Curl force dynamics

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Abstract

Curl forces, governing the acceleration of particles in Newtonian dynamics, are position-dependent but not derivable from a potential. A physical example is forces exerted on small particles by light. Although nonconservative, curl forces are not dissipative because volume in the position-velocity state space is preserved. When the force has rotational symmetry, for example when generated by a single optical vortex, particles spiral outwards and escape, even with an attractive gradient force, however strong. Without rotational symmetry, and for dynamics in the plane, possible constants of motion can be sought numerically using the Volume of Section (VoS): dots representing times along an orbit satisfying a condition (e.g. crossing the x axis), in the three-dimensional space of the remaining variables. For some curl forces, e.g. optical fields with two opposite-strength vortices, dots in the VoS lie on surfaces, indicating a hidden constant of motion. For other curl forces, e.g. those from four vortices, the dots form ‘dust cloud’ patterns, apparently exploring volumes in an unfamiliar kind of chaos, suggesting that no constant of motion exists. The dynamics generated by optical vortices could be studied experimentally.

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1. Introduction

This concerns Newtonian dynamics driven by forces $F(r)$ depending on position $r$ (but not velocity), whose curl is not zero so they are not derivable from a scalar potential \[1\]. Thus the acceleration is

$$\ddot{r} = F(r), \quad \nabla \times F \neq 0.$$ \hspace{1cm} (1.1)

Motion governed by curl forces is nonconservative: the work done by $F(r)$ depends on the path. It is also non-dissipative, because \[1\] the flow preserves volume in the position-velocity state space ($r, v = \dot{r}$): there are no attractors. In the absence of a potential, there is usually no underlying hamiltonian or lagrangian structure, so Noether’s theorem does not apply: the link between symmetries and conservation laws is broken, as elementary examples \[1\] have demonstrated.

Without a hamiltonian, there is no conserved energy. Our aim here is to investigate whether there are other conserved functions of the variables ($r, v$). Alternatively stated, we ask about the dimensionality of regions in state space explored by typical orbits. The extreme case, of most interest, would be where there are no conserved quantities at all, and motions explore regions of full dimensionality densely. We will not be able to answer these questions definitively using analytical arguments, but will present suggestive numerics indicating rich structures that deserve to be explored further.

Although the usefulness of curl forces as physical models has been the subject of dispute in engineering mathematics \[2\], their applicability in optics is not in doubt and their nonconservative nature has been recognized \[3-8\]. In this paper we will use optical curl forces as examples, to illustrate more general curl force dynamics. We consider the
force on a small polarizable particle in a monochromatic light field $\psi(r)$; if $a$ is the ratio of imaginary and real parts of the polarizability [8], the optical force is proportional to

$$\mathbf{F} = -\nabla |\psi|^2 + a \text{Im} \left[ \psi^* \nabla \psi \right].$$

(1.2)

The second term is a curl force if

$$\nabla \times \text{Im} \left[ \psi^* \nabla \psi \right] = \text{Im} \left[ \nabla \psi^* \times \nabla \psi \right] \neq 0.$$  

(1.3)

We should dispel a possible confusion. In optics, the term ‘curl force’ has sometimes [9, 10] been used in a different sense from (1.1), to denote forces that are the curl of a vector potential $\mathbf{A}$. To relate the two terminologies, we first note that any $\mathbf{F}$ can be separated into its curl-free and divergence-free parts, given respectively by a scalar and a vector potential:

$$\mathbf{F} = \mathbf{F}_{\text{grad}} + \mathbf{F}_{\text{curl}}, \text{ where } \mathbf{F}_{\text{grad}} = -\nabla \phi, \quad \mathbf{F}_{\text{curl}} = \nabla \times \mathbf{A},$$

$$\Rightarrow \quad \nabla \times \mathbf{F}_{\text{grad}} = 0, \quad \nabla \cdot \mathbf{F}_{\text{curl}} = 0.$$  

(1.4)

In this representation, the curl condition (1.1) is

$$\nabla \times \mathbf{F} = \nabla \times \mathbf{F}_{\text{curl}} = \nabla \nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A} \neq 0.$$  

(1.5)

When this holds, we call $\mathbf{F}_{\text{curl}}$ a pure curl force.

