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SOLUTION OF THE RAMAN-NATH EQUATION FOR LIGHT DIFFRACTED BY ULTRASOUND AT NORMAL INCIDENCE

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Synopsis

Estimates are derived for the number of diffracted beams which appear with appreciable intensity when light at normal incidence traverses ultrasonic waves in a liquid. The estimates are used to justify an approximate method for solving the Raman-Nath equation with an analogue computer; this approach yields results for all values of the parameters attained in experiments so far. Numerical results are compared with the measurements of Klein and Hiedemann, showing for the first time that even outside the Bessel-function limit the Raman-Nath equation leads to predictions in quantitative agreement with experiment.

1. *Introduction.* In view of the fact that the Raman-Nath theory of ultrasonic diffraction at normal incidence[1]), published nearly thirty years ago, has formed the basis of a large number of theoretical investigations of the phenomenon [e.g. refs 2 and 3], it is rather surprising that it has never been developed sufficiently to be compared with experiment outside the range where the Bessel-function expressions are valid. This development is carried out in the present paper. Numerical results are obtained with the aid of an analogue computer and compared with experiment in section 3; the theory behind the method of calculation is presented in section 2.

We can state the theory of Raman and Nath as follows: if light of wave number k at normal incidence traverses a vessel of depth D filled with liquid irradiated by ultrasound of wave number U which causes its refractive index to deviate by a maximum of μ_1 from its mean value μ_0, then the intensity of the n'th diffracted light beam emerging from the vessel is

$$I_n(x) = |\psi_n(x)|^2,$$

where the amplitude $\psi_n(x)$ satisfies the Raman-Nath differential-difference equation

$$2\frac{\partial \psi_n}{\partial x} - \psi_{n-1} + \psi_{n+1} = i\rho n^2 \psi_n;\ \psi_n(0) = \delta_{n,0}, \tag{1}$$

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involving the two parameters

$$x = k\mu_1 D$$

and

$$\rho = \frac{U^2}{\mu_0\mu_1 k^2}.$$

In experiments so far, x has ranged from 0 to about 20, and ρ from 0.01 to 100. A detailed account of the basis of the theory and the solutions of (1) is given in ref. 4.

2. *Truncation of the Raman-Nath equation.* If we make the stipulation

$$\psi_n = 0 \quad \text{when} \quad |n| > N, \tag{2}$$

the Raman-Nath equation (1) reduces to a finite set of linked differential equations for the amplitudes ψ_{-N} through ψ_{+N}. The solutions of the truncated set of equations will not be the exact amplitudes, but will provide an approximation to them which improves as N increases. The physical interpretation of this procedure is very simple: for a particular set of values of the parameters x and ρ, only a finite number M of spectra appear on each side of the central beam ($2M + 1$ spectra in all) with non-negligible intensity. If $N \geq M$ then the amplitudes we discard according to (2) are negligible anyway, and the approximation is an excellent one. It is obviously good enough to let N equal M.

In section 3 the truncated set of equations will be solved by analogue computation; before this can be done, however, we must find some estimate for the number M, otherwise we should have to rely on trial and error, or experiment, to tell us where to truncate the equations.

To derive basic inequalities satisfied by the amplitudes, we multiply the Raman-Nath equation and its complex conjugate by ψ_n^* and ψ_n respectively, and add the two resulting equations; this gives

$$2\frac{\partial}{\partial x}(\psi_n\psi_n^*) + \psi_n^*\psi_{n+1} - \psi_{n-1}^*\psi_n + \psi_{n+1}^*\psi_n - \psi_n^*\psi_{n-1} = 0.$$

If this is summed over n from $-\infty$ to $+\infty$ the terms in $n - 1$ and $n + 1$ cancel in pairs, leaving

$$2\frac{\partial}{\partial x}\sum_{n=-\infty}^{\infty}|\psi_n(x)|^2 = 0,$$

i.e.

