Catastrophe and Fractal Regimes in Random Waves

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1. Introduction

This is an account of two recent contributions to the theory of intensity fluctuations in random waves. Detailed treatments have been published elsewhere [1,2]; my purpose here is to give simplified outlines of the rather subtle arguments involved, and bring out the sharp contrasts between the two régimes considered.

Waves often acquire randomness in their wave functions ψ and intensities $I(\equiv|\psi|^2)$ by encounter with a random structure S. Familiar examples are starlight passing through turbulent atmosphere, sunlight reflected and refracted by water waves, and sound, radio and radar reflected by landscapes. The randomness of ψ is not related to that of S in any simple manner. In particular, if S is a spatially fluctuating refractive index with Gaussian randomness, ψ will not usually be a Gaussian random function of position. The most interesting statistics are those describing the intensity fluctuations. These are the moments I_n of the probability distribution of $|\psi|^2$, namely

$$I_n \equiv \langle I^n \rangle = \langle |\psi|^{2n} \rangle, \qquad (1)$$

where <> denotes averaging over the ensemble of S. (If ψ were a Gaussian random wave, I_n would be given by $I_1 n!$ for all problems to be considered here.)

In this paper I shall consider two random wave régimes where the effect of S on ψ cannot be approximated by perturbation theory. The régimes are distinguished by the absence or presence in S of detail on length scales close to the wavelength λ of ψ.

If S does not possess such detail, i.e. if it appears smooth on the wavelength scale, then methods based on geometrical optics can be employed. It is well known [3,4,5] that these lead to waves dominated by caustics (envelopes of the rays) which generically take the form of the catastrophes classified by Professor THOM [6]. In this 'diffraction catastrophe' régime, to be discussed in Section 2, ψ is characterised by violent fluctuations whose statistics are highly non-Gaussian and for which the moments I_n scale with λ according to 'critical exponents' that depend on the hierarchy of catastrophes.

If S does possess detail over a wide range of scales that includes λ, methods based on geometrical optics cannot be used. Instead, S can be modelled by a 'fractal' [7], that is, by a hierarchical structure with no length scale at all, whose Hausdorff-Besicovitch measure dimension D is not an integer. I call the corresponding waves 'diffractals' and discuss this régime in Section 3. The statistics of diffractals obey scaling laws very different from those for diffraction catastrophes, and involve integrals and asymptotic behaviour unfamiliar in wave theory.

Mathematically, it will be useful to think of the difference between these régimes in terms of the shortwave limit $\lambda \to 0$. If S is smooth on fine scales, then geometrical optics becomes valid as $\lambda \to 0$. But if S is a fractal, then as λ gets smaller, ever-finer levels of structure are exposed and the geometrical optics limit is never attained.

2. Random Diffraction Catastrophes

When λ is small enough, waves ψ diffracted by a smooth structure S can be described in terms of the rays of geometrical optics. The rays envelop caustics (focal manifolds) on which the intensity I rises to large values. On wavelength scales, the caustics are decorated with 'diffraction catastrophes' [8] characteristic of their topological type. In the language of catastrophe theory, the rays, waves, and caustics exist in the 'control space' $\underline{C}$. Now, it is crucial to the argument that when S is random $\underline{C}$ has many dimensions, corresponding not just to the spacetime point $\underline{r},t$ where ψ is measured but to the random variables $\underline{V}$ specifying the members of the ensemble of S. Therefore we can write the wave as $\psi(\underline{r},t;\ \underline{V})$, and the ensemble averages in (1) as

$$I_n = \int\ldots\int d\underline{V}\ P(\underline{V})|\psi(\underline{r},t;\ \underline{V})|^{2n}, \tag{2}$$

where $\underline{r},t$ is fixed and where $P(\underline{V})$ is the density of realisations of S over its ensemble. The high dimensionality of $\underline{V}$ means that generically the caustics contain catastrophes of high codimension.