The separation (1.4) is not unique, because a gradient $\nabla \phi_1$ can be added and subtracted from each part:

$$\mathbf{F}_{\text{grad}1} = \mathbf{F}_{\text{grad}} - \nabla \phi_1, \quad \mathbf{F}_{\text{curl}1} = \mathbf{F}_{\text{curl}} + \nabla \phi_1.$$  

(1.6)

To maintain the separation, $\phi_1$ must be related to a new vector potential $\mathbf{A}_1$ by
\[ \nabla \times \mathbf{A}_1 = \nabla \phi_1, \quad \text{i.e.} \quad \nabla^2 \phi_1 = 0, \quad \nabla \times \nabla \times \mathbf{A}_1 = \nabla \nabla \cdot \mathbf{A}_1 - \nabla^2 \mathbf{A}_1 = 0. \quad (1.7) \]

For motion in the plane \( \mathbf{r}(x, y) \), \( \mathbf{A}_1 \) can be chosen to lie in the perpendicular direction \( \mathbf{e}_z \), and its magnitude – the ‘stream function’ \( \mathbf{A}_1 \) – satisfies

\[ \nabla \times \mathbf{A}_1 \mathbf{e}_z = \left( \partial_y \mathbf{A}_1, -\partial_x \mathbf{A}_1 \right) = \nabla \phi_1 = \left( \partial_x \phi_1, \partial_y \phi_1 \right), \quad (1.8) \]

whose solution (see e.g. [11]) is

\[ \mathbf{A}_1(x, y) = C - \int_0^x dx' \partial_y \phi_1(x', 0) + \int_0^y dy' \partial_x \phi_1(x, y'). \quad (1.9) \]

An optical curl force, defined by (1.2) and (1.3), is also a pure curl forces, in the sense defined by (1.4) and (1.5), if

\[ \nabla \cdot \text{Im} \left[ \psi^* \nabla \psi \right] = \text{Im} \left[ \psi^* \nabla^2 \psi \right] = 0, \quad \text{e.g. if } \nabla^2 \psi = A\psi (A \text{ real}). \quad (1.10) \]

So, optical curl forces generated by fields satisfying the Helmholtz or Laplace equations are also pure curl forces.

Our examples, illustrating what we think are general features of curl force dynamics, will be concerned with curl forces in the plane, usually generated by optical fields with one or more vortices. The conservative (i.e. gradient-force) counterpart of this dynamics has been extensively studied (also in three dimensions) e [12-14].

The structure of the paper is as follows. Section 2 concerns forces with rotational symmetry. We show the unexpected result that in a large class of cases the existence of a curl force means that the particle always escapes to infinity; even an arbitrarily strong accompanying attractive gradient force fails to restrain it. Section 3 concerns forces that do not
possess rotational symmetry. The state space \((r, v)\) is four-dimensional. As a numerical tool for displaying possible constants of motion, we introduce (section 3.1) the Volume of Section (VoS), by analogy with the Poincaré surface of section familiar in hamiltonian dynamics. We give two examples of curl force dynamics, associated with several optical vortices of zero total strength. One (section 3.2) indicates the existence of a hidden constant of motion (hidden, in the sense that we do not know its functional form). The other (section 3.3) indicates no constants at all, i.e. orbits exploring a 4D region of the state space, in what seems an unfamiliar kind of chaos. Readers interested only in this latter case should look at the ‘dust clouds’ in figure 9.

Most curl forces are nonhamiltonian, but for completeness we mention a special class \([15]\) that are, although they will not play a role in this paper. Hamiltonians anisotropic in the momenta can generate curl forces satisfying (1.1), because when there is a potential the acceleration is not its gradient.

2. Rotational symmetry: separability, and inevitable escape from a single optical vortex

It is convenient to write the most general rotationally symmetric dynamics in the plane \(r=(x, y)=r(\cos \theta, \sin \theta)\), of the type (1.1), in the form

\[
\dot{r} = -g(r)e_r + \frac{h(r)}{r}e_\theta.
\]

The first term is a radial gradient force (with potential given by the integral of \(g(r)\)), attractive if \(g(r)>0\), and the second is an azimuthal curl force. In terms of the angular momentum
\[ J = r^2 \dot{\theta}, \quad (2.2) \]

the dynamical equations for the evolution \( r(t) \) and \( J(t) \) are

\[
\begin{align*}
\dot{r} &= \frac{J^2}{r^3} - g(r), \quad \dot{J} = h(r). \quad (2.3)
\end{align*}
\]

This is already a reduction from four freedoms in (1.1) (i.e. \( x, y, v_x, v_y \)) to three (i.e. \( r, J, \dot{r} \)), reflecting the fact that changing the starting azimuth \( \theta(0) \) simply rotates the trajectory. We assume that \( h(r) \) has the same sign for all \( r \) – positive, say. Then (2.3) shows that the angular momentum always increases – an obvious consequence of the torque associated with the azimuthal force, as well as illustrating the non-applicability of Noether’s theorem (angular momentum not conserved, although there is rotational symmetry).