$$\sum_{n=-\infty}^{\infty}|\psi_n(x)|^2 = \text{const} \equiv C_0 = \sum_{n=-\infty}^{\infty}|\psi_n(0)|^2 = 1, \tag{3}$$

where the boundary condition on ψ_n at $x = 0$ has been introduced. This equation represents energy conservation in the diffraction process. Now all the terms in the sum are positive, and on physical grounds, or by (1), we have that

$$|\psi_n|^2 = |\psi_{-n}|^2,$$

so (3) implies

$$\left.\begin{aligned} |\psi_0| &\leq 1 \\ |\psi_n| &\leq 1/\sqrt{2} \qquad (n \neq 0) \end{aligned}\right\} \tag{4}$$

Now in order to get an estimate of M we need some information about the decrease of $|\psi_n|$ with increasing $|n|$, and the basic bounds (4) do not provide it. If, however, we write the Raman-Nath equation (1) in the form of an inequality,

$$|\psi_n| \leq \frac{1}{\rho n^2}\left(2\left|\frac{\partial \psi_n}{\partial x}\right| + |\psi_{n-1}| + |\psi_{n+1}|\right), \tag{5}$$

then we can use (4) in the right-hand side to get a set of bounds which does show the convergence of the sequence of $|\psi_n)$. To do this we need basic bounds, analogous to (4), for $|\partial\psi_n/\partial x|$. These are easy to get, because the Raman-Nath equation, having no coefficients depending on x, is satisfied by $|\partial^r\psi_n/\partial x^r|$ as well as by ψ_n, so an argument exactly parallel to that leading to (3) gives

$$\sum_{n=-\infty}^{\infty}\left|\frac{\partial^r \psi_n(x)}{\partial x^r}\right|^2 = C_r = \sum_{n=-\infty}^{\infty}\left|\frac{\partial^r \psi_n(0)}{\partial x^r}\right|^2.$$

The values of the derivatives at $x = 0$ can be found from the Raman-Nath equation by an iterative process; we have

$$\frac{\partial^r \psi_n(0)}{\partial x^r} = \tfrac{1}{2}\left[i\rho n^2 \frac{\partial^{r-1}\psi_n(0)}{\partial x^{r-1}} + \frac{\partial^{r-1}\psi_{n-1}(0)}{\partial x^{r-1}} - \frac{\partial^{r-1}\psi_{n+1}(0)}{\partial x^{r-1}}\right]$$

and

$$\psi_n(0) = \delta_{n,0}.$$

Thus

$$\begin{aligned} \frac{\partial \psi_n}{\partial x}(0) &= \tfrac{1}{2}(i\rho n^2 \delta_{n,0} + \delta_{n-1,0} - \delta_{n+1,0}) \\ &= \tfrac{1}{2}(\delta_{n-1,0} - \delta_{n+1,0}), \end{aligned}$$

and

$$C_1 = \sum_{n=-\infty}^{\infty}\left|\frac{\partial \psi_n}{\partial x}\right|^2 = \tfrac{1}{4}\sum_{n=-\infty}^{\infty}(\delta_{n-1,0} - \delta_{n+1,0})^2 = \tfrac{1}{2},$$

so that

$$\left.\begin{aligned} \left|\frac{\partial \psi_0}{\partial x}\right| &\leq \frac{1}{\sqrt{2}} \\ \left|\frac{\partial \psi_n}{\partial x}\right| &\leq \tfrac{1}{2} \qquad (n \neq 0). \end{aligned}\right\} \tag{6}$$

If (6) and (4) are substituted into (5), the result is

$$|\psi_n| \leq \frac{1}{\rho n^2}[1 + \sqrt{2}] = \frac{2.414}{\rho n^2} \quad (|n| > 1). \tag{7}$$

This inequality will give a value for M if we can decide just how small $|\psi_n|$ must be for it to be judged negligible. Taking account of the stated accuracy of recent experiments[5]) we choose 0.03 as the level below which an intensity can be neglected, *i.e.* $|\psi_n|^2$ to be at most 3% of $\sum_n |\psi_n|^2$. An upper bound for M is thus given by the solution of

$$\left(\frac{2.414}{\rho n^2}\right)^2 = 0.03,$$

and we have

$$M < M_1 = \frac{3.73}{\sqrt{\rho}}. \tag{8}$$

From the manner of its derivation this result is only applicable for $M_1 > 1$.

