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Uniform approximation: a new concept in wave theory

M. V. Berry

Recent advances in the mathematical theory of asymptotic approximation have given rise to the beginnings of a comprehensive theory of the propagation of short waves. Expressions for the wave functions can be constructed solely from the pattern of rays or paths occurring in the corresponding geometrical or classical problems, and do not show the infinite divergences or discontinuities characteristic of earlier geometrical theories. In this article a series of wave problems is treated descriptively. Cylindrical waves are examined first both on account of their conceptual simplicity and because the exact wave functions are fully understood. Next the rainbow and glory scattering of plane waves from spheres is considered. The third example, somewhat out of character with the others but showing the wide applicability of techniques of uniform approximation, concerns the calculation of scattering lengths, and finally a brief account of the classical problems of diffraction from edges and curved surfaces is given.

1. Introduction

Probably no mathematical structure is richer, in terms of the variety of physical situations to which it can be applied, than the equations and techniques that constitute wave theory. Eigenvalues and eigenfunctions, Hilbert spaces and abstract quantum mechanics, numerical Fourier analysis, the wave equations of Helmholtz (optics, sound, radio), Schrödinger (electrons in matter), Dirac (fast electrons) and Klein-Gordon (mesons), variational methods, scattering theory, asymptotic evaluation of integrals (ship waves, tidal waves, radio waves around the earth, diffraction of light)—examples such as these jostle together to prove the proposition. We shall be interested in one aspect of this menagerie of separate yet related matters. We ask: what is the connection between the concepts of wave theory and logically more primitive *geometrical* notions of *ray, particle path*, etc.?

One would perhaps have expected the nature of this connection to have been fully hammered out long ago by Lord Rayleigh and others when the subject

Dr Berry is Lecturer in Physics at the University of Bristol.

was in its infancy. Indeed, in the early years of this century Runge (see Born & Wolf[1]) showed that a short-wave asymptotic solution of the Helmholtz wave equation could be constructed, in which the surfaces of constant phase of the wave—*the wave fronts*—were cut orthogonally by the *rays* or *paths* arising when the same system was treated geometrically or classically, while the intensity was proportional to the local *concentration of rays*. But this partial clarification of the situation had two basic flaws: in a *mathematical* sense the wave field thus constructed could not possibly be anything like an accurate representation of the true one in regions where the ray pattern has *caustics* or *foci*, since there the density of rays is infinite, and all the formulae diverge. Secondly, there is the *physical* problem that the Runge theory does not explain the *diffraction* which occurs even for quite short waves, because there are for example within the shadows of objects no rays with which to construct a wave field.

This second problem has been dealt with by Keller in his *geometrical theory of diffraction*; the best introduction to a series of beautiful papers on this subject is Keller.[2] Essentially one extends the class of rays by relaxing the minimization conditions in the principles of Fermat or Hamilton, thus including *diffracted rays* from edges and tips of objects, *creeping rays* around surfaces, and *imaginary rays* on the (geometrically) dark side of caustics. With the aid of these new types of ray, *diffracted wave fields* can be constructed according to the Runge procedure; proportionality constants giving the strength of these fields relative to those generated by ordinary rays can be found by solving exactly the appropriate canonical problems—straight edge, cylindrical surface, circular caustic, etc. The geometrical theory of diffraction then states that the field at a point is the superposition of the fields on all the rays, of whatever type, through the point.

But the wave fields given by this procedure still diverge in singular regions of the ray pattern, so our first problem remains. Until recently, *transitional approximations* were used, which are valid in the immediate neighbourhood of the singular region; the earliest and most famous of these was given by Airy[3] in his theory of the rainbow. Later it was discovered that for each different type of singular region the wave field has a characteristic functional form; Felsen[4] has drawn up a list of transitional approximations for a wide variety of situations.

Although these transitional approximations are valid in regions determined by the ray pattern, they do not in themselves involve variables defined by the rays. This is an unsatisfactory conceptual situation, but it can be resolved with the aid of the recently developed mathematical *theory of uniform approximation*. We shall not go into the techniques of the theory here—basically, they involve the judiciously chosen mapping of one variable on another—but we shall state its results and clothe them with physical meaning. In mathematical terms, we have a function of two variables, $f(x,\alpha)$, and we wish to approximate it for large α. Using traditional techniques of asymptotic analysis, we obtain various different approximate forms for the function, each valid for a restricted range of x, and diverging at the boundaries (x_n, say) of these ranges. What a uniform approximation does is to provide a function which, while simpler than $f(x,\alpha)$ is just complicated enough to describe the different forms of variation in the ranges of x, as well as the behaviour near and at the points x_n.

When applied to the wave problems we have been discussing, this body of mathematics enables us to state a new heuristic principle: under short wave conditions, wave fields can be described everywhere using *only quantities characterizing the rays* (in Keller's extended sense) of the corresponding geometrical situation; the action functions on these paths or rays, however, do not generally appear in sinusoidal or exponential functions (as in the Keller theory), but as arguments of higher functions which are characteristic of the way in which the rays concentrate in the critical regions. Thus, to take three examples, a simple caustic surface involves an Airy function; an axial caustic needs a zero order Bessel function; and quantum problems involving reflections between two dynamical turning points require parabolic cylinder functions.

In this article we shall examine several wave problems, the aim being to lay bare the connections just outlined. First we consider the circular cylindrical caustic; conceptually this is the simplest instance where a uniform approximation is necessary. Then we look at waves round transparent spheres and show how uniform approximations provide a natural characterization of the meteorological phenomena of rainbows and glories and their molecular analogues. Thirdly, the scattering lengths used in nuclear experiments and metal physics are examined from our point of view, and finally we consider very briefly recent developments in the classical wave problems involving diffraction by edges and smooth bodies.

2. Cylindrical waves

The simplest waves that can exist in an infinite free space are those described by complex exponential functions of the co-ordinates. Depending on the conceptual stance one adopts, these can variously be called plane waves, momentum eigenfunctions, or the irreducible representations of the translation group. The rays for these waves are straight and parallel, and none of the difficulties of shadows, ray concentration, etc., arise; we may say that the wave function is its own asymptotic approximation. This is not true, however, of the next most simple set of waves, involving the two-dimensional rotation group; these are *cylindrical waves*. Let us use a quantum-mechanical notation, bearing in mind that the results are applicable to linear waves of any type. Then for particles of mass m and energy E, the relevant solutions of Schrödinger's equation with well defined angular momentum L are, using polar co-ordinates r, ϕ:

$$\psi_L(r, \phi) = e^{iL\phi/\hbar} J_{L/\hbar}\left(\frac{(2mE)^{\frac{1}{2}} r}{\hbar}\right), \tag{1}$$

where $J_n(x)$ is the Bessel function of the first kind. For the wave function to be single valued with respect to circuits about the origin, as the principles of wave mechanics demand, L must be an integral multiple of Planck's constant $\hbar$. We wish to find an approximation for equation (1) which is valid under nearly classical conditions, when $L \gg \hbar$.