Consider first the case $\lambda = 0$. Then there is no diffraction at all, and I is infinity on the caustics. It follows from the conservation of energy, and can also be shown directly, that the infinities of I are integrable, so that the first moment $I_1 = \langle I\rangle$ exists. But higher powers of I cause the integrals in (2), and hence the moments I_n, to diverge. Therefore this simple argument based on geometrical optics confirms what has long been known [9,10], that the non-Gaussian strong fluctuations in ψ originate in focusing. However, it is too crude to account for the finite moments actually measured [11].

In practice several factors prevent I_n being infinite. Two examples are spatial and temporal incoherence of the source, as with the finite angular size and polychromaticity of the sun, which blur caustics refracted onto the sea bed [12,13]. Most fundamental, however, is the finite value of λ, which causes the divergences of I_n (as in, say, twinkling starlight) to be softened by diffraction.

A measure of this effect is the set of 'critical exponents' ν_n, defined by

$$\nu_n = \lim_{\lambda\to 0}\frac{d(\log I_n)}{d(\log 1/\lambda)},\ \text{i.e.}\ I_n \propto \lambda^{-\nu_n}, \tag{3}$$

which shows just how the I_n diverge as $\lambda \to 0$. Now I shall determine ν_n in terms of the catastrophes, using a simplified version of the argument in [1].

The main result will be that as n increases the exponents depend on catastrophes of ever higher codimension. Here is a simple physical argument to show why this must be so in the most familiar case where the randomness of S is stationary and averages are measured experimentally by time-averaging the intensity at fixed position $\underline{r}$: over long times, diffraction catstrophes of arbitrarily high codimension can pass arbitrarily close to $\underline{r}$. The higher-order catastrophes are rare but give rise to large localised fluctuations in the intensity. However, large rare fluctuations in I are precisely what dominate high moments I_n, which therefore depend on high-order catastrophes, as stated.

To calculate the exponents ν_n a scaling argument is employed. Consider the integral (2) for I_n. For small λ it will be dominated by those values of $\underline{V}$ lying on caustics. The integral will split into contributions $I_n(j)$ from the different catastrophes, here labelled j. Each contribution scales differently with λ and can be written

$$I_n(j) \propto \lambda^{-\nu_{nj}}, \tag{4}$$

where ν_{nj} is thus the exponent governing the contribution of the j'th catastrophe to the n'th moment.

What determines ν_{nj} are two aspects of the architecture of the j'th diffraction catastrophe. Firstly, there is the 'singularity index' β_j which describes how the wave amplitude $|\psi|$ on the caustic singularity diverges as $\lambda \to 0$:

$$|\psi|_{\text{on the caustic}} \propto \lambda^{-\beta_j}. \tag{5}$$

β_j is a measure of the strength of the diffraction catastrophe. And secondly, there is an index γ_j which describes how the diffraction pattern shrinks onto the caustic as $\lambda \to 0$:

$$\begin{array}{l}\text{hypervolume of maximum of}\\ \text{the diffraction catastrophe}\\ \text{in control space}\end{array} \propto \lambda^{\gamma_j} \tag{6}$$

γ_j is a measure of the extent of the diffraction catastrophe. The index β_j has been known for some time [14,15], but γ_j is a new quantity, introduced in [1] and recently shown [16] to be an invariant of the j'th catastrophe (related to the Jacobian of a diffeomorphism of control space).

In [1] the indices γ_j and β_j were determined from the diffraction integral representing ψ for the j'th catastrophe. Here the general procedure is illustrated in the simplest case of the 'fold' diffraction catstrophe where ψ is an Airy function:

$$\psi(V) = \frac{\text{constant}}{\lambda^{1/2}} \int_{-\infty}^{\infty} dS \; e^{2\pi i(S^3/3 + VS)/\lambda}. \tag{7}$$

The factor in brackets in the exponent is Thom's standard 'potential function' for the fold, where V is the single control parameter and S the 'state variable' (which usually represents postion on some initial wavefront from which the wave diffracts to $\underline{r}$, t). An obvious change of variables gives

$$\psi(V) = \frac{\text{constant}}{\lambda^{1/6}} \int_{-\infty}^{\infty} dS' \; e^{i(S'^3/3 + [V(\frac{2\pi}{\lambda})^{2/3}]S')}. \tag{8}$$

This shows that in terms of the standard Airy diffraction function, namely

$$\psi_{\text{standard}}(V') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dS' \; e^{i(S'^3/3 + V'S')} \equiv \text{Ai}(V'), \tag{9}$$

$|\psi|$ rises to $O(\lambda^{-1/6})$ on the caustic itself (V = 0) and the width of the first bright fringe is $O(\lambda^{2/3})$. Therefore $\beta = 1/6$ and $\gamma = 2/3$ for the fold diffraction catastrophe.