It is possible to get better bounds for M by using the inequality (7) in (5), instead of the basic bounds (4). To do this consistently we also need an estimate for $|\partial\psi_n/\partial x|$ which is better than its basic bound (6). This can easily be found, as can the basic bound for $|\partial^2\psi_n/\partial x^2|$, which is also required. The inequality for $|\psi_n|$, obtained after some calculation, is

$$|\psi_n| \leq \frac{1}{\rho^2 n^4}\left[(\rho^2+3)^{\frac{1}{2}} + 2 + 2(1+\sqrt{2})\left(1 + \frac{3n^2-1}{(n^2-1)^2}\right)\right] \quad (|n| > 2). \tag{9}$$

For $|n|$ greater than about 4 the relative error introduced by neglecting the last term in (9) is less than 10%, and we shall neglect this term; the resulting inequality for M is

$$M < M_2 = \frac{2.51}{\sqrt{\rho}}\left(1 + \frac{(\rho^2+3)^{\frac{1}{2}}}{6.828}\right)^{\frac{1}{4}} < \frac{2.67}{\sqrt{\rho}} \quad (\rho \lesssim 1). \tag{10}$$

This is a better estimate than (8) (for $\rho < 1$), whereas (9) is not necessarily better than (7).

A slightly better bound is obtained at the cost of much calculation by iterating the whole process again. The result is

$$M < M_3 = \frac{2.16}{\sqrt{\rho}}\left[1 + \frac{6(\rho^2+3)^{\frac{1}{2}} + (\rho^4 + 33\rho^2 + 10)^{\frac{1}{2}}}{17.656}\right]^{\frac{1}{8}}$$
$$< \frac{2.44}{\sqrt{\rho}} \quad (\rho \lesssim 1). \tag{11}$$

The trend of (8), (10) and (11) leads us to estimate the number of spectra visible on each side of the central beam as being the nearest integer to

$$M = \frac{2}{\sqrt{\rho}} \; (\rho \lesssim 1) \tag{12}$$

Now that we know the value of M, and hence the best place to truncate the Raman-Nath equations, the next step is to solve the resulting finite set of linked differential equations; this is done in the next section.

3. *Numerical solution of the Raman-Nath equation and comparison with experiment.* We have to find the amplitudes $\psi_n(x)$, defined by (1) and (2), with N equal to the nearest integer to $2/\sqrt{\rho}$. The fact that the equations have one-point boundary conditions makes it natural to try to solve them by analogue computation, with x as the independent variable.

It is necessary to separate the amplitudes into their real and imaginary parts, because an analogue machine can operate only with real quantities. If we put

$$\psi_n(x) = A_n(x) + iB_n(x),$$

and use the symmetry relation

$$\psi_n(x) = (-)^n \psi_{-n}(x),$$

we get

$$\left.\begin{aligned}
\frac{\partial A_0}{\partial x} &= A_1 \\
\frac{\partial B_0}{\partial x} &= -B_1 \\
\left.\begin{aligned}
\frac{\partial A_n}{\partial x} &= -\tfrac{1}{2}(\rho n^2 B_n - A_{n-1} + A_{n+1}) \\
\frac{\partial B_n}{\partial x} &= -\tfrac{1}{2}(-\rho n^2 A_n - B_{n-1} + B_{n+1})
\end{aligned}\right\} & 0 < n < N \\
\frac{\partial A_N}{\partial x} &= -\tfrac{1}{2}(\rho N^2 B_N - A_{N-1}) \\
\frac{\partial B_N}{\partial x} &= -\tfrac{1}{2}(-\rho N^2 A_N - B_{N-1}) \\
A_n(0) = \delta_{n,0} \quad & B_n(0) = 0
\end{aligned}\right\} \tag{13}$$

The required quantities are

$$I_n(x) = A_n^2(x) + B_n^2(x).$$

In an analogue machine the various terms in (13) are represented by voltages, and it is essential that these remain bounded as x varies. We showed in section 2 that $|\psi_n|$, $|\partial\psi_n/\partial x|$ and $\rho n^2|\psi_n|$ are all bounded by constants of order unity, and it is a trivial matter to show that the proofs hold for the solutions of the truncated equations as well as for those of the infinite set. All terms in (13) are thus bounded. Since the independent variable x is represented in the computer by time, the important consequence of this boundedness is that the machine solution will always be inherently stable however long it computes. This is to be contrasted with the digital-computer methods recently devised by Gill[6]), Hargrove[7]) and Hance[8]), where stability for large x has to be built in at the cost of greatly complicating the programme.