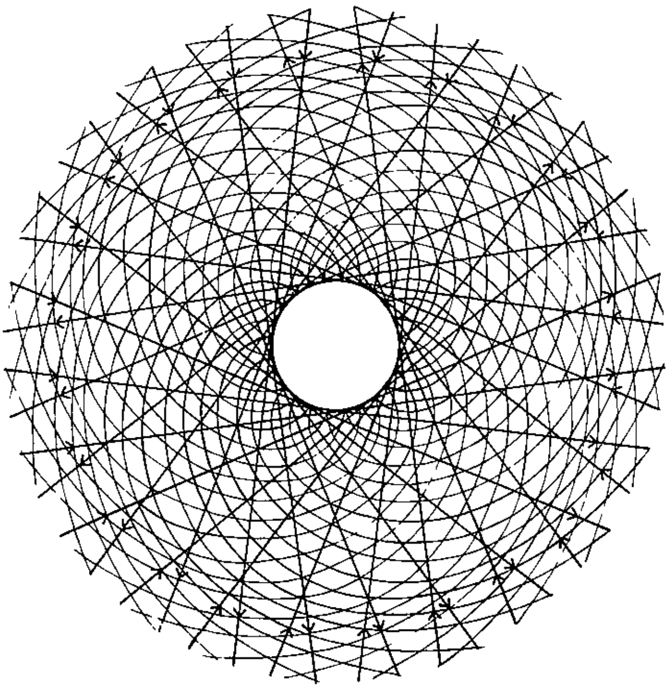

Fig. 1. Rays and wave fronts for cylindrical wave.

The classical paths corresponding to this problem are simply all those straight lines—straight because we are dealing with propagation in free space—which correspond to the fixed angular momentum L, i.e. which have the impact parameter:

$$b(L) = L/(2mE)^{\frac{1}{2}}.$$

They are sketched in Fig. 1. The most noticeable feature of the pattern is the caustic circle at $r = b$, where the concentration of rays (and hence the classical particle density) is infinite. No real rays reach points within the circle, and points outside are reached by the two tangent rays to the circle. The orthogonal trajectories of these rays are the lines of constant action, or *wave-fronts*, and take the form of two sets of spirals meeting at cusps on the caustic; these are the *involutes* of the caustic, and can be imagined as generated by the ends of pieces of string unwinding from it. Since this is a time-independent situation we are to imagine a constant stream of particles travelling along the rays; there are as many particles moving radially outward as inward, and the net current is purely angular, so that a mote placed among the particles would be wafted round in a circle. The density of particles at any point—that is, the classical limit of the intensity of the wave given by equation (1)—is proportional to the density of rays $\rho(r, \phi)$. There are two rays through each point, and $\rho(r, \phi)$ is the same for each of them in this problem, so that

$$|\psi_L(r,\phi)|^2 \approx 2\rho(r,\phi). \tag{2}$$

A simple calculation based on Fig. 1 gives:

$$\left.\begin{array}{ll}\rho(r,\phi) = \dfrac{C}{(r^2-b^2(L))^{\frac{1}{2}}} & (r > b) \\ 0 & (r < b)\end{array}\right\}, \tag{3}$$

where C is a proportionality constant.

We can construct an expression exhibiting wave properties by using the geometrical theory of diffraction, which introduces a *phase* along each ray, which for this quantum problem is just the classical action S, measured in units of Planck's constant $\hbar$. This is the line integral of momentum along the path, measured from an arbitrary wave front, and in our case the momentum is constant along each ray (although it varies in direction from ray to ray), so that

$$S = (2mE)^{\frac{1}{2}} \times \text{distance along ray.}$$

Referring to Fig. 2, we take the wave front emanating from $\theta = 0$ as the zero of action, and it is a simple matter to derive

$$\left.\begin{array}{l}S_1(r,\phi) = -(2mE(r^2-b^2))^{\frac{1}{2}} + L\phi + L\arccos\dfrac{b}{r} \\ S_2(r,\phi) = +(2mE(r^2-b^2))^{\frac{1}{2}} + L\phi - L\arccos\dfrac{b}{r}\end{array}\right\}. \tag{4}$$

According to the Runge theory, the wave outside the caustic has the form

$$\psi_L(r,\phi) \approx (\rho(r,\phi))^{\frac{1}{2}}[\alpha e^{iS_1(r,\phi)/\hbar} + \beta e^{iS_2(r,\phi)/\hbar}]. \tag{5}$$

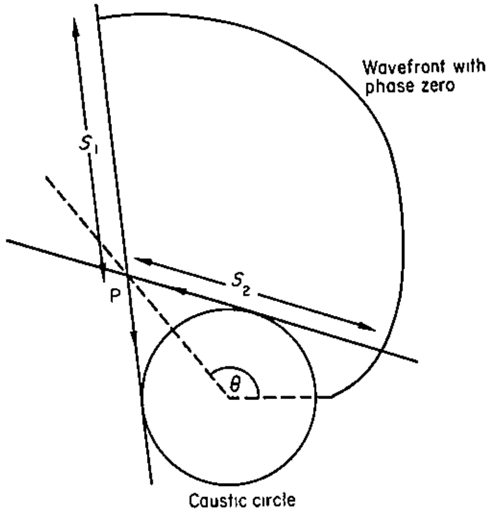

Fig. 2. Calculation of action function for cylindrical wave.

The beauty of this example is that the Bessel functions appearing in equation (1) are well understood; in particular, we know all their properties under short-wave conditions, when $L/\hbar \gg 1$. The approximation corresponding to equation (5) is

$$J_n(x) \approx \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (x^2-n^2)^{-\frac{1}{4}} \cos\left((x^2-n^2)^{\frac{1}{2}} - n \arccos\frac{n}{x} - \frac{\pi}{4}\right).$$

$$(n \gg 1,\ x \gg n). \qquad (6)$$

By comparison with equations (3), (4) and (5), we can determine the coefficients C, α, β, as:

$$C = \hbar/[2\pi(2mE)^{\frac{1}{2}}],$$

$$\alpha = e^{-i\pi/4},$$

$$\beta = e^{+i\pi/4}.$$

The coefficients α and β, taken together, amount to the 'phase jump of $\pi/2$ through a caustic' occurring in classical diffraction theory.

