Distribution of Modes in Fractal Resonators

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1. Introduction

I shall present a conjecture which if correct would greatly extend our understanding of the distribution of modes in vibrating systems. Consider a region R of D-dimensional space, with a boundary ∂R which is d-dimensional, where $d = D-1$. Let eigenfunctions ψ_n and eigenvalues k_n in R satisfy

$$\left.\begin{aligned} (\nabla^2 + k_n^2)\,\psi_n &= 0 \text{ in } R \\ \psi_n &= 0 \text{ on } \partial R \end{aligned}\right\} . \tag{1}$$

Let M_D be the measure of R (e.g. the volume if $D = 3$) and let m_d be the measure of ∂R (e.g. the area if $D = 2$). Then it is known [1,2,3] that the asymptotic mode number $N(k)$, defined as the number of modes with $k_n < k$ (where $k \to \infty$) is given (after a smoothing whose technical details need not concern us) by

$$N(k) = \frac{M_D k^D}{(D/2)!(4\pi)^{D/2}} - \frac{m_d k^d}{4(d/2)!(4\pi)^{d/2}} + \ldots \tag{2}$$

My conjecture is as follows: formula (2) remains valid when the resonator R and/or its boundary ∂R are fractals [4] (that is if D and/or d are not integers) provided D and d are interpreted as the Hausdorff-Besicovitch (fractal) dimensions of R and ∂R, and M_D and m_d are the Hausdorff D- and d-measures [4] of R and ∂R. I shall list some examples indicating the vast scope of this generalisation, and then present a discussion of its meaning and a plausibility argument for its correctness.

2. Examples

These fall into three classes. I: ∂R is a fractal, R is not. II: R is a fractal, ∂R is not. III: R and ∂R are both fractals.

Class I Here D is an integer but d is not. The first example is oscillations of water in a lake, where R (the lake) has $D = 2$, while ∂R (the lake's coastline) has $1 < d < 2$ (usually $d \simeq 1.3$,[4]). A model for this might be the Koch drum, where R is a planar membrane ($D = 2$) and ∂R the Koch snowflake curve [4] ($d = \log 4/\log 3 = 1.262$). Another example is vibrations of the whole Earth, where R (the Earth's matter) has $D = 3$ and ∂R (the Earth's surface) has $2 < d < 3$. Another example is the acoustic modes of a concert hall with fractally irregular walls ($D = 3$, $2 < d < 3$). Finally there are the oscillations of fluid in sponges or fractally porous rock ($D = 3$, $2 < d < 3$).

Class II Here d is an integer but D is not. The first example is the Weierstrass guitar, by which I mean the vibrations of a (long!) wire bent into a segment of a fractal curve such as the Weierstrass function [4,5] or the graph of one-dimensional Brownian motion [4]. In this case $d = 0$ (because the boundary is simply two points) and $1 < D < 2$. The second example is the Mandelbrot drum of the first kind, by which I mean vibrations of a rigid elastic sheet (e.g. fibre-glass) moulded to fit a fractal surface ($2 < D < 3$) and bounded by a smooth curve ($d = 1$).

Class III Here neither D nor d is an integer. The first example is the Mandelbrot drum of the second kind, which is the same as the drum of the first kind just defined except that its boundary is fractal, as in the case, for instance, of the surface of an island bounded by its coastline ($d = D - 1$). Another example is vibrations of the material of sponges or fractally porous rock (as opposed to the Class I vibrations of their fluid contents), for which $2 < D < 3$. A model for such resonators is the shivering Sierpinski sponge ($D = \log 20/\log 3 = 2.727$). The final set of examples in this class concerns fractal networks: the elastic vibrations of a tree, for instance, have $D < 3$ (because the branches are nearly volume-filling) and $d \lesssim 2$ (because the leaves are nearly area-filling); another problem of this sort is the waves described by Schrödinger's equation in networks [6] such as Cayley trees [7,8], to model the behaviour of electrons in disordered media.

3. Discussion

The fact that the terms in the formula (2) for $N(k)$ retain their meaning when D and/or d are fractional does not, of course, ensure that the formula can be validly employed for fractals. However, I shall now give a scaling argument strongly suggesting that at least the k-dependence of (2) is correct.

