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SOME GEOMETRIC ASPECTS OF WAVE MOTION: WAVEFRONT DISLOCATIONS, DIFFRACTION CATASTROPHES, DIFFRACTALS

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ABSTRACT. On the finest scales, waves $\psi = \rho e^{i\phi}$ are dominated by the singularities of their phase ϕ, called 'wavefront dislocations'. On the largest scales, when the wavelength is small in comparison with diffracting objects ('geometrical-optics limit'), waves are dominated by caustics (focal surfaces); stable caustics can be classified as catastrophes; caustics are decorated with characteristic interference patterns, called 'diffraction catastrophes'. When waves encounter fractals (objects with an infinite hierarchy of length scales) the short-wave limit is not geometrical optics; this is a new regime, and such waves are called 'diffractals'.

1. INTRODUCTION. In this article I describe wave motion unconventionally, in terms of the contrast and interplay between three morphologies. These are distinguished by the scale on which a wave is explored, and the scales of variation of diffracting objects or refracting media, in comparison with the wavelength λ.

The first morphology, to be discussed in section 2, occurs on the scale of λ, where the principal features are the wavefronts, or surfaces of constant phase. Wavefronts are dominated by their singularities, in the form of lines in space called 'wavefront dislocations' [1] by analogy with the similar structures that interrupt the regular stacking of atomic planes in crystals. It is typical for waves to be dislocated in this way.

The second morphology, to be discussed in section 3, occurs when λ is small in comparison with all scales of variation of the waves' environment. In the limit $\lambda \longrightarrow 0$, geometrical optics gives a description of wave fields in terms of families of rays. These are dominated by their focal singularities, called caustics, which are envelopes of the ray families, typically in the form of surfaces in space. Stable caustics are classified by the catastrophe theory of Thom [2] and Arnol'd [3]. When λ is small but not zero, asymptotic approximations to wave functions can be obtained in terms of a hierarchy of interference patterns decorating the caustics; these are the 'diffraction catastrophes'.

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The third morphology, to be discussed in section 4, occurs when waves propagate in environments characterised by a hierarchy of scales extending infinitely small. Mandelbrot [4] uses the term 'fractals' to describe such structures in terms of a Hausdorff-Besicovitch dimension that need not be an integer. A fractal never appears smooth, no matter how short are the waves used to explore it. Therefore the short-wave limit is not geometrical optics. For waves of this sort I introduced the term 'diffractals' [5].

Most of the material reviewed here has been published elsewhere, so I shall confine myself to making assertions for which justification can be found in the papers cited, and to emphasising mathematical problems raised by this 'morphological' approach to wave physics.

2. WAVEFRONT DISLOCATIONS. Let $\psi(X)$ be a complex scalar wave function, i.e. a smooth map from spacetime X = (x, y, z, t) to the complex plane whose polar coordinates are the wave amplitude ρ and the wave phase ϕ. Thus

$$\psi(X) = \rho(X)\, e^{i\phi(X)}, \tag{1}$$

where ρ and ϕ are real. The wavefronts are the hypersurfaces $\phi =$ constant (mod 2π). Let C be a simple closed curve in X (fig. 1) and let C' be the image of C in ψ. We define the dislocation strength S_C enclosed by C as the winding number of C' about the origin $\psi = 0$, i.e.

$$S_c = \frac{1}{2\pi}\int_c d\phi . \tag{22}$$

The dislocations themselves are those regions in X through which C must pass in order for S_C to change. Obviously $\psi = 0$ on a dislocation. Therefore $\rho = 0$ and ϕ is undefined, so that dislocations are singularities of the wavefronts, as is clear from the example in fig. 1. Because the vanishing of ψ requires two conditions (Re ψ = Im ψ = 0), dislocations have codimension two, and so occur typically as lines in a plane or points in space (which move unless ψ is monochromatic).

For all waves except those in quantum mechanics, the quantity of direct physical significance is not ψ but Re ψ. Dislocations correspond to births and deaths of crests, troughs and zeros of Re ψ (crests are loci of $\phi =$ mod 2π, troughs are loci of $\phi = \pi$ mod 2π). This can be seen by referring to fig. 1. Consider the waves passing points A, B, C. The graphs of Re ψ as a function of time are shown on fig. 2. Clearly the wave through C has one more crest and two more zeros than that through A, and these features are born at the point B through which the dislocation passes. In fact dislocations were discovered in this way [1] during an experiment in which ultrasonic waves were reflected from a rough surface in a laboratory simulation of the echo-sounding of the Antarctic glacier bed by radio waves (see also [37]).