Within the caustic circle the action functions (4) take complex values; according to the Keller theory of diffraction we say that points there are reached by two *complex rays*, and we can construct a wave field as before. But only the ray with action S_1 contributes, since a little analysis shows that the other ray would give a contribution growing exponentially large inside the caustic, and our wave equation (1) is regular there. Thus the wave is:

$$\psi_L(r,\phi) \approx |\rho(r,\phi)|^{\frac{1}{2}} \gamma e^{iS_1(r,\phi)/\hbar}, \tag{7}$$

which, if compared with the known approximation complementary to equation (6), namely:

$$J_n(x) \approx (2\pi)^{-\frac{1}{2}}(n^2-x^2)^{-\frac{1}{4}} e^{((n^2-x^2)^{1/2}-n \operatorname{arc\,cosh} n/x)}.$$

$$(n \gg 1,\ x \ll n), \tag{8}$$

leads to the identification of the 'diffraction coefficient' γ as unity.

Considering now the *intensity* $|\psi(r,\phi)|^2$, we see from equation (7) that (in the classical limit $\hbar \to 0$) there are no particles inside the caustic. Outside the caustic, equation (5) gives:

$$|\psi_L(r,\phi)|^2 \approx 4\rho(r,\phi)\cos^2\left[\frac{(S_2-S_1)}{2\hbar}-\frac{\pi}{4}\right](r > b). \tag{9}$$

This is a simple two-wave interference pattern (such as one gets in Young's double-slit experiment). But the classical formula (2) is not simply the limit of this result as $\hbar \to 0$; we have to add to the mathematics the *physical principle* that no measuring instrument can detect infinitely fast oscillations, so that the observed limit is the *average value* of equation (9). The average of $\cos^2 x$ is $\frac{1}{2}$, so that equation (9) does in fact reduce to equation (2) under the right assumptions. The classical limits of all wave problems involving more than just one family of rays (i.e. practically every wave problem that there is) are the result of an average of this kind.

Now let us examine the wave field near the caustic circle where our earlier approximations diverge. From equation (1) we see that we require an approximation to a Bessel function whose argument is nearly equal to its order. Such a transitional approximation, valid for a range midway between those of equations (6) and (8), is:

$$J_n(x) \approx (2/n)^{\frac{1}{3}} Ai[(2/n)^{\frac{1}{3}}(n-x)].$$
$$(n \gg 1,\ x \approx n). \tag{10}$$

In this formula, $Ai(x)$ is the *Airy function*, sketched in Fig. 3; this function was introduced by Airy in 1838 precisely to describe wave propagation near caustics. The corresponding form of equation (1), namely:

$$\psi_L(r,\phi) \approx e^{iL\phi/\hbar}\left(\frac{2\hbar}{L}\right)^{\frac{1}{3}} Ai\left[\left(\frac{2}{\hbar L}\right)^{\frac{1}{3}}(L-r(2mE)^{\frac{1}{2}})\right], \tag{11}$$

shows that, very near the caustic, the wave function is of order $\hbar^{\frac{1}{3}}$, whereas equations (5) and (7) show that away from $r = b$ it is of order $\hbar^{\frac{1}{2}}$ multiplied by an exponential that either oscillates or decays.

The transitional approximation (11) exhibits this change in the asymptotic dependence of $\psi_L(r, \phi)$ on $\hbar$ for different regions of r, but the form of the oscillations and exponential decay is not the correct one as given by equations (5) and (7). But because the Airy function has the correct *qualitative* behaviour, it

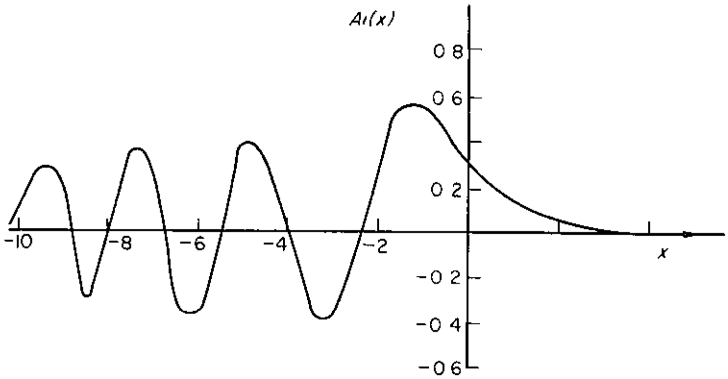

Fig. 3. The Airy function.

is possible, by the choice of an appropriate variable, to map the behaviour of the Bessel functions onto it, and obtain a *uniform approximation*. The result, first derived by Langer[5] is:

$$J_n(x) \approx \left\{\frac{4\left[\frac{3}{2}\left((x^2-n^2)^{\frac{1}{2}}-n\arccos\frac{n}{x}\right)\right]^{\frac{2}{3}}}{x^2-n^2}\right\}^{\frac{1}{4}} Ai\left\{-\left[\frac{3}{2}\left((x^2-n^2)^{\frac{1}{2}}-n\arccos\frac{n}{x}\right)\right]^{\frac{2}{3}}\right\}.$$

$$(n \gg 1). \qquad (12)$$

The argument of the Airy function is positive inside the caustic, zero on it and negative outside. In the appropriate ranges of x we can invoke known properties and asymptotic forms of this function to show that it reduces, as it must, to equations (6), (8) and (10). Equation (12) expresses a complicated function of the two variables n and x in terms of a simpler function of a *combination* of n and x; it is valid when n is large, irrespective of the value of x.

From our present point of view, what is important is that we can re-write equation (12) in terms of the classical functions (3) and (4), i.e.

$$\psi_L(r,\phi) \simeq \left[\frac{3}{4\hbar}(S_2-S_1)\right]^{\frac{1}{6}}(2\pi\rho)^{\frac{1}{2}}e^{i(S_1+S_2)/2\hbar}Ai\left(-\left[\frac{3}{4\hbar}(S_2-S_1)\right]^{\frac{2}{3}}\right). \quad (13)$$

The precise manner in which the classical concepts enter is interesting. The sum and difference of the actions along the two contributing paths occur in equation (13), but the *topology* of these paths—the fact that they form a caustic, etc.—must also be known, since it is through this that the characteristic Airy-function dependence is involved. Now suppose that, instead of moving in free space, the particle is subject to a cylindrically symmetric field of force. Then under a wide range of conditions the paths and constant-action surfaces are similar to those in Fig. 1 except that the rays are curved. The angular eigenfunctions are now not given by Bessel functions or indeed in general by any tabulated functions, but it can be shown by techniques of uniform approximation that the formula (13) applies unchanged to this case, provided of course that we interpret the action and density of paths as those occurring in the appropriate classical problem.