Knowing β_j and γ_j, the corresponding contribution $I_n^{(j)}$ is estimated from the integral (2) as follows:

$$\begin{aligned} I_n^{(j)} &\propto [\text{maximum value of } |\psi|]^{2n} \times \begin{array}{l}\text{control space extent of}\\ \text{diffraction maximum}\end{array} \\ &\propto \lambda^{-2n\beta_j} \times \lambda^{\gamma_j}, \end{aligned} \tag{10}$$

whence (4) gives

$$\nu_{nj} = 2n\beta_j - \gamma_j. \tag{11}$$

It is thus established how each catastrophe contributes to each moment. Summing the contributions gives, for the n'th moment:

$$I_n \longrightarrow \sum_j C_{nj} \lambda^{-\nu_{nj}} \quad \text{as } \lambda \to 0, \tag{12}$$

where the sum is over all catastrophes j. The dominant term is clearly the one with the largest ν_{nj}, so that from (3) the critical exponents are given by

$$\nu_n = \max(\nu_{nj}). \tag{13}$$

This is the main result. To actually work out the ν_{ni}, and hence the ν_n, is a tedious exercise which I claim to be the most elaborate dimensional analysis (scaling) ever performed. What emerges are sets of rational numbers ν_n that depend only on the dimensionality d of the physical space in which ψ propagates. The reason for this is that although the codimension of the contributing catastrophes can be arbitrarily large the 'corank,' i.e. the number of state variables, cannot exceed d-1 since this is the dimensionality of wavefronts. Therefore for waves in two space dimensions only the corank-one 'cuspoid' catastrophes can contribute, and in three space dimensions only corank-one and corank-two catastrophes contribute; the extra singularities make ν_n larger in the latter case. Apart from this dimension-dependence the ν_n are universal: they do not depend on the details of S, only on the fact that it is smooth and its randomness is described by many variables $\underline{V}$.

Table 1 shows the first few ν_n for d = 2 and d = 3, together with symbols representing the dominant catastrophes; in more familiar terms [3], A_2 is the fold, A_3 the cusp, D_4 the elliptic and hyperbolic umbilics, and E_6 the symbolic umbilic. The value $\nu_2 = 0$ does not mean that the second moment I_2 is not singular as $\lambda \to 0$, only that its divergence is slower than any power of λ^{-1}. In fact, $I_n \propto \log(\lambda^{-1})$, as explicit (and elaborate) analysis [17,18,19] reveals.

Table 1 Critical exponents ν_n, and contributing catastrophes, for $2 \leq n \leq 5$

n	2	3	4	5
ν_n(2 space dimensions)	0	1/3	3/4	5/4
dominant catastrophe	A_2	A_2	A_3	A_3
ν_n(3 space dimensions)	0	1/3	1	5/3
dominant catastrophe	A_2	A_2 and D_4	D_4	D_4 and E_6

These values of ν_n constitute testable predictions about the wavelength-dependence of the moments I_n of such random waves. Usually I_n is measured as a function of other parameters, such as distance from a turbulent medium or strength of turbulence (see Section 3), but I am trying to arrange for direct measurements of ν_n to be made on the basis of the definition (3).

In this exposition I have not mentioned the serious problems [1] that arise in three space dimensions when n > 5, from the appearance of singularities with 'modality.' These singularities, which lie beyond Thom's classification, are discussed by ARNOL'D [15]. Making plausible assumptions it was possible to calculate the critical exponents up to ν_{13}.