A further reduction is possible, to two freedoms. Generalizing our earlier analysis [1], to include the radial force in (2.1), we transform the independent time variable \( t \) to \( J \), so we now write \( r(J) \) instead of \( r(t) \). We have, using (2.3),

\[
\frac{dr}{dJ} = r' = \frac{\dot{r}}{\dot{J}} = \frac{\dot{r}}{h(r)}. \quad (2.4)
\]

Thus we get the \( r(J) \) dynamics

\[
\begin{align*}
\dddot{r} + (r')^2 \frac{\partial}{\partial r} \log[h(r)] = \frac{\left(\frac{J^2}{r^3} - g(r)\right)}{h(r)^2}. \quad (2.5)
\end{align*}
\]

This has already achieved the separation to two freedoms: \( r, r' \), but we can make the equation simpler by defining the new radial variable
Thus we seek the dynamics \( R(J) \) rather than \( r(J) \) or the original \( r(t) \). The relevant derivatives are

\[
R' = r'h(r), \quad \text{and} \quad R'' = r''h(r) + (r')^2 \frac{\partial}{\partial r} h(r),
\]

leading to the final equation

\[
R'' = \left( \frac{J^2}{r^3} - g(r) \right) \frac{1}{h(r)}.
\]

Explicitly,

\[
R = R(J), \quad r = r(R(J)), \quad R'' = \frac{d^2R(J)}{dJ^2}.
\]

Once (2.8) has been solved for \( R(J) \), \( r(J) \) can be determined by inverting (2.6). Then the time variable can be reinstated from the second equation in (2.3):

\[
t(J) = \int_{J_0}^{J} \frac{dJ_1}{h(r(J_1))}.
\]

Thus we have \( r(t) \) and \( J(t) \). Finally, the azimuth can be found from (2.2)

\[
\theta(t) = \int_{t_0}^{t} \frac{dt_1 J(t_1)}{r^2(t_1)}.
\]
This completes the separation of the radial and angular dynamics, reducing the original four-freedom system to two, i.e. $R, R'$. The reduced dynamical equation (2.8) is of hamiltonian form:

$$H(R, P, J) = E(J) = \frac{1}{2} P^2 + U(R, J),$$

where

$$U(R, J) = \frac{J^2}{2r(R)^2} + \int_{\text{const.}} r^{(R)} dr' g(r').$$

This is $J$ dependent, and since $J$ corresponds to time, the energy $E$ is not conserved; in fact it always increases:

$$\frac{dE(J)}{dJ} = \frac{\partial U}{\partial J} = \frac{J}{r} \geq 0.$$ (2.13)

The repulsive part $\dot{J}^2/2r(R)^2$ of the potential $U(R)$ in (2.12) always increases. Although this argument is formulated in the $(R, J)$ plane, in the original $(r, t)$ plane it carries the unexpected consequence that the particle always recedes from the origin, however strongly attractive the radial force $g(r)$ is (except when it is a hard wall, in which case the particle spirals ever closer to the wall, ever faster).

An important special case of rotational dynamics is generated by optical forces from a single isotropic vortex of order $m$, whose light wave is

$$\psi(r) = (x + iy)^m = r^m \exp(aim).$$ (2.14)

According to (1.2), the radial and azimuthal forces in (2.1) are

$$g(r) = r^{2m-1}, \quad h(r) = ar^{2m},$$ (2.15)

and the curl force is a pure curl force according to (1.10). Thus (2.8) simplifies to
\[ \rho'' = B \left[ J^2 \rho^{-(2m-3)/(2m+1)} - \rho^{-1/(2m+1)} \right], \]

where \( \rho = \frac{2m+1}{a} R, \quad B = \frac{2m+1}{a^2}. \) \hfill (2.16)

This is a variant of the Emden-Fowler equation [16, 17], with two source terms instead of one; a general solution in closed form seems unavailable. The variables are related by

\[ \rho = r^{2m+1}, \quad J = a \rho^{2m/(2m+1)}, \] \hfill (2.17)

and the Hamiltonian (2.12) simplifies to

\[ H(\rho, p_\rho, J) = E(J) = \frac{1}{2} p_\rho^2 + U(\rho, J), \quad \text{where} \]

\[ U(\rho, J) = \frac{1}{2} B(2m+1) \left( J^2 \rho^{-2/(2m+1)} + \frac{\rho^{2m/(2m+1)}}{m} \right). \] \hfill (2.18)

The potential, whose perpetual increase causes the particle to escape, is illustrated in figure 1.