The result (13) can be extended to cover waves corresponding to ray patterns with non-circular caustics and force fields which do not have circular symmetry, provided the appropriate topology of Fig. 1 is retained. The densities $\rho_1(\mathbf{r})$ and $\rho_2(\mathbf{r})$ on the two rays through each point are not now the same, and the appropriate generalization of equation (13) is:

$$\psi(\mathbf{r}) \approx \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{i(S_1+S_2)/2\hbar}\left\{(\rho_1{}^{\frac{1}{2}}+\rho_2{}^{\frac{1}{2}})\left[\frac{3}{4\hbar}(S_2-S_1)\right]^{\frac{1}{6}} Ai\left[-\left(\frac{3}{4\hbar}(S_2-S_1)\right)^{\frac{2}{3}}\right]+\right.$$
$$\left.+i(\rho_1{}^{\frac{1}{2}}-\rho_2{}^{\frac{1}{2}})\left[\frac{3}{4\hbar}(S_2-S_1)\right]^{-\frac{1}{6}} Ai'\left[-\left(\frac{3}{4\hbar}(S_2-S_1)\right)^{\frac{2}{3}}\right]\right\}, \quad (14)$$

where Ai' is the derivative of the Airy function.

To be an acceptable wave function, equation (14) must be single-valued; this corresponds to requiring an integral number of wavelengths to fit round the caustic (or, in other words, for the actions S_1 and S_2 to change by an integral multiple of h during a circuit of the caustic) and is the generalization of the quantum condition that $L/\hbar$ in equation (1) must be an integer.

A rigorous mathematical treatment of much of the matter in this section is give by Ludwig.[6]

3. Rainbows and glories

The mathematics of waves propagating through and around spheres has many and varied applications. To understand the transmission of short radio waves, for instance, we must consider the earth as a diffracting obstacle. Again, the title of this section comes from the beautiful phenomena of meteorological optics that occur when light hits raindrops. Finally, the quantum elastic collision of two atoms or molecules can be treated in a good approximation by calculating what happens when a plane wave scatters off a radially directed field of force. There are many other examples.[7] These problems involving spheres are more realistic than those we considered in Section 2 which involved cylinders; we treated those first because they are next to the plane wave in conceptual simplicity.

In rigorous treatments of these spherical problems the wave field is usually expressed as an infinite series of angular eigenfunctions—'partial waves'—which are three-dimensional generalizations of the cylindrical $\psi_L(r, \phi)$ of Section 2. To fix our ideas, let us think of the incident wave as plane (this case applies directly to the second and third of the above examples; to treat the first, incident waves from a point source are appropriate); furthermore, in practice we often have to restrict ourselves to examining the *angular distribution* of the field far from the sphere. If the scatterer extends over more than a few incident wavelengths, we are justified in looking for a *semi-classical* expression for the wave

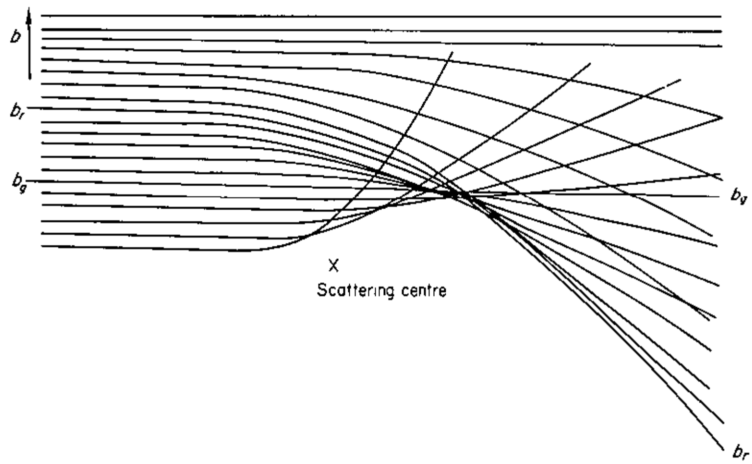

Fig. 4. Typical pattern of particle paths in interatomic force field.

field. We know from Section 2 that those angular eigenfunctions whose order exceeds several units can be written to a good approximation in a form like equations (13) or (14). But if we do this and try to sum the series numerically, we encounter great difficulties, since for short waves very many terms are needed, and they alternate in sign. Further, such a procedure is conceptually unsatisfactory since it involves analysing the problem into a series of wave patterns like those of Fig. 1, and makes no reference to the actual pattern of paths resulting from the incident plane wave.

This pattern depends on the nature of the diffracting sphere, and can be extremely complicated. We shall consider a type of pattern relevant in many situations in atomic physics, where the scattering force is attractive at large distances and repulsive at short range; it is shown in Fig. 4. The rays are defined by their *impact parameter b*, which is the distance the ray would be from the centre of the sphere at its closest approach if it were undeflected. The full three dimensional pattern is generated by rotating this figure about the $b = 0$ ray.

For large values of b the rays pass by out of range of the sphere, and are undeflected. Slightly closer rays enter the outer regions, are attracted, and emerge deflected by an angle $\Theta(b)$ which we shall define as negative. Nearer the origin the repulsive core of the field of force comes into play, and Θ decreases in magnitude, passes through zero and for very small b—corresponding to particles aimed straight at the scattering centre—the path almost returns on itself, and $\Theta(b)$ tends to $+\pi$. This behaviour of the deflection function is illustrated in Fig. 5.

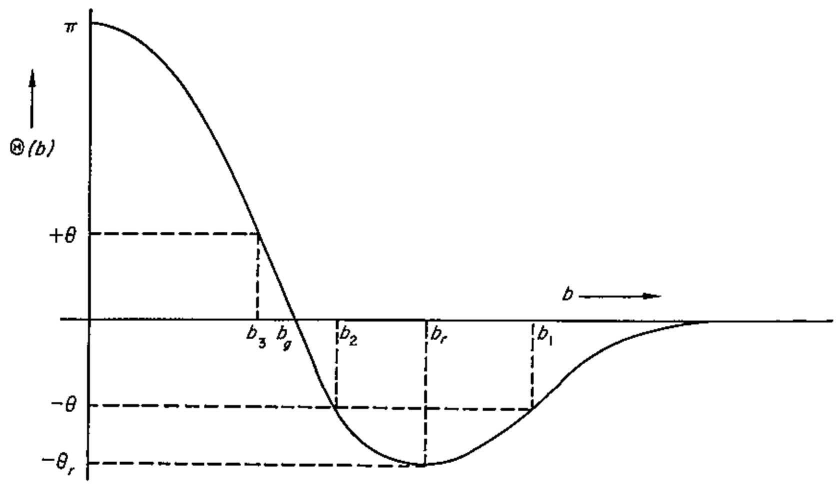

Fig. 5. Classical deflection function for rays of Fig. 4.

