Proc. Roy. Soc. Lond. A. **314**, 143–152 (1970)
Printed in Great Britain

The interpretation of optical projections

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(*Communicated by C. R. Burch, F.R.S.—Received* 25 *October* 1968—
Revised 9 *June* 1969)

The first question considered is whether a series of projections (such as X-ray shadowgraphs) of a three-dimensional object can be used to infer unambiguously the structure of the object, it is shown that this is the case. Secondly, the relations between the statistics of a statistically isotropic object and the statistics of its projection are considered.

1. INTRODUCTION

A very general method of studying systems is to illuminate them with waves of some kind, and then to try to interpret the emerging radiation. There are at present no inversion methods which take into account all the wave phenomena, such as diffraction, refraction, absorption and polarization, which must occur to some extent in any practical case. However, in two important classes of experiment, we can record on a photographic plate, placed beyond the object to be examined, a blackness dependent at each point of the plate on *the line integral of a density function* along a straight ray through the point. X-ray shadowgraphs (as used, for instance, in medical diagnosis) provide one example of this, where the density function is related to the imaginary part of the X-ray refractive index. Another case is provided by phase-contrast microscopy, in which a quarter-wave plate is used to convert phase objects (nearly transparent living cells, for example) into amplitude images; here the density function is the real part of the optical refractive index. When the mechanism of wave propagation has one of these simple forms, i.e. when straight-line ray propagation may be assumed, with either the amplitude or phase of the ray controlled by a line integral through the medium, then it is possible to invert the data and calculate the density function of the object.

We illuminate the object with a parallel beam along direction $\mathbf{\Omega}$, and specify its three-dimensional density function by $\rho(\boldsymbol{r})$. We specify positions in space by a coordinate ξ along $\mathbf{\Omega}$ and a two-dimensional vector $\boldsymbol{R}_{\Omega}$ perpendicular to $\mathbf{\Omega}$. Points on the photograph are then specified by $\boldsymbol{R}_{\Omega}$ (see figure 1), and for the simple systems we are interested in, the exposure at any point on the photograph by $\mathbf{\Omega}$ is determined by the 'opacity function'

$$F(\mathbf{\Omega}; \boldsymbol{R}_{\Omega}) = \int_{-\infty}^{\infty} \rho(\xi, \boldsymbol{R}_{\Omega})\, \mathrm{d}\xi. \tag{1}$$

The actual optical density D of the photograph thus depends on F, since the exposure E at any point is proportional to e^{-F} in the case of absorption. We assume conditions such that F can be deduced uniquely from D. Our problem is to invert (1) and get an explicit formula for the density $\rho(\boldsymbol{r})$ in terms of the measurable opacity function $F(\boldsymbol{\Omega}; \boldsymbol{R}_\Omega)$. In § 2 we give the general formula, applicable to any structure; this involves taking photographs over a range of incident directions $\boldsymbol{\Omega}$. Then, in § 3, we treat the case where the medium is statistically homogeneous in character, and show how to calculate its autocorrelation function from granularity measurements on a single photograph. No attempt is made in this paper to give rigorous justifications for every step of the mathematical argument.

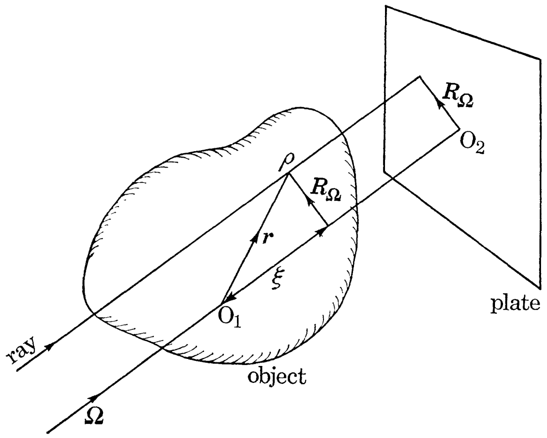

FIGURE 1

2. General structures

We introduce the Fourier transform $\rho^*(\boldsymbol{q})$ of our density function, defined by

