lecture 1

The singularities of light: intensity, phase, polarization

Geometry dominates modern optics, in particular through understanding light in terms of its singularities. There are different levels of description in optics, each characterised by different singularities. Analogous considerations apply to other types of wave: quantum, acoustic, elastic, water….

The coarsest level is geometrical optics, in which light fields are described by families of rays. Here the singularities are caustics: focal lines and surfaces, that is, the envelopes of ray families, on which the intensity diverges. These bright-light singularities are classified by the mathematics of catastrophe theory [1-4], providing a list of the geometric forms of caustics that are stable under generic perturbations. Many phenomena are described by caustics: rainbows [5], the bright lines of focused sunlight on the bottoms of swimming-pools, bright-edged shadows of floating insects [6], and twinkling starlight [7, 8] (whose statistics involve a competition between singularities [9]). In wave optics, these singularities are smoothed by diffraction, which decorates them with rich and ubiquitous interference patterns [10, 11], described by a new class of special functions (‘diffraction catastrophes’ [12]),
represented by oscillatory integrals (chapter 36 of [13], [14, 15]. In white light, caustics display interesting colours [16, 17].

Wave optics, when represented by complex scalar wavefunctions, also introduces the additional concept of phase, which has its own singularities. Equivalent terms for phase singularities are optical vortices, nodal manifolds or wavefront dislocations [18, 19]. On phase singularities, the light intensity is zero, so these are the singularities of dark light. Geometrically, they are lines in space, or points in the plane, around which the phase changes by a multiple (generically 1) of $2\pi$. Phase singularities are complementary to caustics, not only because the former are dark (zero intensity) and the latter are bright (infinite intensity), but also in the sense of Bohr: caustics are prominent features in the short-wave asymptotic regime, in which phase singularities are too close to be clearly resolved – because these are fine-scale features, clearly discernable only in the opposite case of high magnification, where caustics are smoothed out and so are no longer distinct features. Phase singularities can form intricate patterns, for example as fine detail in diffraction catastrophes [10, 11, 20] and near spiral phase plates [21, 22]. They can organise the coloured interference patterns formed by white light [23-25]. They occur in all types of quantum [26-32] or classical (e.g. acoustic [33] and tide [34-38]) waves and have been extensively reviewed [39-41]. In three dimensions, phase singularity lines can be linked and knotted [42-47].

Incorporating the vector (electromagnetic) nature of light brings further singularities, corresponding to the new physical property thereby introduced, namely polarization. Polarization singularities are lines in three dimensions, of two types [48-51]: C singularities, on which the polarization is purely circular, and L singularities, on which the
polarization in purely linear. In direction space, polarization singularities play a central role in crystal optics [52-57] (notably conical refraction [58, 59]), and in the pattern of polarized light in the blue sky [60]. The C and L lines are different for the electric and magnetic fields [51], but coincide for paraxial fields [61].

Historically, all three levels of singularity can be considered to have originated in the same decade: the 1830s [62].

As well as representing physics at each level, these optical and wave geometries illustrate the idea of asymptotically emergent phenomena [63]. The levels form a hierarchy, with each deeper level of theory eliminating the earlier singularities and generating new ones. Consequences of this approach are predictions that the phase singularities of scalar light will have quantum cores [64, 65], and large momentum transfers to small particles [66].

References

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Superoscillation and weak measurement

Contrary to naive intuition, band-limited functions can oscillate arbitrarily faster than their fastest Fourier component, over arbitrarily long intervals [1]. There is no contradiction with the uncertainty principle, because where such ‘superoscillations’ occur the functions are exponentially weak [2-5]. In random waves with wavenumber $k$, such as typical monochromatic optical fields, superoscillations are unexpectedly common [6]: for substantial fractions of the domain (one-third in two dimensions [7, 8]), the local wavenumber (modulus of phase gradient [9]) exceeds $k$. Superoscillations are related to ‘supergain’ and ‘superdirectivity’ in antenna theory [10, 11]. They have implications for signal processing [1], and raise the possibility of sub-wavelength resolution microscopy without evanescent waves [12-16].

In quantum mechanics, where the concept originated, superoscillations correspond to weak measurements [17-19], in which pointer shifts [20, 21] can correspond to ‘weak values’ of observables (e.g. photon momenta [9, 22, 23]) far outside the range represented in the quantum state (i.e. outside the spectrum of the operator representing the observable). For typical quantum weak measurements, ‘superweak’ values are unexpectedly common [24, 25].

For relativistic waves, superoscillations correspond to superluminal group velocities [26]. An application is to a recent (now discredited) experiment [27, 28] suggesting superluminal speeds for neutrinos; although in principle a weak measurement of neutrino speed could lead to a superluminal result without violating causality (analogous to an earlier
demonstration for light [29]), analysis showed [30] that the effect is too small to explain the speed that was claimed (see also [31-33]).

References


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