelerate in free space without spreading [1], it was recognised [2] that this implies that beams of light in vacuum can bend and propagate without diffraction. The implication was based on the analogy between the time-dependent Schrödinger equation and the paraxial wave equation. A great variety of beams, nonparaxial as well as paraxial, and following a variety of trajectories, have since been discovered theoretically and created in the laboratory [3-8].

It was understood from the start that what is accelerating or bending is not individual geometrical rays in the family underlying the beam, or the centroid of the beam (when this is transverse-normalized, by apodization or other amplitude modulation [2]); both of these must follow linear trajectories. Instead, it is the caustic, on which the rays are focused, that bends. The caustic is the envelope of the family of rays corresponding to the beam. Caustics are curved even though the rays are straight. It is this connection with caustics that I wish to emphasize here, in the context of their classification by catastrophe theory, and associated exact solutions of the paraxial wave equation.

The classification, originally by Thom [9] and extended by Arnold [10-13], concerns stable singularities of smooth gradient maps depending on one or more parameters; the number is the codimension. This mathematics applies to geometrical optics, where the gradient maps describe families of rays because of the Fermat-Hamilton principle, and the singularities are caustics [14-16]. In this paper, the parameters are coordinates in three dimensional space.
The central feature of the catastrophe classification (fold, cusp, swallowtail, hyperbolic and elliptic umbilic, etc), distinguishing this from other classification schemes such as aberration theory, is that the caustics it describes are stable. Under any smooth reversible deformation of the ray family (a diffeomorphism) the caustics listed as catastrophes also deform smoothly; in particular they do not change their topology. Contrast this with the ideal point focus of a lens in elementary optics; under any change in the ray family, this explodes into complicated caustic surfaces whose nature depends on the deformation: the point focus is not stable.

Of course, the stable caustics are structures that emerge as singularities in the geometrical-optics approximation to beams satisfying wave equations. But as is well understood [17] the catastrophe-classified caustics are skeletons of the waves, decorated by interference patterns whose local behaviour – the diffraction catastrophes - is also stable. Their intricate forms have been studied in detail [18-21] and they now form a class of special functions (chapter 36 of [22]), of which the Airy and Pearcey [23] functions are the simplest members.

Each caustic singularity, and its corresponding diffraction catastrophe, is described by a canonical form generated from an elementary polynomial [13]. Smooth deformation of this polynomial generates deformed caustics – other representatives of the same universality class (fold, cusp, etc.). In this paper, caustics will sometimes appear in their canonical forms and sometimes as deformations of these.

The most familiar Airy beam is commonly realised in three dimensions as the product of two Airy functions [2]. Its caustic consists of two smooth surfaces - fold catastrophes with codimension 1 - meeting
at a finite angle along an edge that is parabolically curved. Although this corresponds to an exact solution of the paraxial wave equation, the transverse sections, in which two smooth curves meet at a finite-angled corner, are not stable in the sense described above. Section 2 explains why, and shows how under typical symmetry-breaking perturbation this beam becomes the codimension-3 hyperbolic umbilic catastrophe, which is stable.

Also briefly recalled in section 2 is the complementary elliptic umbilic catastrophe; this is also easily realised paraxially, as was understood theoretically and generated in an experiment [19], in which the equivalent of today’s spatial light modulator was a sculpted droplet of water.

Section 3 builds on existing studies of beams based on cuspoid catastrophes. Following the pioneering study [4] of the Pearcey beam, corresponding to the cusp catastrophe with codimension 2, a more general class of paraxial exact waves is presented; some are curved and some are not. For the codimension 3 swallowtail, whose different sections have been studied as the initial sections of optical beams [8], an exact paraxial propagating wave is presented; it is a distorted version of the canonical form.

Airy-related beams possess an important self-healing property [4, 24-26]: although a scattering obstacle placed in the beam causes some local disturbance, subsequent propagation is hardly affected, in the sense that the wave closely resembles the beam in the absence of the obstacle. The interpretation in terms of rays and caustics provides an immediate elementary understanding [27] of self-healing: the rays touching the
caustic beyond the obstacle are different from the rays that were disturbed by the obstacle, and have never encountered it.