![Figure 1. Potential in (2.18) for m=2 and different J](image)

In an alternative expression of the dynamics (2.1), position is denoted by the complex variable

\[ z = x + iy. \] \hfill (2.19)
Now the evolution (1.1) is

$$\ddot{z} = \left( -g(|z|) + ih(|z|)/|z| \right) \frac{z}{|z|}. \quad (2.20)$$

In this representation, the one-vortex dynamics takes the form of a stationary nonlinear Schrödinger equation:

$$\ddot{z} = |z|^{2m-2} (-1 + ia) z. \quad (2.21)$$

Although a general solution seems unavailable, a particular solution can be found, representing a particle escaping to infinity at time \(t=t_0\) while spiralling logarithmically (figure 2). For \(m>1\), this is

$$z(t) = \frac{C \exp \left( iq \log \left( \frac{t_0}{t_0 - t} \right) \right)}{(t_0 - t)^{1/(m-1)}}, \quad (2.22)$$

in which

$$q = \frac{m+1 + \sqrt{(m+1)^2 + 4ma^2}}{2a(m-1)}, \quad C = \left( \frac{q(m+1)}{a(m-1)} \right)^{1/(2m-2)}. \quad (2.24)$$
Figure 2. One-vortex track (2.22), for \(0 \leq t \leq 0.99 t_0\), \(t_0=1\), \(m=2\), \(a=1\), corresponding to initial conditions \((x_0, y_0, v_{x0}, v_{y0})=(3.269, 0, 3.269, 11.642)\)

This solution corresponds to the initial conditions

\[
x(0) = \frac{C}{(t_0)^{1/(m-1)}}, \quad y(0) = 0, \quad v_x(0) + iv_y(0) = \frac{C(1+i(m-1)q)}{(m-1)(t_0)^{m/(m-1)+iq}}.
\] (2.25)

For the excluded case \(m=1\), (2.21) is a linear equation, whose spiralling solutions are

\[
z(t) = c_+ \exp(it\sqrt{1-ia}) + c_- \exp(-it\sqrt{1-ia}).
\] (2.26)

3. More optical vortices: seeking chaos

3.1 The Volume of Section

What distinguishes dynamics under curl forces from hamiltonian dynamics? A fundamental difference would be motion exploring a full-dimension region of the \((r, v)\) state space. For motion in the \(r=(x, y)\) plane, this would be exploration of a four-dimensional region. As a numerical tool for investigating this possibility, we introduce the Volume of Section (VoS), defined as follows. We select times \(t_n\) along an orbit \(r(t)\) satisfying some condition, for example \(x(t_n)=0\). At such times, we plot the other three variables, for example \((y(t_n), v_x(t_n), v_y(t_n))\), and examine the dot patterns in this three-dimensional space – the volume of section – after long times. There are four natural choices for the VoS, corresponding to \(x(t_n)=0, y(t_n)=0, v_x(t_n)=0, v_y(t_n)=0\); we will denote these by VoSx, VoSy, VoSv_x, VoSv_y. Although the full dynamics in 4D state space is volume-preserving, the absence of underlying symplectic structure means that the 3D map between successive dots on each VoS does not preserve volume.
This contrasts with the area-preserving 2D Poincaré map on a 3D constant-energy hypersurface in hamiltonian dynamics.

If there is one conserved quantity, as in a hamiltonian system, the dots in each VoS will lie on a surface, and if the dynamics is chaotic the surface will be partly or wholly filled. If there is an additional constant, as in an integrable system, the dots will lie on a curve. And in the situation we are seeking, when there is no conserved quantity, the dots will fill a volume.