The solution was performed on the PACE TR48 analogue computer at the Department of Theoretical Physics at the University of St Andrews. With the machine used to full capacity it would have been possible to calculate the intensities I_0 through I_{18}. Nobody seems ever to have reported seeing a higher number of spectra than this, so that a TR48 is sufficient to solve the ultrasonic diffraction problem under all experimental conditions. In the only comprehensive set of measurements, performed by Klein and Hiedemann[5]), ρ always exceeded about 0.1, and under these conditions $|\psi_n|$ is negligible whenever $|n| > 2/(0.1)^{\frac{1}{2}} = 6.323$. The solutions were thus computed for $N = 6$.

Using an XY plotter, curves showing the intensities I_n of the diffracted beams versus x were drawn for values of ρ ranging from 1.0 to 0.1 at intervals of 1.0; the range of x was 0 to 29. It was found that those spectra for which

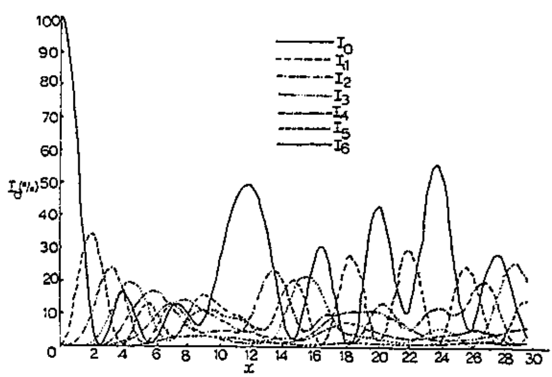

Fig. 1. Solution of Raman-Nath equation for $\rho = 0.1$.

$|n|$ exceeded $2/\sqrt{\rho}$ were so faint that their intensities on the plotted curves were indistinguishable from the x axis, so that the conclusions of section 2 are verified. Sample curves from the set, for $\rho = 0.1$, $\rho = 0.4$, and $\rho = 1.0$, constitute figures 1, 2 and 3. They show quite clearly the changing nature of the diffraction pattern as ρ varies: when ρ is very small the intensities

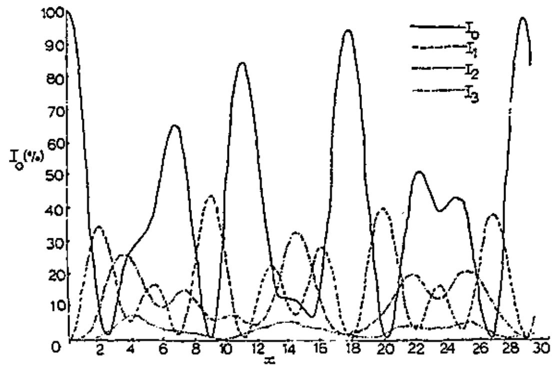

Fig. 2. Solution of Raman-Nath equation for $\rho = 0.4$.

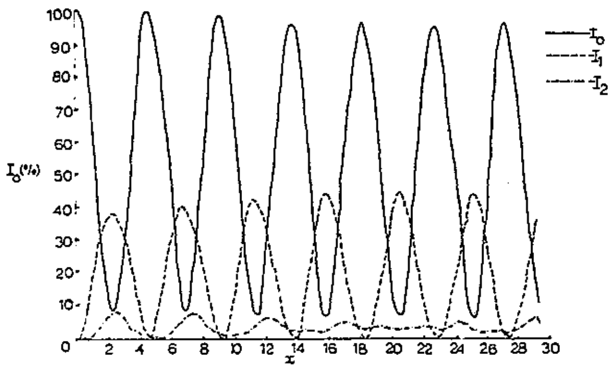

Fig. 3. Solution of Raman-Nath equation for $\rho = 1.0$.

resemble Bessel functions for x less than about 4, while for $\rho \sim 1$ the pattern is nearly periodic, and fewer lines are visible, as would be expected[3]). Figures 1 and 3 are almost identical with curves calculated in 1936 by Extermann and Wannier[9]) with enormous labour.