Here I choose to represent the caustics and diffraction catastrophes by exact solutions of the paraxial wave equation, namely

$$2i\partial_z \psi(x,y,z) + \left(\partial_x^2 + \partial_y^2\right)\psi(x,y,z) = 0. \quad (1.1)$$

The beam propagates in the z direction from an initial wave in the plane $z=0$, and $x$ and $y$ are transverse coordinates, represented in units of wavelength/2\pi. The solution propagating from an initial wave $\psi(x,y,0)$ is the Fresnel transform

$$\psi(x,y,z) = \frac{-i}{2\pi} \int \int dxdy' \psi(x',y',0) \exp\left(i \frac{(x-x')^2 + (y-y')^2}{2z}\right). \quad (1.2)$$

Such solutions are approximations to solutions of the non-paraxial Helmholtz or Maxwell equations, among whose exact solutions are also waves whose corresponding rays possess stable curved caustics, for example circular beams [28, 29] based on the Bessel wave $J_n(r) \exp(in\phi)$ (in polar coordinates). More generally the standard diffraction catastrophes represent waves locally near caustics, as uniform approximations in the asymptotic regime of short wavelength [17, 30] (see also section 36.12 of [22]).

In practice, beams with caustics are frequently gauss-modulated, and it is not difficult to find exact paraxial representations of such ‘structured gaussian beams’. For the purposes of this paper, it is not necessary to consider such embeddings, even though they lead to deep and subtle analogies in which caustics play a role [31, 32].
As a preliminary, we recapitulate the original Airy beam [2], in 3D $xyz$ space but independent of $y$. The initial wave, propagating according to (1.2), is, in its canonical form,

$$
\psi(x, y, 0) = \text{Ai}(x) = \frac{1}{2\pi} \int ds \exp \left( i \left( \frac{1}{3} s^3 + sx \right) \right)
$$

$$
\Rightarrow \psi(x, y, z) = \exp \left( \frac{1}{6} iz \left( -\frac{1}{6} z^2 + x \right) \right) \text{Ai} \left( x - \frac{1}{4} z^2 \right).
$$

The caustic is a parabolically curved sheet. To see this, we consider the Fourier transform with respect to $x$, whose phase is the action as a function of the momentum variable $k_x$:

$$
\overline{\psi}(k_x, y, 0) = \exp \left( \frac{i}{6} k_x^3 \right) \Rightarrow \text{action } S(k_x, y) = \frac{i}{6} k_x^3.
$$

This representation is convenient because although there is a caustic in $x$ there is no caustic in $k_x$ (in mathematical language, the projection of the Lagrangian manifold in phase space [33-35] is singular onto $x$ but not onto $k_x$). Standard canonical transformation theory gives the initial $x$ and $k_y$ as

$$
\begin{align*}
x &= -\partial_{k_x} S(k_x, y) = -k_x^2, \\
y &= +\partial_{y} S(k_x, y) = 0,
\end{align*}
$$

and hence the straight rays as

$$
\begin{align*}
X(k_x, z) &= -k_x^2 + k_x z, \\
Y &= y.
\end{align*}
$$

These are shown in figure 1a. Their envelope is the familiar parabolic caustic, on which the argument of Ai is zero, with two rays crossing on the bright side, corresponding to the interference fringes where the argument of Ai is negative, and no rays on the dark side where the argument is positive. In space, this is the claimed curved sheet, invariant along $y$. 
A slight generalization, obtained by adding a phase factor to the initial wave, is

\[
\psi(x, y, 0) = \exp\left(\frac{i}{2} ax^2\right) \text{Ai}(x)
\]

\[
\Rightarrow \psi(x, y, z) = \exp(i\gamma) \text{Ai}(\xi),
\]

where

\[
\xi = \frac{1}{1 + az} \left( x - \frac{z^2}{4(1 + az)} \right),
\]

\[
\gamma = \frac{ax^2}{2(1 + az)} + \frac{xz}{2(1 + az)^2} - \frac{z^3}{12(1 + az)^3}.
\]

The rays are given parametrically in terms of \( k_x \) by

\[
x(k_x, z) = \frac{ak_x - 1 \pm \sqrt{ak_x - \frac{1}{4}}}{a^2} + k_xz \left( |k_x| - \frac{1}{4a} \right), \quad Y = y.
\]