3.2 Two vortices

To explore the possibilities, we need to choose suitable forces $F(r)$. As we have seen, the optical curl force from a single vortex always leads to escape, even in the presence of an attractive gradient force. The escape is associated with the continuous increase of angular momentum caused by the torque from the curl force. It is natural to try to avoid this by exploring the curl force from two vortices of opposite strength, so the net torque is zero. The simplest such optical field, representing two vortices on the $x$ axis, at $x=+1$ with strength $-1$ and at $x=-1$ with strength $+1$, is

$$\psi_1(r) = (x-1+iy)(x+1-iy).$$ \hfill (3.1)

This generates a curl force. Although it is not a pure curl force, because (cf. (1.10)) $\nabla^2\psi_1 = 4$, it can easily be made so, for example by adding $-2y^2$.

We note in passing that the associated gradient force,

$$F_{\text{grad}}(r) = -\nabla|\psi_1(r)|^2 = -\left\{x(r^2 - 1), y(r^2 + 1)\right\},$$ \hfill (3.2)

is integrable as well as hamiltonian. Of course the energy
\[ E_1 = \frac{1}{2}(v_x^2 + v_y^2) + |\psi_1(r)|^2 = \frac{1}{2}(v_x^2 + v_y^2) + (r^2 + 1)^2 - 4x^2 \] (3.3)

is conserved. And, as can easily be confirmed, the following quantity is also conserved:

\[ K_1 = (r \times \mathbf{v} \cdot \mathbf{e}_z)^2 - 4v_y^2 - 8y^2 \left( r^2 + 2 \right). \] (3.4)

Figure 3 shows the track of an orbit in the \((x,y)\) plane, with a pattern clearly illustrating the integrability.

These conserved quantities are destroyed when the gradient force is combined with the corresponding curl force according to (1.2):

\[ \mathbf{F}_{1\text{curl+grad}}(r, \alpha) = -4 \left\{ x(r^2 - 1), y(r^2 + 1) \right\} + 2a \left\{ -2xy, x^2 - y^2 - 1 \right\}. \] (3.5)

Figure 4 shows the pattern of directions of this force, inspiralling to the two vortices (circles), with a stagnation point (square) on the negative \(y\) axis, and asymptotically attractive.
This is a promising choice for a curl force, because it fails the
‘anisotropic hamiltonian’ test derived elsewhere (equation (2.6) in [15].
Figure 5 shows an orbit generated by this force, in the (x,y) and (v^x,v^y)
(hodograph) planes. It appears irregular, indicating that the system is not
integrable: there is at most one constant of motion. Of course this cannot
be energy because the force is non-conservative.
To investigate whether in fact there is such a constant of motion, we show in figure 6 the corresponding VoS patterns. The dots clearly lie on surfaces, indicating that a constant of motion, that is, a conserved function of $r$ and $v$, associated with the force (3.5), does exist. We do not know what this constant is; the complicated form of the surfaces suggests that it is not a simple function.
3.3 Four vortices

We have investigated two-vortex fields more general than (3.5), in which the gradient force is not of optical type, suggesting curl force dynamics with no constants of motion. But forces in which both the curl and gradient parts are optical are not only conceptually simple but could also be explored experimentally. For this reason, we now consider the optical force from four alternating-sign vortices arranged on a square:
\[ \psi_2 = (x+1+i(y+1))(x+1-i(y-1))(x-1+i(y-1))(x-1-i(y+1)). \] (3.6)

Again we create the corresponding curl + gradient force according to (1.2):

\[ F_{2\text{curl+grad}} = -8\left(x\left(r^6 + 4x^2 - 12y^2\right), x\left(r^6 + 4y^2 - 12x^2\right)\right) + 8a\left(x\left(r^2(x^2 - 3y^2) + 4\right), -y\left(r^2(y^2 - 3x^2) + 4\right)\right). \] (3.7)

Figure 7 shows the pattern of directions of this force, inspiralling to the four vortices (circles), with three stagnation points (squares) on the x axis, and asymptotically attractive.

Figure 8 shows an orbit generated by this force, in the \((x,y)\) and in the \((v_x,v_y)\) (hodograph) planes. As with the two-vortex orbit in figure 5,
this looks irregular, indicating that the dynamics is not integrable, and again raising the question of whether any constant of motion exists.

Figure 8. Orbit in the \((x, y)\) plane (left) and the \((v_x, v_y)\) (hodograph) plane (right) for the four-vortex force \(F_{\text{curl+grad}}\) (equation (3.7)), for \(a=1.8, (x_0, y_0, v_{x0}, v_{y0})=(0, 0, -0.1, -0.1)\), and \((0 \leq t \leq 75)\).