Analogue solution of the Raman-Nath equation is compact – the TR48 is a desk-top machine; it is economical of time – patching the truncated set (13) for $N = 6$ takes less than an hour; finally, it is sufficiently accurate – the noise level in all the calculations never exceeded 1%. It seems difficult to image a more convenient way of computing the intensities.

In the experiments of Klein and Hiedemann[5]) the zero-order intensity I_0 was measured as a function of x ($x = 0$ to 9), not at constant ρ but at constant ρx($\rho x = 0.94$, 1.26, 1.48). To compare the theory with these measurements a further series of curves was computed. These showed the zero-order intensity only, and covered the range $\rho = 0.1$ to $\rho = 2.0$ in such a way that the interval in $1/\rho$ between successive curves was 0.5; the range of x was 0 to 15. Unfortunately an XY plotter was not available for this series of curves; they were recorded by switching the computer into REP-OP mode, displaying the solution on a high-precision TEKTRONIX oscilloscope, and photographing the trace with a LAND POLAROID oscilloscope camera. This almost certainly resulted in a loss of accuracy, estimated at 5% maximum.

The photographs were used to construct curves of I_0 versus x for the three relevant values of ρx; the experimental curves were replotted on the same graphs, and the three resulting comparisons are shown in figures 4, 5 and 6.

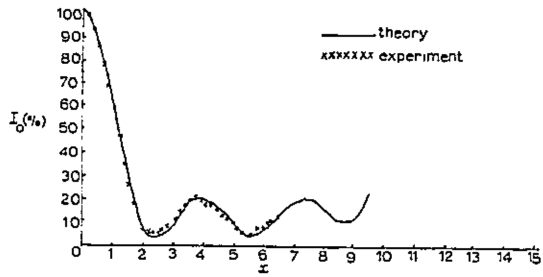

Fig. 4. Comparison of theory with experiments of Klein ($\rho x = 0.94$)

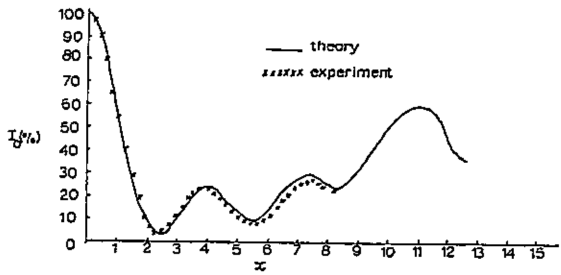

Fig. 5. Comparison of theory with experiments of Klein ($\rho x = 1.26$)

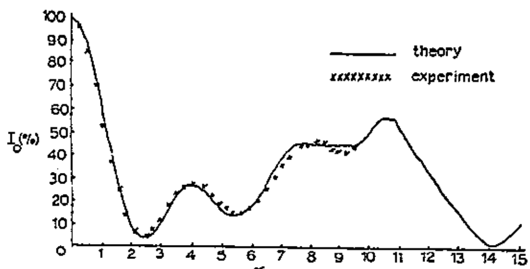

Fig. 6. Comparison of theory with experiments of Klein ($\rho x = 1.48$)

There are some slight discrepancies, but on the whole the theoretical curves lie well within the 3% tolerance given by Klein and Hiedemann; the differences are almost certainly due to the method of recording the computer solutions. These satisfying results represent the first demonstration of the correctness of the Raman-Nath theory outside the parameter range where the Bessel-function expressions are valid.

4. *Relation to other work.* The method of truncation is implicit in the work of Nath[2]), who calculated I_0 and I_1 analytically, taking $N = 1$, Nath drew curves for the case $\rho = 1$, for which the neglected intensity I_2 reaches a maximum of about 0.1 (see figure 3). Later, Van Cittert[10]) solved the truncated equations numerically for the case $\rho = 1$, $N = 2$.

More recently, Gill[6]) derived expressions for the Laplace transforms of the amplitudes in the form of infinite continued fractions whose successive convergents represent the solutions obtained by truncation for successive values of N. This continued fraction is derived independently in ref. 4, where it is used as a method of linking together the various analytic approximate solutions of (1).

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