$$\rho(\boldsymbol{r}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}\boldsymbol{q} \exp(\mathrm{i}\boldsymbol{q} \cdot \boldsymbol{r})\rho^*(\boldsymbol{q}) \tag{2}$$

and introduce coordinates $\boldsymbol{q} = (q_\xi, \boldsymbol{Q}_\Omega)$,

so that $\boldsymbol{q} \cdot \boldsymbol{r} = q_\xi \xi + \boldsymbol{Q}_\Omega \cdot \boldsymbol{R}_\Omega$.

When we insert these, together with (2), into (1), the ξ integral can be evaluated, leaving a δ-function of q_ξ, so that

$$F(\boldsymbol{\Omega}; \boldsymbol{R}_\Omega) = \frac{1}{\sqrt{(2\pi)}}\int \mathrm{d}\boldsymbol{Q}_\Omega \exp(\mathrm{i}\boldsymbol{R}_\Omega \cdot \boldsymbol{Q}_\Omega)\rho^*(0, \boldsymbol{Q}_\Omega).$$

This can be inverted to give

$$\rho^*(0, \boldsymbol{Q}_\Omega) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}\boldsymbol{R}_\Omega F(\boldsymbol{\Omega}; \boldsymbol{R}_\Omega) \exp(-\mathrm{i}\boldsymbol{R}_\Omega \cdot \boldsymbol{Q}_\Omega). \tag{3}$$

This result shows that we can derive the Fourier transform of the density function anywhere on a plane through the origin of $\boldsymbol{q}$-space perpendicular to $\boldsymbol{\Omega}$, from the single photograph of corresponding $\mathbf{\Omega}$. If we rotate $\mathbf{\Omega}$ in any fixed plane through an angle π, the planes on which we know $\rho^* (\boldsymbol{q})$will sweep the entire space, enabling us via (2) to reconstruct the density $\rho(\boldsymbol{r})$. The density function is thus *overdetermined* by the totality of possible photographs.

To proceed further with the analysis, let us choose the fixed plane, through which we rotate $\mathbf{\Omega}$, as the plane perpendicular to the axis of z; we can then specify $\mathbf{\Omega}$ by the single angle ψ. All rays for all $\mathbf{\Omega}$ then clearly lie in planes perpendicular to the z axis, the z values along any ray are constant and have the same value within the object and on the photographic plate. Thus the problem is now essentially two-dimensional, sections of the object perpendicular to z may be treated independently, and we may regard F, ρ and ρ^*, for a given z, as functions of two polar coordinates:

$$F = F(\psi; p),$$
$$\rho = \rho(r, \phi),$$
$$\rho^* = \rho^*(q_r, q_\phi).$$

The Fourier transform ρ^* is determined along a line through the origin perpendicular to $\mathbf{\Omega}$, this line has the angular coordinate

$$q_\phi = \psi + \tfrac{1}{2}\pi.$$

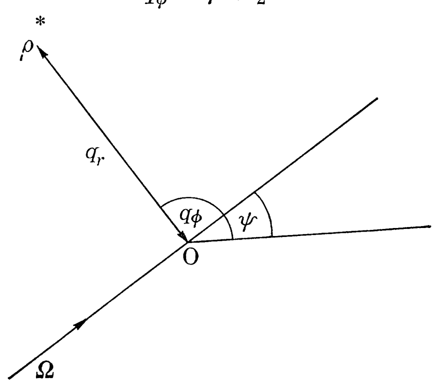

FIGURE 2. Coordinates in the q_r, q_ϕ plane in ρ^* space.