The two singular features of this system of rays, which are responsible for the breakdown of the simple semiclassical procedure under short-wavelength conditions, are more clearly seen by examining the *wave-fronts*, as in Fig. 6. Most prominent is the system of cusps lying on a cone of semi-angle θ_r, this being the negative deflection of the most attracted ray with impact parameter b_r which just brushes the repulsive region of the force field. This cone is a *caustic* of the ray pattern of Fig. 4 exactly analogous to the circular caustic considered in Section 2. The second feature is rather less obvious: there is a value b_g of the impact parameter, somewhat smaller than b_r, for which there is no net deflection. In three dimensions these rays form a circular cylinder, of radius b_g, and the corresponding wave fronts are *toroidal* in form. Points off the cylinder axis are illuminated by just two rays, but points on the axis receive rays from all azimuth angles. Thus an *axial caustic* arises, which persists indefinitely as one recedes from the scattering centre, and demands special treatment. It is obviously also possible to have an axial caustic in the *backward* direction.

Now let us see what effect these two different types of caustic have on the amplitude $f(\theta)$ of radiation scattered at an angle θ to the forward direction. What is actually observed is the intensity $|f(\theta)|^2$ by measurements taken far

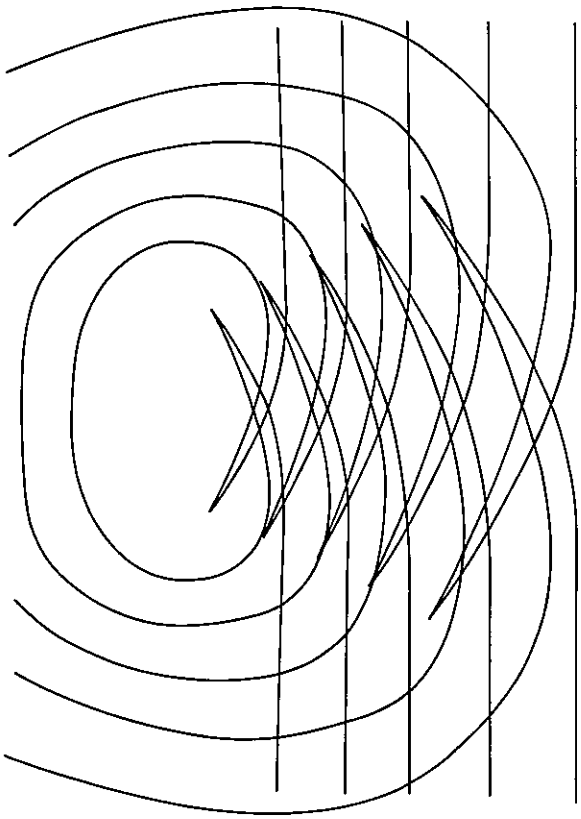

Fig. 6. Wave fronts for rays of Fig. 4.

from the scattering centre. In semi-classical cases we expect that the amplitude at θ will contain contributions only from rays emerging at that angle, i.e. when the *deflection* Θ equals $\pm\theta$ (for force fields which cause rays to wind several times around the origin before emerging, we also get contributions if $\Theta = \pm\theta - 2m\pi$ where m is integral). We expect these contributions to involve the actions $S(\theta)$ along the various rays (of course the actions along these infinite rays are themselves infinite; what $S(\theta)$ denotes is the difference in action between the actual rays and hypothetical undeflected ones). For angles θ where the contributing rays have impact parameters b which are not close to the singular values

b_r and b_g, the contributions are independent, and, for instance, if $0 \ll \theta \ll \theta_r$ we see on referring to Fig. 5 that we can write:

$$f(\theta) = f_1(\theta) + f_2(\theta) + f_3(\theta),$$

while if $\theta \gg \theta_r$ only the ray 3 on the 'repulsive' branch contributes, so that:

$$f(\theta) = f_3(\theta).$$

The contributions $f_i(\theta)$ take the form dictated by the geometrical theory of diffraction, namely:

$$f_i(\theta) \approx \alpha_i [\rho_i(\theta)] e^{iS_i(\theta)/\hbar}, \tag{15}$$

where $\rho_i(\theta)$ is the density of the ith set of rays at angle θ and α_i is a phase constant.

But if we choose an observation angle such that b_1, b_2 or b_3 is near to b_r or b_g, then one or more of the path-density factors ρ_i becomes infinite, and we can no longer treat the rays as contributing independently. Let us first consider what happens in the neighbourhood of θ_r. The scattered intensity then exhibits *rainbow oscillations*, the term arising from the same behaviour observed when light refracted through atmospheric water droplets emerges near the angle of maximum deflection around 138° to the forward direction (in nature the oscillations are seen as *supernumerary* bows). As θ approaches θ_r from below, the values b_1 and b_2 get closer and the action-difference between the two rays diminishes, so that the angular oscillations caused by their interference become slower. We then write:

$$f(\theta) = f_r(\theta) + f_3(\theta),$$

where $f_r(\theta)$ is the rainbow amplitude which reduces to $f_1 + f_2$ when $\theta \ll \theta_r$. According to our principles of uniform approximation, $f_r(\theta)$ must have the form characteristic of waves near a caustic, so we can take over all the theory of Section 2, and assert that, apart from a constant factor, the rainbow amplitude is given just by equation (14) with $\mathbf{r}$ replaced by θ. Thus, instead of solving a difficult quantum problem involving Schrödinger's equation and an intricate summation of perhaps thousands of terms, we need only solve the equivalent classical problem, and use the Airy-function form appropriate to the resulting pattern of paths.

The resulting intensity is shown schematically in Fig. 7; the rainbow pattern is characterized by the strong maximum near θ_r, with slow oscillations on one side and exponential decay into the shadow on the other. Superimposed on the Airy-function form are rapid interference oscillations between $f_r(\theta)$ and the repulsive amplitude $f_3(\theta)$. This behaviour has been observed in molecular beam experiments (Hundhausen & Pauly[8]).