This theory has tantalising analogies with the study of critical phenomena in statistical mechanics [20]. Here, critical behaviour ("$T \to T_c$") emerges as $\lambda \to 0$. Our analogue of the incorrect 'mean field theory' is geometrical optics. Like mean field theory, geometrical optics can be generated by a quadratic approximation in the exponent of an integrand (of a diffraction integral rather than a functional integral). And, just as in statistical mechanics, the correct behaviour is embodied in a series of 'universal' exponents. However, there is a serious difference: in the random waves problem the real work of calculating the exponents is made possible not by the 'renormalisation group' technique but by the Thom-Arnol'd classification of stable singularities of gradient maps.

Nevertheless, the analogies are sufficiently close to prompt the following question: is there a 'critical space dimensionality' d_c, analogous to (4) in statistical mechanics, beyond which all (or perhaps only some) ν_n are infinite, so that geometrical optics ('mean field theory') is valid? This would mean that catastrophes j of corank > d_c-1 and very high codimension would give contributions ν_{nj} that increase indefinitely with j (instead of reaching a maximum and then decreasing, as in cases so far studied), so that the dominant catastrophes are those of infinite codimension and I_n increases faster than any power of λ^{-1} as $\lambda \to 0$. At present singularity theory is not sufficiently developed to enable this question to be answered.

3. Random Diffractals

If the diffracting structure S is a fractal, ψ is a diffractal. Virtually nothing is known about diffractals. They constitute a new régime in wave physics, with potential to describe a wide range of phenomena from the sighing of the forest through the reflection of radio waves by landscapes to the propagation of light in fluids near their critical points.

In [2] I set up and solve what must be the simplest diffractal problem: free-space propagation of an initially-plane wave on which S has imposed a random fractal deformation of the wavefront at z = 0. Only propagation in two space dimensions x,z is considered. The initial wavefront is the fractal curve z = h(x), so that to a good approximation the diffraction problem has boundary condition

$$\psi(x\,,0) = e^{-2\pi i h(x)/\lambda} \tag{14}$$

Using diffraction theory, the propagation of ψ in the z direction can be studied, and the development of intensity fluctuations as a function of z can be followed by averaging over the ensemble of random wavefronts h(x).

To carry out this programme, h must be specified more precisely. It is here taken to be a Gaussian random function whose graph has fractal dimension D lying between 1 and 2 and whose correlations are described by the r.m.s. increment of h over distance X by

$$\sqrt{\langle[h(x+X) - h(x)]^2\rangle} = L^{D-1}\;|X|^{2-D}. \tag{15}$$

The distance L (called the 'topothesy' of the wavefront) is the separation of points on the graph of h whose connecting chord has r.m.s. slope unity; L is a measure of the strength of the wavefront deformation. Both the variances $\langle h^2\rangle$ and $\langle(\partial h/\partial x)^2\rangle$ are infinite, but the existence of the average [15] is all that is required for diffracted statistics to be well defined. Fig. 1 shows a computed random function with D = 1.5. These graphs have the property of being self-similar under magnification [7] provided x and h are scaled in suitable ratio.

Diffraction theory shows that the intensity moments I_n (Eq. (1)) depend on D and also on one other parameter ζ which incorporates z, λ and L as follows:

$$\zeta = \frac{2\pi z}{\lambda}\left(\frac{2\pi L}{\lambda}\right)^{(D-1)/(2-D)} \Big/ 2^{1/(4-2D)} \; . \tag{16}$$

This is the most important of several diffractal scaling laws derived in [2]. It turns out that even for this apparently simple problem it is prohibitively difficult to calculate moments higher than the second (the first moment I_1 is unity for all ζ, as follows easily from (14)). I_2 is given by the following double integral:

$$I_2(\zeta) = \frac{4}{\pi\zeta}\int_0^\infty du \int_u^\infty dv\, \cos\frac{uv}{\zeta}\, e^{-[2u^{4-2D} + 2v^{4-2D} - (v+u)^{4-2D} - (v-u)^{4-2D}]} \; . \tag{17}$$

This integral has been studied by other authors (although not with diffractal interpretation) in connection with the propagation of laser beams [21] and radio [22] through turbulence; my results in [2] complement and extend theirs. The behaviour of the second-moment curves $I_2(\zeta)$ as D varies from 2 (extreme fractal with the graph of h just area-filling) to 1 (marginal fractal with the graph of h almost smooth) is summarised on Fig. 2.