Figure 9 shows the corresponding VoS dot patterns for this four-vortex dynamics. They are very different from the two-vortex VoS patterns in figure 6. It is hard to interpret these ‘dust cloud’ patterns as anything other than irregularly exploring a four-dimensional region in the state space. If correct, this means that curl forces can generate an unfamiliar kind of chaos – contrasting with hamiltonian chaos, where because of energy conservation a hypersurface is explored. Preliminary computations support the conjecture that orbits with nearby initial conditions separate exponentially. The dust cloud patterns appear to possess a complicated structure; there are ‘holes’, almost or totally devoid of dots, and some regions appear denser than others, almost hinting that the orbit would condense onto a chaotic attractor – which of course it cannot do because the full dynamics is 4D volume-preserving.
4. Concluding remarks

This study reveals a variety of structures in orbits governed by rather simple curl forces, including those exerted on small particles from optical waves with vortices. For a single optical vortex, the orbits spiral outwards and always escape. When there are two vortices, it sometimes appears that there is a hidden constant of motion, even though there is no conserved energy. Most interesting are other cases, for example four vortices, where it seems there are no conserved quantities at all. Further
computations, not reported here, show that these different behaviours are not exceptional.

We regard this as an exploratory study, raising several questions:

- Are the dust cloud patterns such as those in figure 9 typical in situations where orbits are bounded?
- How can their structures be characterised?
- For optical fields with an infinite periodic array of vortices, with total strength zero in each unit cell, can motion under the associated curl plus gradient forces be chaotic and explore the full state space dimensionality – and, if so, is this typical or exceptional? (This would extend previous studies [13, 14] of orbits in optical lattices under conservative forces.)
- Are periodic orbits dense, as in hamiltonian systems? This is not a trivial question because, as we discussed elsewhere (section 4 of [15], the nonconservative nature of curl forces imposes strong restrictions on the forms of periodic orbits.
- Where numerics strongly suggests a constant of motion, as in the case considered in section 3.2 and illustrated in figure 6, is there any analytical way, general for curl forces (1.1), to establish its existence and characterise it?
- Can the curl force dynamics we have identified theoretically be seen experimentally, in the motion of small polarizable particles in vacuum, governed by forces from optical fields with several vortices?

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References


Figure captions

Figure 1. Potential in (2.18) for $m=2$ and different $J$

Figure 2. One-vortex track (2.22), for ($0 \leq t \leq 0.99 t_0$), $t_0=1, m=2, a=1$, corresponding to initial conditions $(x_0,y_0,v_{x0},v_{y0})=(3.269, 0, 3.269, 11.642)$

Figure 3. Orbit in $(x,y)$ plane for gradient (i.e. not curl) force $F_{1\text{grad}}$ (equation (3.2)) from two vortices, for $0 \leq t \leq 100$ and with $(x_0,y_0,v_{x0},v_{y0})=(1/2, 1/2, 1/2, 1/2)$

Figure 4. Two-vortex force $F_{1\text{curl+grad}}$ (equation 3.5)) for $a=2$

Figure 5. Orbit in the $(x, y)$ plane (left) and the $(v_x,v_y)$ (hodograph) plane (right) for the two-vortex force $F_{1\text{curl+grad}}$ (equation (3.5)), for $a=2$, $(x_0,y_0,v_{x0},v_{y0})=(1/2,0,-1,2)$, and $0 \leq t \leq 1000$

Figure 6. Two-vortex Volume of Surface patterns corresponding to figure 5, for ($0 \leq t \leq 50000$): (a) VoSx; (a) VoSy; (a) VoS$v_x$; (a) VoS$v_y$

Figure 7. Four-vortex force $F_{2\text{curl+grad}}$ (equation 3.9))

Figure 8. Orbit in the $(x, y)$ plane (left) and the $(v_x,v_y)$ (hodograph) plane (right) for the four-vortex force $F_{2\text{curl+grad}}$ (equation (3.7)) , for $a=1.8$, $(x_0,y_0,v_{x0},v_{y0})=(0,0,-0.1,-0.1)$, and ($0 \leq t \leq 75$)

Figure 9. Four-vortex Volume of Surface ‘dust cloud’ patterns corresponding to figure 8, for ($0 \leq t \leq 10000$): (a) VoSx; (a) VoSy; (a) VoS$v_x$; (a) VoS$v_y$