The caustic, given by the zero \( \xi=0 \) of the Airy function, is now asymptotically straight for \( a\neq0 \) as illustrated in figure 1b. Because of the denominators \( 1+az \) in \( x \) in (1.7), the beam is no longer diffraction-free: with increasing distance \( z \), the Airy fringes get broader.
2. The unstable curved-edge beam and the stable umbilic beams

The Airy beam in space is usually implemented as the following product:

\[
\psi(x,y) = \text{Ai}(x) \text{Ai}(y)
\]

\[
\Rightarrow \psi(x,y,z) = \exp\left(\frac{i}{2}z\left(-\frac{1}{3}z^2 + x + y\right)\right) \text{Ai}\left(x - \frac{1}{4}z^2\right) \text{Ai}\left(y - \frac{1}{4}z^2\right). \tag{2.1}
\]

The caustic in the initial plane \(z=0\) is two half-lines meeting at a corner where \(x=y=0\), as in figure 2a. This propagates to two sheets with the corner expanding to the parabolic edge \(x=y=z^2\), as shown in figure 3a.
Figure 2. (a) caustic in plane section of separable 3D Airy beam (2.1); (b) section of the hyperbolic umbilic when the initial Airy wave is not separable, as in (2.3)

Now, neither a point in the plane where two caustic lines meet at a finite angle, nor an edge in space where two surfaces meet at a finite-angled corner, is on the list of stable singularities, indicating that this caustic arises from a special, i.e. untypical, circumstance. It is not hard to guess what this is: in the initial wave (2.1), the $x$ and $y$ dependences are separable. A generic initial wave will not be separable, resulting in a caustic that is stable under all smooth deformations, in which the corner in figure 2a is a special section of one of the catastrophes.

The only catastrophe with this property is the hyperbolic umbilic. Away from $z=0$, its section is as shown in figure 2b: a smooth outer curve and a cusped inner curve. The full caustic surface is shown in figure 3b. Decorating this surface is the hyperbolic umbilic diffraction catastrophe, given in its canonical form by
$$\psi^{(H)}(\xi, \eta, \zeta) = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt \exp\left(i(s^3 + t^3 + \zeta st + \xi s + \eta t)\right).$$  \hspace{1cm} (2.2)

The architecture of $\psi^{(H)}$ has been studied in detail [20].

To investigate whether the corner of figure 2a indeed propagates into the hyperbolic umbilic, we introduce a nonseparable initial phase, proportional to $(x+y)^2 = x^2 + 2xy + y^2$ (slightly more convenient than the apparently simpler phase $xy$, but the result is essentially the same). It is not hard to show from (1.2) that the exact paraxial wave is

$$\psi(x, y, 0) = \exp\left(\frac{i}{2}a(x+y)^2\right)\text{Ai}(x)\text{Ai}(y)$$

$$\Rightarrow \psi(x, y, z) = \frac{3^{2/3}}{(2\pi)^2} \frac{\exp(i\gamma)}{\sqrt{1+2az}} \psi^{(H)}(\xi, \eta, \zeta),$$  \hspace{1cm} (2.3)

where

$$\xi = 3^{1/3} \left\{ \frac{x(1+az)-ayz}{1+2az} + \frac{z^2\left(a^2z^2-1\right)}{4\left(1+2az\right)^2} \right\},$$

$$\eta = 3^{1/3} \left\{ \frac{y(1+az)-axz}{1+2az} + \frac{z^2\left(a^2z^2-1\right)}{4\left(1+2az\right)^2} \right\},$$

$$\zeta = \frac{3^{2/3}az^2}{1+2az};$$  \hspace{1cm} (2.4)

$$\gamma = \frac{a(x+y)^2}{2(1+2az)} + \frac{z(x+y)(1+az)}{2(1+2az)^2} + \frac{z^3(-2 + az)(1+az)^2}{12(1+2az)^3}. $$

The two-sheeted caustic corresponding to this wave is easily generated by solving $\xi$ and $\eta$ for $x$ and $y$ and using the parametric representation [22]

$$\xi = -\frac{1}{12} \zeta^2 \left(\exp(2\tau) \pm 2 \exp(-\tau)\right), \quad \eta = -\frac{1}{12} \zeta^2 \left(\exp(-2\tau) \pm \exp(\tau)\right).$$  \hspace{1cm} (2.5)
An important structural feature is the cusped edge of the inner branch of the caustic - the ‘rib’. Unlike the finite-angled edge in figure 3a, the cusp is stable under perturbation. Its equation is

\[ x = \frac{1}{4} z^2 (1 - 2az), \]

indicating that the rib slopes towards increasing \( x \) and \( y \) until \( z = 1/2a \) where it has an inflection before bending inwards.