The second singular region arises at small angles because of the axial caustic in the forward direction. Then b_1 recedes to infinity, while b_2 and b_3 (corresponding to slightly attracted and repelled rays emerging near $\theta = 0$) coalesce at b_g. The contribution from b_1 to the near forward intensity arises (see Fig. 4) from those distant rays which are only slightly attracted by the long-range tail of the force field, and depends sensitively on the precise way the force diminishes with distance.[12] But we shall consider the characteristic behaviour contributed by the two rays near b_g. Thus we write, for the neighbourhood of the forward direction:

$$f(\theta) = f_g(\theta) + f_1(\theta),$$

and we call the behaviour of $f_g(\theta)$ the *glory effect*. The name arises by analogy with another meteorological effect, namely the strong backscattering of light

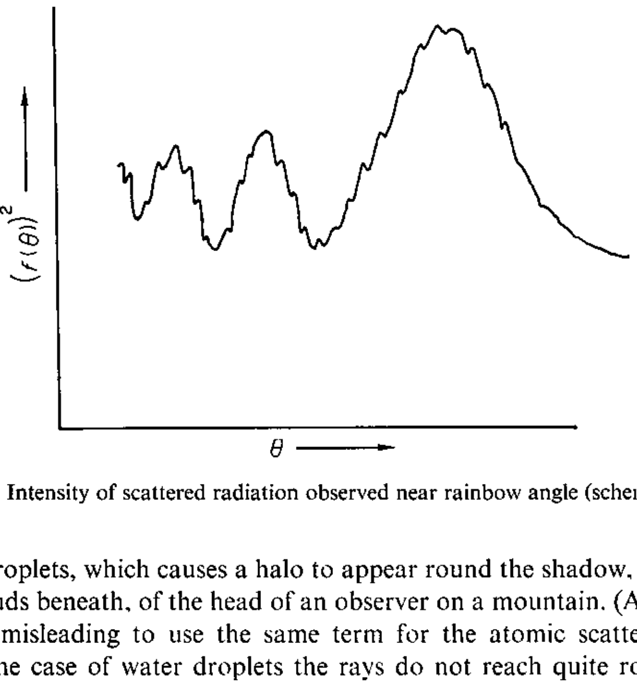

Fig. 7. Intensity of scattered radiation observed near rainbow angle (schematic).

by water droplets, which causes a halo to appear round the shadow, cast by the sun on clouds beneath, of the head of an observer on a mountain. (Actually it is somewhat misleading to use the same term for the atomic scattering effect, since[7] in the case of water droplets the rays do not reach quite round to the backward direction, so there is no axial caustic, and the true glory is more difficult to explain.)

Principles of uniform approximation tell us that to describe in a mathematical way the glory behaviour of the scattering amplitude $f(\theta)$ we must first find a simple situation where an axial caustic occurs alone, without the complicating factor of the conical caustic causing rainbow effects. Such a case is provided by a *ring source*, of radius a, emitting particles of energy (Fig. 8). The scattering amplitude, as observed at a distant point P with polar angle θ relative to the forward direction threading the ring, is easily calculated to be

$$f_g(\theta) \propto J_0\left(\frac{(2mE)^{\frac{1}{2}}\, a \sin\theta}{\hbar}\right). \tag{16}$$

This zero-order Bessel function cannot be further approximated (in contrast to the very-high-order cylindrical wave functions of Section 2, which we approximated by Airy functions).

To use equation (16) in our problem we have to write it in a form which only involves actions and densities along rays. First of all, we realize that the difference in action between the two paths (called 2 and 3 in anticipation of the glory problem) reaching P is:

$$S_2(\theta) - S_3(\theta) = \text{momentum} \times \text{path difference} = (2mE)^{\frac{1}{2}} \times 2a \sin\theta. \tag{17}$$

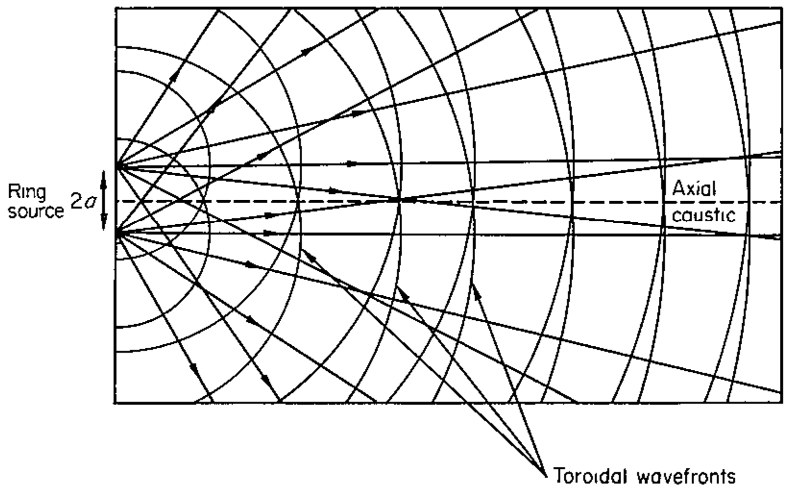

Fig. 8. Rays and wave fronts from ring source.

Next we realize that $a(2mE)^{\frac{1}{2}}/\hbar$ is the number of particle wavelengths fitting into the ring, which is large under semi-classical conditions, so θ only has to be a little different from zero for the argument of the Bessel function to be large, and we can then use the asymptotic approximation:

$$J_0(x) \approx \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos(x - \pi/4). \tag{18}$$
$$(x \gg 1).$$

But far away we must be able to write $f_g(\theta)$ as a sum of two independent components, i.e. as:

$$f_g(\theta) \approx \alpha_2\, \rho_2^{\frac{1}{2}}\, e^{iS_2/\hbar} + \alpha_3\, \rho_3^{\frac{1}{2}}\, e^{iS_3/\hbar}. \tag{19}$$

We can do this using equations (16), (17) and (18) if we identify:

$$\left.\begin{aligned} \alpha_2 = e^{i\pi/4}, \quad \alpha_3 = e^{-i\pi/4} \\ \rho_2 = \rho_3 \propto \frac{1}{\sin\theta} \end{aligned}\right\}. \tag{20}$$

Thus we can write equation (16) by analogy with equation (13), in the form:

$$f_g(\theta) \approx [\rho(S_2 - S_3)]^{\frac{1}{2}} e^{i(S_2+S_3)/\hbar} J_0\left(\frac{S_2 - S_3}{2\hbar}\right). \tag{21}$$

Finally we must generalize to the case where the densities along the contributing rays are not the same, and we get[12] the analogous result to equation (14), namely

$$f_g(\theta) \propto e^{i(S_2+S_3)/2\hbar}(S_2 - S_3)^{\frac{1}{2}}\left[(\rho_2^{\frac{1}{2}} + \rho_3^{\frac{1}{2}}) J_0\left(\frac{S_2 - S_3}{2\hbar}\right) - \right.$$
$$\left. - i(\rho_3^{\frac{1}{2}} - \rho_2^{\frac{1}{2}}) J_1\left(\frac{S_2 - S_3}{2\hbar}\right)\right]. \tag{22}$$

This formula shows in detail how the intensity, which away from $\theta \approx 0$ takes its classical values independent of $\hbar$, rises to a high value of order $\hbar^{-1}$ at $\theta = 0$ in the presence of a glory; this is a stronger enhancement than in the rainbow case, where the intensity at θ_r is of order $\hbar^{-\frac{1}{3}}$.