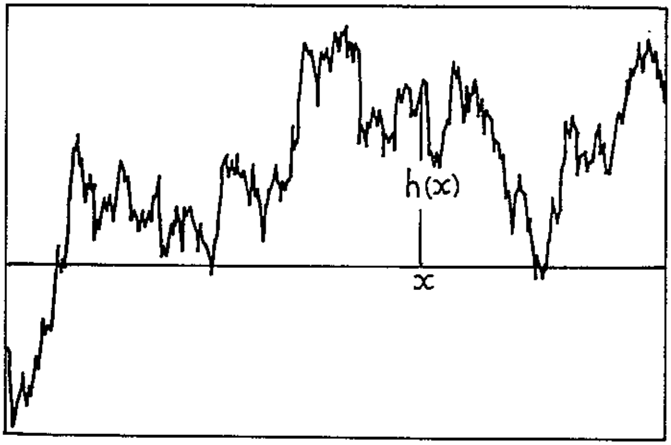

Fig. 1 Random fractal wavefront h(x) with D = 1.5 (computed by Z. V. Lewis)

In all cases $I_2(0) = 1$ (no intensity fluctuations near the initial wavefront where (14) shows that ψ is purely phase-modulated), and $I_2(\infty) = 2$ (Gaussian intensity fluctuations far from the initial wavefront). For intermediate values of ζ there may or may not be a maximum where $I_2 > 2$. In the case of the 'rougher' fractals $D \geq 1.5$ there is no maximum. For the 'less rough' fractals $D < 1.5$, however, there is a weak maximum in $I_n(\zeta)$. This can be regarded as an anticipation of the very strong maximum ($I_2 \propto \log \lambda^{-1}$) that occurs when the initial wavefront is smooth and which arises from diffraction catastrophes as explained in Section 2; this highly non-Gaussian second moment is illustrated in the bottom curve on Fig. 2.

The fractal and ordinary rêgimes are separated by the marginal case $D \to 1$. As shown on Fig. 2 the decay to Gaussian fluctuations is extremely weak for this case and takes the form of a term $(\log \zeta)^{-1}$. This asymptotic behaviour emerges from the analysis [2] as the result of an accumulation of power-law decays which is unprecedented in wave theory as far as I am aware.

4. Discussion

In this work I have tried to extend the boundaries of conventional random wave theory by studying two extreme rêgimes. In both cases the form of the intensity probability distribution is unknown and certainly not Gaussian. All we have is some information about the moments I_n. In the 'diffraction catastrophe' case of waves encountering smooth random structures S it was shown in Sec. 2 that the I_n obey universal scaling laws (3) as $\lambda \to 0$. However, this is very far from being a complete description of the statistics. For a start, each power $\lambda^{-\nu_n}$ is multiplied by a coefficient that depends on the measure of the dominant catastrophe in the space $\underline{V}$ of random variables of the ensemble of S, and this in turn depends on the nature of S--it is not universal. And then there is the question of the intensity correlations between different points rather than the fluctuations at a single point; it seems [17 - 19] that these correlations are characterised by several length scales, but there has been no analysis of the limit $\lambda \to 0$.

For diffractals the situation is just as bad. We know nothing about higher moments $I_{n>2}$. And we do not know whether there is any 'universality' about the behaviour of I_2 summarised on Fig. 2. It is probably a reasonable approximation to consider a random fractal S as imposing a random fractal deformation on a wavefront, but the randomness need not be Gaussian and the deformation need not be of