However, it is more natural to regard the coordinate p on the photographic plate as having both positive and negative values than to restrict it to positive values only, and both sides of the plate contribute to the determination of ρ^* along a line perpendicular to a given $\mathbf{\Omega}$. We therefore choose the rather unusual range of variable as:

$$\left.\begin{matrix} -\infty < p < +\infty \\ -\infty < r < +\infty \\ -\infty < q_r < +\infty \end{matrix}\right\} \qquad \left.\begin{matrix} -\tfrac{1}{2}\pi \leqslant \psi \leqslant \tfrac{1}{2}\pi \\ 0 \leqslant \phi \leqslant \pi \\ 0 \leqslant q_\phi \leqslant \pi. \end{matrix}\right\}$$

These coordinates are shown in figure 2, and in this system (3) becomes:

$$\rho^* (q_r, q_\phi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}p F(\psi; p) \exp(-\mathrm{i}pq_r). \tag{4}$$

To get ρ, we now need to invert (4), using a version of (2) in suitable coordinates. However, care is needed in defining the area elements in the integration, and we must use the fact that, for any $A(\boldsymbol{q})$

$$\int \mathrm{d}\boldsymbol{q} A(\boldsymbol{q}) = \int_0^{\pi} \mathrm{d}q_\phi \int_0^\infty q_r \,\mathrm{d}q_r [A(q_r, q_\phi) + A(-q_r, q_\phi)].$$

We also note that in two dimensions

$$\boldsymbol{q} \cdot \boldsymbol{r} = q_r r \cos(q_\phi - \phi).$$

These results, together with (4) and (1), give the convergent integral

$$\rho(r,\phi) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \mathrm{d}p \int_0^{\pi} \mathrm{d}q_\phi \int_0^\infty q_r \,\mathrm{d}q_r [\exp(\mathrm{i}q_r r\cos(q_\phi - \phi) - \mathrm{i}q_r p) + \exp(-\mathrm{i}q_r r\cos(q_\phi - \phi) + \mathrm{i}q_r p)] F(\psi; p)$$
$$= \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \mathrm{d}p \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \mathrm{d}\psi \int_0^\infty q_r \,\mathrm{d}q_r F(\psi;p)\cos[q_r(r\cos(q_\phi - \phi) - p)].$$

Taken by itself, the q_r integral is divergent, but we can replace

$$q_r \cos[q_r(r\cos(q_\phi - \phi) - p)] \quad \text{by} \quad -\partial \sin[q_r(r\cos(q_\phi-\phi) - p)]/\partial p$$

and integrate by parts over p. The resultant q_r integral is easy to evaluate, and we obtain as our final result for ρ (restoring the z coordinate) the principal value of the double integral

$$\rho(z,r,\phi) = \frac{1}{2\pi^2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \mathrm{d}\psi \int_{-\infty}^{\infty} \mathrm{d}p \frac{\partial F(\psi; z, p)/\partial p}{r\sin(\phi - \psi) - p}. \tag{5}$$

This formula involves all the rays, not merely those going through the point (z, r, ϕ). However, on each photograph (specified by ψ) there is a large contribution from the point $p = r\sin(\phi - \psi)$, which corresponds to the ray through (z, r, ϕ) on that photograph (figure 3). If we think of F as a function in three dimensions, the major contribution to $\rho(\boldsymbol{r})$ comes from the circles $p = r\sin(\phi - \psi)$.

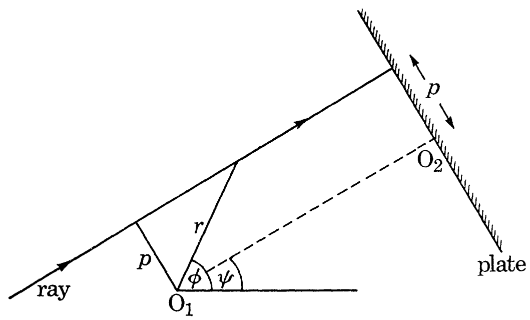

FIGURE 3. Coordinates in plane of constant z, showing a ray through (z, r, ϕ). O_1 is the origin of object coordinates, and O_2 is the origin of coordinates on the plate.