Historically, the systematic development of semi-classical methods for scattering from force fields was pioneered by Ford & Wheeler[9] in a series of beautiful papers. They recognized the importance of rainbow and glory effects and derived transitional approximations valid very close to θ_r for rainbows and 0 or π for glories; of course the transitional approximation for the *optical* rainbow was derived in 1838 by Airy.[3]

4. Scattering lengths

At very low energies the scattering of particles from an attractive field of force no longer involves the many thousands of partial waves and strong angular dependence just discussed. Instead, the fact that the free space wavelength (λ = Planck's constant/momentum = $\hbar/(2mE)^{\frac{1}{2}}$) is much larger than the scattering region means that the scattered radiation is isotropic, and consists only of the *S*-wave—the angular eigenfunction corresponding to angular momentum zero. Scattering processes of this kind are involved in nuclear physics and the electron theory of metals.

The amount of scattered radiation depends very sensitively on the form of the potential $V(r)$ describing the variation of the force field with distance. We characterize this radiation by the *scattering length a*, this being half the radius of the equivalent geometrical-optics hard-sphere scatterer (so that the total cross-section seen by radiation scattered away from the forward direction is $4\pi a^2$). The calculation of the scattering length for a given potential involves a simple Schrödinger equation, soluble in only a few seconds of computer time, but it is always advantageous to have analytic formulae which show exactly what features of $V(r)$ affect a.

It is a little surprising that such a long-wavelength process can be treated by semi-classical methods at all. To see that it can, we first realize that, beyond the range of the scatterer, the particles travel in free space; such propagation is classical, even down to zero energy. Secondly, if the potential $V(r)$ is large and negative then the wavelength $\lambda = h/(2m(E-V))^{\frac{1}{2}}$ is small *within the scatterer* even

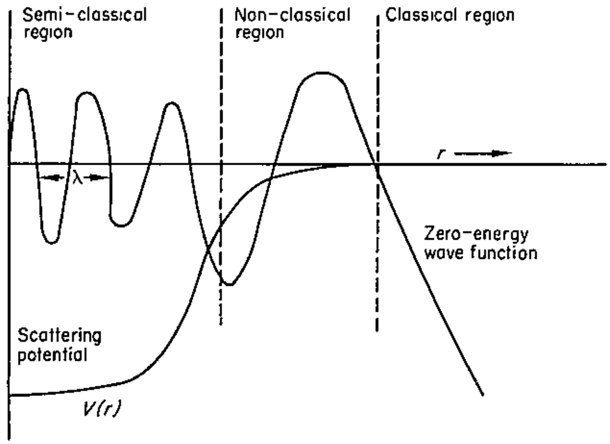

Fig. 9. Scattering potential and zero-energy S-wave function.

if it is infinite outside. It is only in the transition region represented by the tail of the scattering potential, where forces act on particles of long wavelength, that the propagation is non-classical (see Fig. 9). Thus we might expect to obtain uniform approximations for the wave function, which involve only the classical action ϕ along a path from infinity aimed directly (since the angular momentum is zero) at the centre of the scatterer, namely:

$$\phi = \int \mathbf{p}\cdot\mathbf{dr} = \int_0^\infty (2m(E-V(r)))^{\frac{1}{2}}\,dr = \int_0^\infty (-2mV(r))^{\frac{1}{2}}\,dr, \qquad (23)$$

since we are treating $E = 0$. But we would expect such approximations to depend also on the *form of the potential tail*, since this determines the exact nature of the non-classical part of the propagation.

Using techniques of uniform approximation developed by Miller & Good[10] and Dingle[11] we can derive expressions for the scattering length (which is simply related to the wave function). For these semi-classical formulae to be good approximations, the number ($\sim\phi/2\pi\hbar$) of wavelengths fitting into the scatterer must be large, but in practice 'large' means anything more than one or two. From the many different forms of potential tail for which approximate forms can be found we shall select two. The first case is where the potential varies smoothly in an otherwise arbitrary manner from $r = 0$ out to a radius $r = R$, when it suddenly goes to zero; this is a generalization of the so-called 'square-well' potential. The wave function simply involves trigonometric functions, and the scattering length is given by:

$$a \approx \frac{-\hbar \tan(\phi/\hbar)}{(-2mV(R))^{\frac{1}{2}}} + R. \tag{24}$$

The second case is where the potential tail falls smoothly to zero according to an inverse power law of the form:

$$V(r) \underset{r\to\infty}{\to} -\frac{\alpha}{r^n}.$$

The wave function now involves Bessel functions of order $1/(n-2)$ and $-1/(n-2)$, and the scattering length is given by:

$$a \approx \frac{\Gamma\left(1-\dfrac{1}{n-2}\right)\left(\dfrac{\alpha\hbar}{(2m)^{\frac{1}{2}}}\right)^{1/(n-2)}}{\Gamma\left(1+\dfrac{1}{n-2}\right)(n-2)^{2/(n-2)}} \, \frac{\sin\left(\dfrac{\phi}{\hbar}+\dfrac{n\pi}{4(n-2)}\right)}{\cos\left(\dfrac{\phi}{\hbar}-\dfrac{n\pi}{4(n-2)}\right)}, \tag{25}$$

where Γ denotes the gamma function (generalized factorial) of analysis.

These formulae take on clear physical meaning if we realize that attractive fields of force are capable of retaining particles in *bound states* with discrete negative energies. As we increase the strength of the force field (specified by ϕ) from zero, there comes a point when a bound state suddenly appears at $E = 0$. On further increasing the attractive forces, the energy of this bound state decreases, and soon a second one squeezes through at $E = 0$, and so on. We expect that our incident particles (which have nearly zero *positive* energy) will interact very strongly with the scatterer when $V(r)$ has a strength such that a bound state just sits at $E = 0$; this corresponds to an infinite value for the scattering lengths. We see from equations (24) and (25) that the exact values of ϕ for which a is infinite depend on the form of the tail of the potential, but in both cases bound states appear at ϕ-values spaced at intervals $\hbar\pi$, so that each bound state introduces an extra node into the zero-energy wave function. There are also values of ϕ, interlaced between those where bound states appear, for which a is zero; this is the Ramsauer–Townsend effect, a resonance phenomenon where the scatterer appears completely transparent to incident particles.

A completely different expression can be obtained for a on the supposition that the force field is weak; this is the *Born approximation*

$$a \approx \frac{1}{\hbar}(2m)^{\frac{1}{2}} \int_0^\infty V(r)\, r^2\, dr.$$

When the potential is strong enough to bind particles, this formula ceases to be valid, but then our semi-classical results (24) and (25) can be used. We can sum up this section by saying that these uniform approximations give mathematical form to the physical processes of resonance transmission and the appearance of new bound states, and also indicate the importance of the long-range tail of the scattering potential.