Several comments can be made. First, this example illustrates how a stable caustic results from a symmetry-breaking perturbation. Second, in this case the caustic and diffraction catastrophe appear not in canonical form (e.g. (2.2) for the diffraction catastrophe) but in the distorted (i.e. non-canonical) form described by the spatial dependence of the parameters in (2.4). Third, as an alternative to the phase-modulated Airy-product initial wave as in (2.3), one can choose as initial wave a stable section of the hyperbolic umbilic (i.e. \( \xi > 0 \) in (2.2)), whose caustic is a
smooth outer curve and a cusped inner curve as in figure 2b; then, as shown in [36], this bends parabolically under paraxial propagation while preserving its form – it is diffraction-free. Fourth, this example recalls the first prediction based on catastrophe theory [37] applied to caustics, for waves reflected from smoothly-deformed surfaces; there too, a symmetry-breaking perturbation caused a finite-angled corner to ‘generify’ into the stable hyperbolic umbilic catastrophe.

The situation is simpler for the elliptic umbilic catastrophe, for which exact paraxial propagation generates a wave [19] that is already in the canonical form for this diffraction catastrophe:

\[
\psi(x,y,0) = 2\pi^2 \left( \frac{2}{3} \right)^{2/3} \text{Re} \left[ \text{Ai} \left( \frac{x + iy}{12^{1/3}} \right) \text{Bi} \left( \frac{x - iy}{12^{1/3}} \right) \right] 
\]

\[
\Rightarrow \psi(x,y,z) = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt \exp \left( i \left( s^3 - 3t^3 + \frac{1}{2} z \left( s^2 + t^2 \right) + xs + yt \right) \right). 
\]

For completeness, the caustic is shown in figure 4.
3. Cusp and swallowtail beams

Diffraction decorating a cusp caustic is described by the Pearcey function [22, 23], depending on two variables, corresponding to the codimension 2 geometry of the cusp in its canonical form:

\[ \text{Pe}(\xi, \eta) = \int_{-\infty}^{\infty} ds \exp \left( i (s^4 + \eta s^2 + \xi s) \right). \] (3.1)

The cusp, consisting of two fold lines meeting at a common tangent, is the semicubical parabola

\[ \eta^3 = -\frac{27}{8} \xi^2. \] (3.2)

The propagation of an initial wave given by \( \text{Pe}(x, y) \) has been studied in detail both theoretically and experimentally [4]. This wave can be embedded in a more general class of analytical Pearcey beams, in which the initial \( \text{Pe} \) is augmented by a phase factor, analogous to (1.7) for \( \text{Ai} \):

\[ \psi(x, y, 0) = \exp \left( i \left( ax^2 + by^2 - cy \right) \right) \text{Pe}(x, y) \]

\[ \Rightarrow \psi(x, y, z) = \frac{\exp(i\gamma)}{\sqrt{1 + 2az} \left( 1 + 2bz \right)^{\frac{1}{4}} \left( 1 + (2b - \frac{1}{2})z \right)^{\frac{1}{4}}} \text{Pe}(\xi, \eta), \] (3.3)

where
\[
\xi = \frac{x(1 + 2bz)^{1/4}}{(1 + 2az)(1 + (2b - \frac{1}{2})z)^{1/4}},
\]
\[
\eta = \frac{y(1 + 2az) + (c - \frac{1}{2})z + z^2(2ac - b)}{(1 + 2az)(1 + 2bz)(1 + (2b - \frac{1}{2})z)},
\]
\[
\gamma = \frac{ax^2 + by^2 - cy - \frac{1}{2}c^2z}{1 + 2az + 1 + 2bz}. \tag{3.4}
\]

In these waves, the cusp in the three-dimensional \(xyz\) space propagates as a rib line (\(\xi=0\) i.e. \(x=0\)) that is usually curved, with \(y(z)\) given by the solution of \(\eta=0\) in (3.4). It is instructive to look at some special cases.