Although our result gives a straightforward way of processing photographic data, it is analytically complicated, and we have not been able to verify it in general without using Fourier transforms. However, we can give a direct verification for objects which have an axis of rotational symmetry, when this is made to coincide with the z axis. For simplicity we revert to two dimensions, omitting the z coordinate. Thus ρ is a function of r only, not ϕ, and

$$r^2 = \xi^2 + p^2.$$

The opacity function F is similarly a function of p only, and

$$F(p) = \int_{-\infty}^{\infty} \rho(r)\,\mathrm{d}\xi = \int_{-\infty}^{\infty} \rho\{\sqrt{(\xi^2+p^2)}\}\,\mathrm{d}\xi.$$

If we insert this into (5), we find, after some reduction, that we have to prove the identity

$$\rho(r) = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\mathrm{d}\psi\int_0^{\infty}\mathrm{d}p\int_{-\infty}^{\infty}\mathrm{d}\xi\,\frac{\partial[\rho\{\sqrt{(\xi^2+p^2)}\}]/\partial p}{r\sin(\phi-\psi)-p}.$$

The angular integral can be evaluated by contour methods, to give

$$\int_{-\pi}^{\pi}\frac{\mathrm{d}\psi}{r\sin(\phi-\psi)-p} = -\frac{2\pi H(p-r)}{\sqrt{(p^2-r^2)}},$$

where H is the unit step function. Thus the angular integral has the effect of restricting the range of the p integral, and we now have to show that

$$\begin{aligned}\rho(r) &= -\frac{1}{\pi}\int_r^{\infty}\mathrm{d}p\int_{-\infty}^{\infty}\mathrm{d}\xi\,\frac{\partial[\rho\{\sqrt{(\xi^2+p^2)}\}]/\partial p}{\sqrt{(p^2-r^2)}}\\ &= -\frac{2}{\pi}\int_r^{\infty}\frac{p\,\mathrm{d}p}{\sqrt{(p^2-r^2)}}\int_p^{\infty}\frac{\partial\rho(r')/\partial r'}{\sqrt{(r'^2-p^2)}}\,\mathrm{d}r'.\end{aligned}$$

Since the formulae are linear, superposition applies and we need prove the result only for a shell at $r = a$:

$$\rho(r) = \delta(r-a).$$

Thus we have to show that

$$\begin{aligned}\delta(r-a) &= \frac{2}{\pi}\frac{\partial}{\partial a}\left\{\int_r^{\infty}\frac{p\,\mathrm{d}p}{\sqrt{(p^2-r'^2)}}\int_p^{\infty}\frac{\delta(r'-a)\,\mathrm{d}r'}{\sqrt{(r'^2-p^2)}}\right\}\\ &= \frac{2}{\pi}\frac{\partial}{\partial a}\left\{\int_r^{a}\frac{p\,\mathrm{d}p\,H(a-r)}{\sqrt{(p^2-r^2)}\sqrt{(a^2-p^2)}}\right\}.\end{aligned} \tag{6}$$

Further application of contour integration shows that

$$\int_r^{a}\frac{p\,\mathrm{d}p}{\sqrt{(p^2-r^2)}\sqrt{(a^2-p^2)}} = \tfrac{1}{2}\pi. \tag{7}$$

Hence (6) is indeed an identity and our basic formula (5) is verified for this case.

It may be observed that the above double integral for $\rho(r)$ is a transformation of a special case ($\mu = \frac{1}{2}$) of equation (2) on page 8 of Bôcher (1926), which equation is equivalent to the solution of Abel's well known integral equation.