This is based on the idea that modes with wave numbers less than k, and hence wavelengths exceeding $\lambda = 2\pi/k$, are unaffected by detail in R and ∂R on scales smaller than λ. Therefore $N(k)$ can be estimated by replacing the fractals R and ∂R by λ-smoothed manifolds R_k and ∂R_k. These smoothed manifolds are not fractals, but have integer dimensionalities equal to the topological dimensions [4] $D_T (\leq D)$ and d_T ($= D_T - 1 \leq d$) of R and ∂R respectively. Now the conventional (integer-dimension) version of (2) can be employed, provided it is realised that the measures M_{D_T} and m_{d_T} are k-dependent. Thus

$$N(k) = C_1 M_{D_T}(k)\, k^{D_T} + C_2 m_{d_T}(k)\, k^{d_T} + \ldots, \qquad (3)$$

where C_1 and C_2 are constants.

As k increases, so do M_{D_T} and m_{d_T}, the laws of increase being

$$M_{D_T}(k) \propto M_D\, k^{D-D_T}, \quad m_{d_T}(k) \propto m_d\, k^{d-d_T}. \qquad (4)$$

(For example, when ∂R is a coastline, $d_T = 1$ and $d \approx 1.3$, and $m_{d_T}(k)$ is the coast's increasing length as measured on the scale λ.) Substitution of (4) into (3) gives precisely the same k-dependence as (2). By continuity it seems very likely that the constants in (2) are also correct.

This argument shows how the wave equation (1) should be interpreted in the cases where R is a fractal (Classes II and III in Section 2). Obviously ∇^2 cannot then be written in D-dimensional coordinates! But it can be written in D_T-dimensional coordinates, so that (1) can be thought of as a wave equation on the smoothed manifold R_k rather than R. (Dr. F.J. Wright has pointed out to me that this procedure is equivalent to solving (1) using the finite-element method, by discretising ∇^2 on a grid sampling R on the scale λ.) A simple illustrative example is the Weierstrass guitar (Section 2), where R_k is a smooth one-dimensional string with length $l(k)$. Then an explicit approximate 'quantum condition' for the eigenvalues, with an obvious origin, is

$$k_n \approx \frac{n\pi}{l(k)}, \quad \text{i.e. } N(k) \simeq \frac{l(k)k}{\pi}, \qquad (5)$$

which agrees with the first term of (2) for a D-dimensional string.

Suppose now that ψ satisfies not the Dirichlet boundary condition as in (1) but the Neumann condition, namely that the normal derivative of ψ vanishes on ∂R. Then the only effect on (2) is to change the sign of the second term from negative to positive. If ∂R is a fractal, this Neumann condition has no meaning if applied literally, and it seems reasonable to interpret it as applying instead to the smoothed boundary ∂R_k.

Next, I want to suggest that (2) with fractional d and/or D is likely to be a useful approximation in many cases where R and ∂R are not fractals. These are cases where R and/or ∂R possess hierarchical structure which does not however extend to infinitely small scales. Then of course the true asymptotic form of N(k) as $k \to \infty$ is given by (2) with integer D and d. However, for k corresponding to wavelengths $2\pi/k$ in the midst of the hierarchy of R and ∂R it is probable that N(k) will be given by (2) with D and d equal to 'pseudo-fractal dimensions' corresponding to the properties of the hierarchy extrapolated to indefinitely small scales.

The conjectured generalisation of (2) to fractal R and ∂R is radical. This can be appreciated by first considering what is known about smooth (non-fractal) resonators. The simplest case is where ∂R is such that (1) is separable. This corresponds to 'integrable' motion [8,9] of the straight 'rays' in R when reflected specularly from ∂R, the integrability being manifested by rays enveloping caustics in R,[10]. For generic smooth resonators the rays are 'non-integrable,' and typically fill R chaotically without forming caustics. However, in my proposed extension to fractal resonators, the reflection of rays <u>is not defined</u> if ∂R is a fractal, and the rays themselves <u>do not exist</u> if R is a fractal. Therefore no 'geometrical-optics' approximation can be invoked, even in the short-wave limit $k \to \infty$.

The eigenfunctions ψ in fractal resonators are examples of what I have called 'diffractals' [11,12]. It is possible that near a fractal boundary ∂R the form of ψ might in some statistical sense resemble the diffractal that evolves from a random fractal wavefront, whose properties are beginning to be understood [11,12].

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