5. Uniform approximations for other problems

The classical problems of diffraction theory are yielding one by one to sustained attack by methods of uniform approximation; this research is being carried out principally at the Courant Institute of New York University. We shall outline briefly what is involved in two such problems.

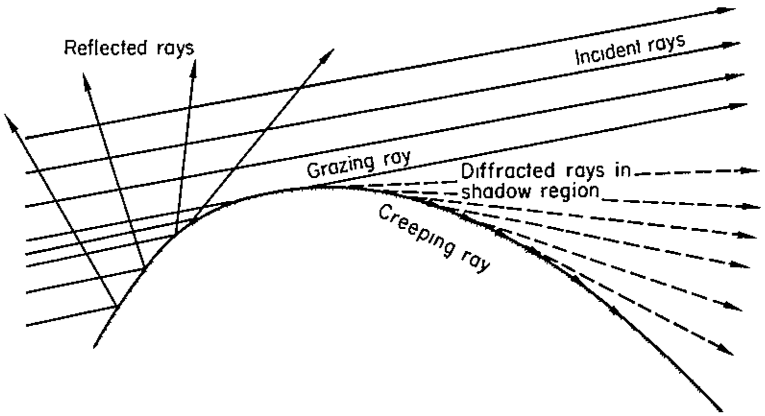

Fig. 10. Rays diffracted from a surface.

We consider first *creeping waves*. The region near an irradiated convex surface (radio waves near the earth, for instance) is divided according to geometrical optics into an illuminated part, containing incident and reflected rays, and a shadow part, these being separated by rays which just graze the surface (Fig. 10). According to Keller's geometrical theory of diffraction, however, some radiation gets into the shadow region in the form of *diffracted rays* shed tangentially from a wave creeping around the surface and attenuating away from the point of incidence of the grazing ray.

It is obvious from Fig. 10 that the surface is a caustic of these diffracted rays, so the formulae of the Keller theory fail there; but uniform approximations can be found which are valid right up to the surface (and which, of course, depend on the precise boundary conditions there).

The second example concerns diffraction by *edges*. According to Keller's theory, each point on an illuminated edge is the source of a cone of diffracted rays whose axis is the edge and whose semiangle equals the angle made with the edge by the incident ray (Fig. 11). It is obvious that the edge itself is a caustic

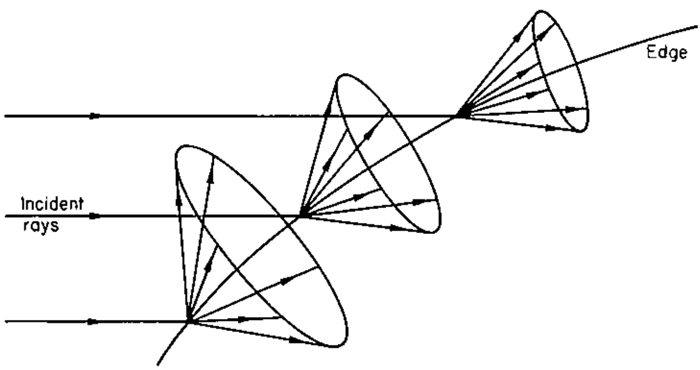

Fig. 11. Rays diffracted from an edge.

of such rays, so a uniform approximation is necessary for a complete description of the short-wave field. It is also possible for these diffracted rays to form caustics away from the edge. For example, a plane obstacle, normally illuminated, will exhibit a cylindrical caustic whose cross-section is the *evolute* of the rim of the obstacle (Fig. 12); such caustics have been observed as bright lines on a screen placed in the shadow of such an obstacle. [If the object is a circular disc the bright line degenerates into the spot whose observation in 1818 by Arago provided important evidence for the wave theory of light; the caustic cylinder in this case becomes an axial line similar to that considered in Section 3 in connection with glory scattering.]

We finish this section with a list of some other problems which have been or could be investigated using the techniques and concepts of uniform approximation.

(i) Profile near the front of an oceanic tidal wave.
(ii) Oscillatory magnetic susceptibility (de Haas–Van Alphen effect) of metal with kidney-shaped Fermi surface.

(iii) 'Whispering-gallery modes'—within large cavities of arbitrary shape.
(iv) Scattering phase shifts in the presence of virtual states bound within the centrifugal potential.
(v) The contribution of the long-range tails of potentials to the near-forward scattering of short waves.[12]
(vi) Structure constants for electron scattering in crystal lattices.

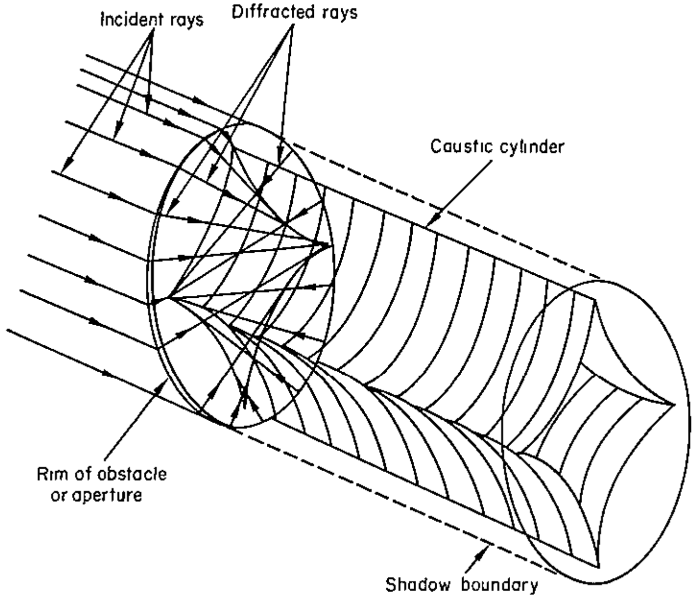

Fig. 12. Caustic cylinder of diffracted rays behind obstacle or aperture.

6. Conclusions

We have found that there is a wide variety of wave phenomena for which the appropriate level of description lies intermediate between the geometrical theory and the full wave formalism. The resulting approximations for the wave fields are valid uniformly throughout the range of the relevant variables (space, time, energy, etc.) and involve for their computation only the path lengths defined along the set of classical paths (extended to include diffracted, creeping and complex rays), and tabulated functions characterizing the geometrical form of the concentration of paths in the problem. This is to be contrasted on the one hand with the divergencies and discontinuities that appear in geometrical theories, and on the other with the notorious difficulties involved in the numerical solution of wave equations with boundary conditions at short wavelengths.

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