3. Statistically defined structures

If the density of the object varies from place to place in a statistical way, then we may be interested not so much in $\rho(\boldsymbol{r})$ as in its mean $\langle\rho\rangle$, its mean square $\langle\rho^2\rangle$ and its autocorrelation function

$$C(t) = \frac{\langle\rho(\boldsymbol{r})\,\rho(\boldsymbol{r}+\boldsymbol{t})\rangle - \langle\rho\rangle^2}{\langle\rho^2\rangle - \langle\rho\rangle^2}, \tag{8}$$

where the fact that C is a function of t instead of $\boldsymbol{t}$ indicates that we are restricting ourselves to statistically isotropic media. (We write t for the length of $\boldsymbol{t}$ and shall use similar notations later.) We shall show how to calculate these functions by measurements on a single photograph taken through a normally illuminated slab of large thickness L. We again assume that the optical density $B(\boldsymbol{R})$ at a point $\boldsymbol{R}$ on the photograph can be used to calculate the exposure function $E(\boldsymbol{R})$ and hence $F(\boldsymbol{R})$, the latter being given by a special case of (1):

$$F(\boldsymbol{R}) = \int_0^L \mathrm{d}\xi\rho(\boldsymbol{R},\xi). \tag{9}$$

It is trivial to calculate the mean of F:

$$\langle F\rangle = \int_0^L \mathrm{d}\xi\langle\rho\rangle = L\langle\rho\rangle. \tag{10}$$

A more interesting quantity is $K(L)$, the fractional variance of F, defined by

$$K(L) \equiv \frac{\langle(F-\langle F\rangle)^2\rangle}{\langle F\rangle^2} = \frac{\langle F^2\rangle - \langle F\rangle^2}{\langle F\rangle^2}. \tag{11}$$

To find this, we need $\langle F^2\rangle$, and from (8) and (9)

$$\begin{aligned}\langle F^2\rangle &= \int_0^L \mathrm{d}\xi\int_0^L \mathrm{d}\xi'\langle\rho(\boldsymbol{R},\xi)\,\rho(\boldsymbol{R},\xi')\rangle \\ &= (\langle\rho^2\rangle - \langle\rho\rangle^2)\int_0^L \mathrm{d}\xi\int_0^L \mathrm{d}\xi' C(|\xi-\xi'|) + \int_0^L \mathrm{d}\xi\int_0^L \mathrm{d}\xi'\langle\rho\rangle^2.\end{aligned}$$

So

$$K(L) = \frac{1}{L^2}\left(\frac{\langle\rho^2\rangle}{\langle\rho\rangle^2} - 1\right)\int_0^L \mathrm{d}\xi\int_0^L \mathrm{d}\xi' C(|\xi-\xi'|). \tag{12}$$

We rewrite the first double integral as follows:

$$\int_0^L \mathrm{d}\xi\int_0^L \mathrm{d}\xi' C(|\xi-\xi'|) = \int_0^L \mathrm{d}\xi\int_{\xi-L}^{\xi} \mathrm{d}\zeta\, C(|\zeta|). \tag{13}$$

The domain of integration is thus the parallelogram shown as OABC in figure 4. In order to produce a simple general result, we shall now assume that the effective range of the autocorrelation function is much less than L, that is to say $C(|\zeta|)$ is negligible for $|\zeta| > \zeta_m$, where $\zeta_m \ll L$. A good approximation to the integral is then obtained by taking the domain of integration to be the rectangle DEFG, so that

$$K(L) \doteqdot \frac{1}{L^2}\left(\frac{\langle\rho^2\rangle}{\langle\rho\rangle^2}-1\right)\int_0^L \mathrm{d}\xi \int_{-\zeta_m}^{\zeta_m} \mathrm{d}\zeta\, C(|\zeta|)$$

$$\doteqdot \frac{2}{L}\left(\frac{\langle\rho^2\rangle}{\langle\rho\rangle^2}-1\right)\int_0^\infty \mathrm{d}\zeta\, C(|\zeta|). \tag{14}$$