In the case studied in [4], the phase factor is absent:

\[
\psi(x, y, z) = \frac{1}{(1 - \frac{1}{2}z)^{1/4}} \text{Pe} \left( \frac{x}{(1 - \frac{1}{2}z)^{1/4}}, \frac{y - \frac{1}{2}z}{\sqrt{1 - \frac{1}{2}z}} \right) \quad (a = b = c = 0). \tag{3.5}
\]

The caustic is

\[
(y - \frac{1}{2}z)^3 = -\frac{27}{8} (1 - \frac{1}{2}z)x^2. \tag{3.6}
\]

The rib is a straight line sloping upwards, with a cusp pointing upwards (i.e. towards increasing \(y\)) for \(z<2\), with its fold lines getting wider as \(z\) increases, and reversing at \(z=2\) (figure 5a) after which the rib points downwards. This reversal is an interesting phenomenon, but is not inevitable: it can be avoided by using the ‘time’ (i.e. \(z\)) reversal symmetry of the paraxial equation (1.2), which implies that for any solution \(\psi(x,y,z)\) the function \(\psi^*(x,y,-z)\) is also a solution. For the Pearcey beam (3.5), the cusp of the conjugate solution points upwards for all \(z>0\), getting narrower as the wave propagates (figure 5b), with the straight rib line sloping downwards.
A case where the rib line is curved (‘accelerating’) is

\[ \psi(x, y, z) = \frac{1}{(1 + \frac{1}{2}z)^{1/4}} \exp \left( \frac{i \left( y^2 - 2y - \frac{1}{2}z \right)}{4 \left(1 + \frac{1}{2}z\right)} \right) \text{Pe} \left( x \left(1 + \frac{1}{2}z\right)^{1/4}, \frac{y - \frac{1}{2}z^2}{\sqrt{1 + \frac{1}{2}z}} \right) \]  

(3.7)

\[ \left( a = 0, b = \frac{1}{4}, c = \frac{1}{2} \right). \]

The caustic (figure 5c) is

\[ \left( y - \frac{1}{4}z^2 \right)^3 = -\frac{23}{8} \left( 1 + \frac{1}{2}z \right)^2 x^2, \]  

(3.8)

getting narrower as \( z \) increases, with the rib curving upwards.

Finally, an interesting case is

\[ \psi(x, y, z) = \frac{1}{(1 + \frac{1}{2}z)^{3/4}} \exp \left( \frac{i \left( x^2 + y^2 - 2y - \frac{1}{2}z \right)}{4 \left(1 + \frac{1}{2}z\right)^3} \right) \text{Pe} \left( \frac{x}{\left(1 + \frac{1}{2}z\right)^{3/4}}, \frac{y}{\sqrt{1 + \frac{1}{2}z}} \right) \]  

(3.9)

\[ \left( a = \frac{1}{4}, b = \frac{1}{4}, c = \frac{1}{2} \right). \]

The caustic (figure 5d) takes the canonical form

\[ y^3 = -\frac{27}{8} x^2, \]  

(3.10)

so the cusped edge is straight and the geometrical caustic is invariant under propagation. But the corresponding wave – the Pearcey beam – is not diffraction-free: it follows from the argument of Pe in (3.9 that the fringes decorating the cusp get broader with increasing \( z \). This case (see also [38]), is analogous to the generalized Airy beam in (1.7) and (1.8). It can also be regarded as analogous to the familiar spreading of a gaussian beam.
Figure 5. Cusped caustics of several Pearcey beams; (a) caustic (3.6) of the wave (3.5) (as studied in [4]); (b) case (a) for the ‘time’ (i.e. $z$ reversed beam; (c) curved caustic (3.8) of the wave (3.7); (d) geometrically propagation-invariant caustic (3.10) of the wave (3.9)