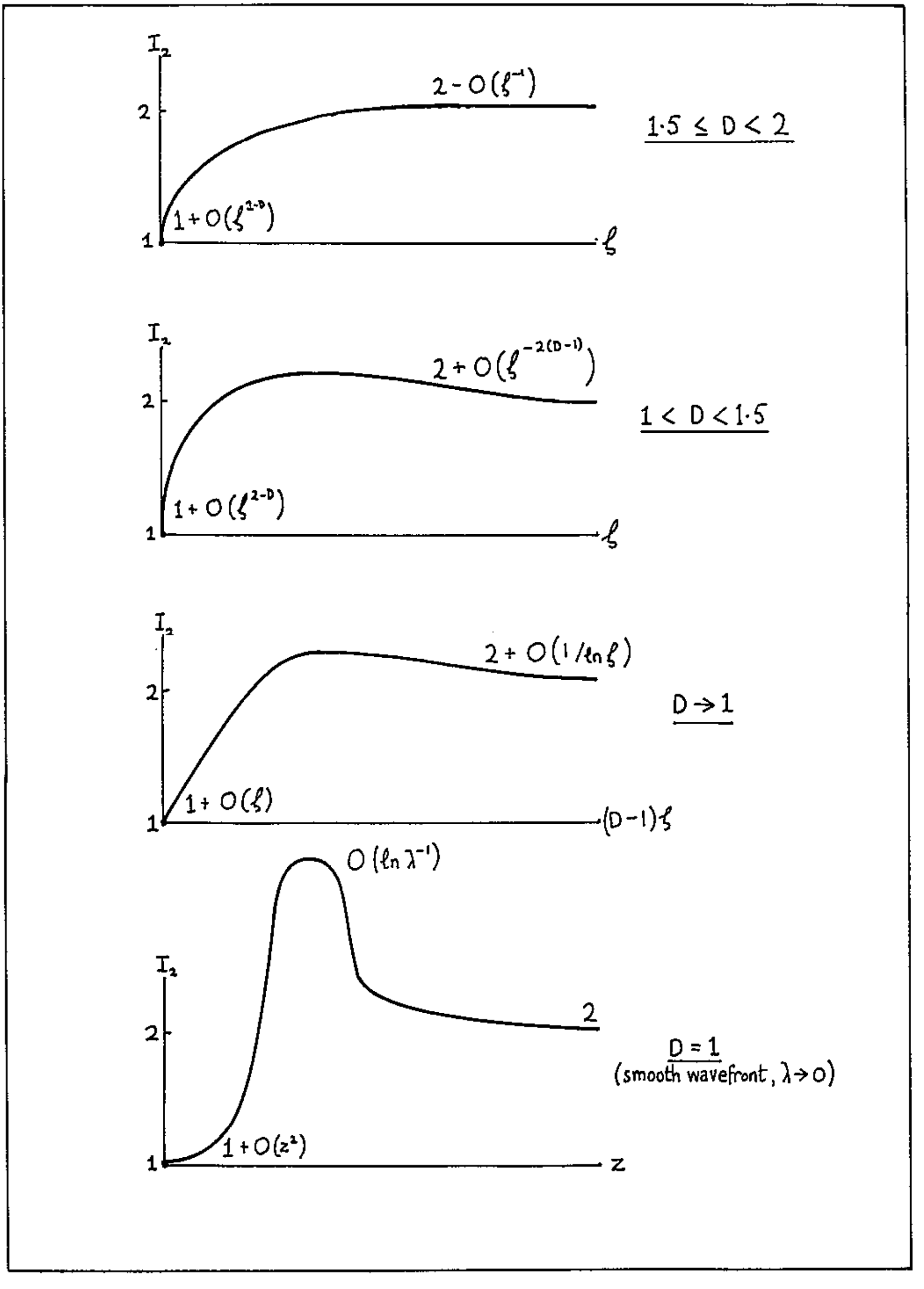

Fig. 2 Development of intensity fluctuations I_2 as a function of propagation distance for three diffractals (top three curves) from initial wavefronts with different fractal dimensions D, and one wave (bottom curve) that develops from a smooth initial wavefront (cf. Sec. 2)

a plane or of a straight line as assumed in Sec. 2. Moreover, we know little about waves within fractals (sound inside a tree, radio waves in the midst of turbulence, high-order modes of oscillation of an inland lake with random fractal boundary, etc.) although a conjecture on this subject is presented in [23].

In conclusion, it appears that the geometrical concepts of catastrophes and fractals can be fruitfully applied to wave physics. This is particularly the case in statistical problems where averaging over an ensemble of different realisations of a system implies that its properties will dominated by those morphologies that are structurally stable, as emphasised by Professor Thom.

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