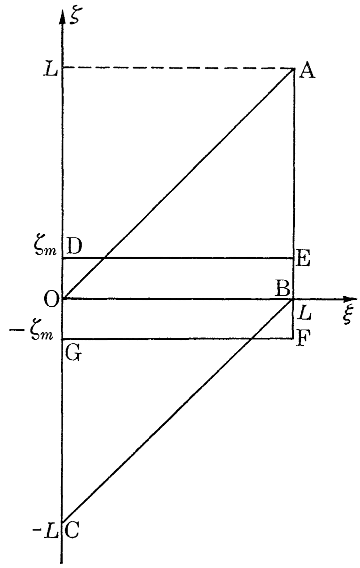

FIGURE 4. Domain of integration of (13).

This decreases as $1/L$, so that the granularity of F slowly disappears as the slab thickness increases. It might appear that this reduction in K would mean that the contrast obtainable with thick slabs would be unmeasurably low. However, it must be borne in mind that the fractional variance V of the exposure function E is given by

$$V \equiv \frac{\langle E^2\rangle - \langle E\rangle^2}{\langle E\rangle^2} = \frac{\langle \exp(-2F)\rangle - \langle \exp(-F)\rangle^2}{\langle \exp(-F)\rangle^2} \tag{15}$$

and this is by no means the same as K, nor can it be expressed in terms of $\langle F\rangle$ and $\langle F^2\rangle$ only. Although an exact calculation of V in terms of ρ involves more statistical parameters of ρ than we have so far used, it is possible to see that in general, although K and E become smaller with increasing L, yet V increases. Thus there is no loss of contrast with increasing thickness, but merely a loss of intensity.

We now consider the autocorrelation function $g(T)$ of F, which is obtainable from a single photograph. In the limit of very small L, such that fluctuations of ρ are small within a distance L, the problem is of course very simple since the photograph is a direct map of a thin slice of the medium. For any value of L, however, we have:

$$g(T) \equiv \frac{\langle F(\boldsymbol{R}+\boldsymbol{T})\,F(\boldsymbol{R})\rangle - \langle F\rangle^2}{\langle F^2\rangle - \langle F\rangle^2}$$

$$= \frac{\int_0^L \mathrm{d}\xi \int_0^L \mathrm{d}\xi' \langle \rho(\boldsymbol{R}+\boldsymbol{T},\xi)\,\rho(\boldsymbol{R},\xi')\rangle - \langle F\rangle^2}{\langle F^2\rangle - \langle F\rangle^2}$$

$$= \frac{\int_0^L \mathrm{d}\xi \int_0^L \mathrm{d}\xi' C[\sqrt{\{T^2+(\xi-\xi')^2\}}]}{\int_0^L \mathrm{d}\xi \int_0^L \mathrm{d}\xi' C(|\xi-\xi'|)} \tag{16}$$

using (12).

Again in cases where L is much greater than the effective range of correlation, we may use a similar method to that leading to (14), whence

$$g(T) \doteqdot \frac{\int_0^\infty \mathrm{d}\zeta\, C\{\sqrt{(T^2+\zeta^2)}\}}{\int_0^\infty \mathrm{d}\zeta\, C(\zeta)}. \tag{17}$$

We know of no general method of inverting (16), but to invert (17), and obtain C in terms of g, we invoke the Hankel transform $C^\dagger(q)$ of C:

$$C(t) = \int_0^\infty qJ_0(qt)\,C^\dagger(q)\,\mathrm{d}q, \tag{18}$$

so that

$$g(T) = \frac{\int_0^\infty q\,\mathrm{d}q\,C^\dagger(q) \int_0^\infty \mathrm{d}\zeta\, J_0\{q\sqrt{(T^2+\zeta^2)}\}}{\int_0^\infty \mathrm{d}\zeta\, C(\zeta)}$$

$$= \frac{\int_0^\infty \mathrm{d}q\, C^\dagger(q)\cos(qT)}{\int_0^\infty \mathrm{d}\zeta\, C(\zeta)}. \tag{19}$$

Thus, apart from a constant factor, the autocorrelation function of F is the Fourier cosine transform of the Hankel transform of the autocorrelation function of ρ.