In the hierarchy of cuspoid diffraction catastrophes, that is, those represented by integrals over a single variable ($s$ in (1.3) and (3.1), the next member above $\text{Ai}$ (codimension 1) and $\text{Pe}$ (codimension 2) is the swallowtail, with codimension 3. In canonical form, its integral representation [22] is

$$S_{\text{w}}(\xi,\eta,\zeta) = \int_{-\infty}^{\infty} ds \exp \left( i (s^5 + \xi s^3 + \eta s^2 + \zeta s) \right), \quad (3.11)$$
and its caustic surface is given parametrically by

\[
\xi = 3\tau^2 (\zeta + 5\tau^2), \quad \eta = -\tau (3\zeta + 10\tau^2).
\]  

(3.12)

As discussed in [8], there are several possibilities for the variables \(x\) and \(y\) in the initial condition \(z=0\); we study one example here. Choosing the initial variables as \(\xi, \eta\) in (3.11) gives the evolution

\[
\psi(x,y,0) = Sw(x,y,0) \\
\Rightarrow \psi(x,y,z) = \exp(i\gamma)Sw(\xi,\eta,\zeta),
\]  

(3.13)
in which, as can be confirmed by direct substitution, the variables are

\[
\xi = x + \frac{1}{6}z\left(y - \frac{1}{2}z - \frac{3}{100}z^3\right),
\]

\[
\eta = y - \frac{1}{2}z\left(1 + \frac{3}{20}z^2\right),
\]

\[
\zeta = -\frac{1}{10}z^2,
\]

\[
\gamma = \frac{1}{10}z\left(x + \frac{1}{20}yz - \frac{1}{20}z^2 - \frac{1}{2500}z^4\right).
\]  

(3.14)
The corresponding caustic is shown in figure 6, extended to negative as well as positive \(z\). It is a distorted (i.e. noncanonical) version of the canonical swallowtail [22]. A characteristic feature for \(z < 0\) is the self-intersection line

\[
y = \frac{1}{2}z\left(1 + \frac{1}{25}z^2\right), \quad x = \frac{1}{300}z^4.
\]  

(3.15)

4. Concluding remarks

My emphasis here has been twofold: on the stability of the caustics underlying Airy and related beams, and on obtaining new classes of exact paraxial wave solutions associated with the caustics. The familiar separable Airy product wave in three dimensions is unstable, and under a
nonseparable perturbation the wave splits into the stable hyperbolic umbilic diffraction catastrophe. More generally, paraxial beams that are deformations of canonical diffraction catastrophes can be generated by including position-dependent phase factors in the initial waveform.

Here I have deliberately restricted consideration to paraxial waves with caustics and diffraction catastrophes whose codimension does not exceed three, and regarded the parameters in the singularities as spatial coordinates. But the hierarchy of stable caustics continues to higher codimension, and the corresponding additional parameters can represent any quantities controlling the ray and wave geometry; for such catastrophes, the spatial coordinates represent special sections of the singularities. Examples of higher singularities that have been studied experimentally are the parabolic and symbolic and $X_9$ catastrophes in light focused by water drops [39], and sections of the butterfly
catastrophe in optical beams [8]. In addition, paraxial beams whose cross-sections contain four [40] or five [41] cusps have been created, though not interpreted as unfoldings of higher catastrophes. The most conceptually far-reaching example of this kind concerns the fluctuations of random short waves, which are predicted [42, 43] to depend on the full hierarchy of singularities.

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**Figure captions**

Figure 1. Rays corresponding to Airy beams for (0≤z≤4). (a) the pure Airy case (1.6); (b) the generalization (1.9) for a=1

Figure 2. (a) caustic in plane section of separable 3D Airy beam (2.1); (b) section of the hyperbolic umbilic when the initial Airy wave is not separable, as in (2.3)

Figure 3. (a) caustic surface for separable Airy beam (2.1); (b) caustic surface for nonseparable hyperbolic umbilic beam (2.3) with a=1/2

Figure 4. Caustic of the elliptic umbilic catastrophe

Figure 5. Cusped caustics of several Pearcey beams; (a) caustic (3.6) of the wave (3.5) (as studied in [4]); (b) case (a) for the ‘time’ (i.e. z reversed beam; (c) curved caustic (3.8) of the wave (3.7); (d) geometrically propagation-invariant caustic (3.10) of the wave (3.9)

Figure 6. Distorted swallowtail caustic of the wave (3.13) and (3.14)