Inverting (19), we have

$$C^\dagger(q) = \text{constant} \times \int_0^\infty \mathrm{d}T\, g(T)\cos(qT),$$

so, from (18),

$$\text{constant} \times C(t) = \int_0^\infty \mathrm{d}q \int_0^\infty \mathrm{d}T\, g(T)\,\cos(qT)\,qJ_0(qt)$$

$$= \int_0^\infty \mathrm{d}T\, g(T) \frac{\partial}{\partial T} \int_0^\infty \mathrm{d}q\, J_0(qt)\sin(qT) = \int_t^\infty \frac{\mathrm{d}T\, g'(T)}{\sqrt{(T^2-t^2)}},$$

where $q\cos(qT)$ has been replaced by $\partial \sin(qT)/\partial T$ and the integration is similar to that performed in (5). The constant factor may be evaluated by noting that $C(0) = 1$, whence

$$C(t) = \frac{\displaystyle\int_t^\infty \frac{\mathrm{d}T\, g'(T)}{\sqrt{(T^2-t^2)}}}{\displaystyle\int_0^\infty \frac{\mathrm{d}T\, g'(T)}{T}}. \tag{20}$$

The integral in the denominator converges at $T = 0$ by virtue of (17) and the fact that $C'(0) = 0$.

A rather tedious argument following closely the verification of (5), and also using the integral (7), can be used to verify (20) by the direct substitution of (17).

Thus unless $C(t)$ fails to decay sufficiently rapidly at large t, a single photograph with large L suffices to determine C via (20), $\langle\rho\rangle$ via (10), and hence $\langle\rho^2\rangle$ via (14).

We now consider the particular case of a Gaussian correlation function

$$C(t) = \exp(-t^2/s^2),$$

and, using (16), we see that for any L:

$$g(T) = \frac{\displaystyle\int_0^L \mathrm{d}\xi \int_0^L \mathrm{d}\xi' \exp(-[T^2+(\xi-\xi')^2]/s^2)}{\displaystyle\int_0^L \mathrm{d}\xi \int_0^L \mathrm{d}\xi' \exp[-(\xi-\xi')^2/s^2]}$$

$$= \exp(-T^2/s^2). \tag{22}$$

Thus the correlation function of F is similar to that of ρ, in this particular case, for all values of L, though the fractional variance $K(L)$ of F tends to the simple limit

$$K(L) \to \frac{s}{L}\left(\frac{\langle\rho^2\rangle}{\langle\rho\rangle^2} - 1\right) \tag{24}$$

only for $L \gg s$.

In the case of the Lorentzian form for C, namely

$$C(t) = \frac{s^2}{s^2+t^2},$$

evaluation of (16) yields

$$g(T) = \frac{2L\tan^{-1}(L/a) + a\ln(a^2/[a^2+L^2])}{2L\tan^{-1}(L/s) + s\ln(s^2/[s^2+L^2])}\,\frac{s}{a}, \tag{25}$$

where $$a = \sqrt{(s^2+T^2)}.$$

4. Conclusion

We have dealt with the inversion of data obtained by passing waves through a structure, under conditions such that we can observe straight line integrals of a density function. In § 2 we show how, by means of an unambiguous procedure, it is possible to find this density function by calculations from photographs taken with the object illuminated in various directions; we do not need prior qualitative knowledge of the structure such as may be possessed for example by radiographers. In § 3 we examine the change in transmission statistics as we take thicker and thicker slabs of a medium whose density varies statistically in space, and show how to calculate the autocorrelation function, mean and mean square of the density, from the statistics of the limiting form of the transmission of a thick slab.

We are grateful to Dr C. R. Burch, F.R.S., Professor F. C. Frank, F.R.S., Dr P. Lloyd and Dr J. F. Nye for several conversations on this subject.

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