(a)

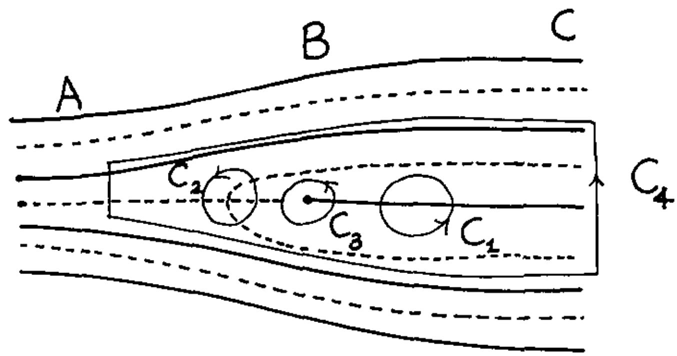
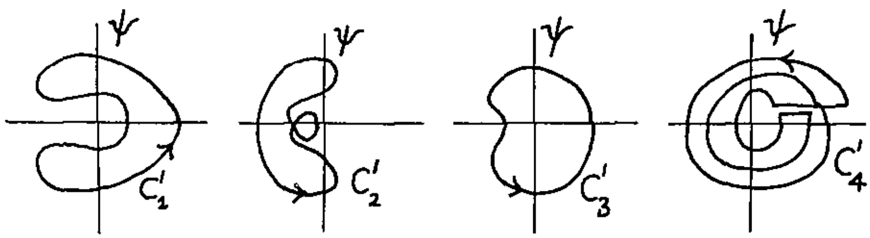


Fig. 1(a). Snapshot of a wave travelling up the page, showing crests (heavy lines) and troughs (dashed lines). The circuits C_1 and C_2 do not enclose any dislocations, while the circuits C_3 and C_4 each enclose a single dislocation.
(b) Images C_1' - C_4' in the ψ plane.

(b)



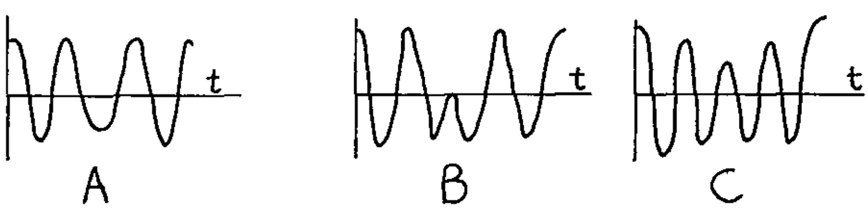


Fig. 2. Graphs of Re $\psi(t)$ at the points A, B, C on fig. 1, showing birth of a crest and two zeros as the dislocation passes B.

Of course ψ must satisfy a wave equation, but it is surprising that this seems to impose no restriction on the conformation, motion and interaction of dislocation lines. In the case of the simple wave equation

$$\Delta\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} , \tag{3}$$

for example, a variety of dislocation morphologies can be exhibited in terms of explicit exact solutions. These solutions can be found in [1] , and take the form of dislocated versions of a 'host' monochromatic plane wave

$$\psi = e^{i\omega(z/c - t)} \tag{4}$$

The simplest case is the isolated 'edge' dislocation line (fig.1) moving rigidly with the host wavefronts; its wave function is

$$\psi = A\left(x + i\beta(z - ct)\right) e^{i\omega(z-ct)} , \qquad (5)$$

where A and β are constants. This corresponds to inserting an extra half-plane of 'crest' wavefront between the crests of the host wave. Dislocations can also be of 'screw' type, about which the wavefronts wind in helicoids, or of mixed 'edge-screw' type. They can be curved, and can form closed loops (holes in wavefronts, or isolated patches of wavefront). Dislocation lines can move rigidly with the host wavefronts, or parallel or perpendicular to them, or they may remain at rest. When dislocations encounter one another, they may annihilate, bounce off one another, or not interact at all.

The most important mathematical questions seem to be: for waves satisfying a given equation, which sorts and strengths of dislocation, and which dislocation interactions, occur stably in the plane, in space and in space-time? A promising approach to these questions has been initiated by Wright [6], who considered not the zeros of the complex ψ but the coalescing extrema of Re ψ as a function of t, using catastrophe theory to investigate stability.

It is surprising that wavefront dislocations have not been extensively studied before. After all, anybody since Huygens could have asked the question: Can wavefronts have edges? Moreover, dislocations are generic features of waves - the air in a sound-filled auditorium is threaded with a tangled 'forest' of dislocation lines. Perhaps this neglect stems from the fact that dislocations are the most delicate features of a wave. To discriminate them, ψ must be studied on length and times scales comparable with λ and the wave period respectively. Eyes and ears cannot do this - they are not 'phase-sensitive' but respond instead to (Re ψ)^2 averaged over a number of cycles.

There have, however, been several partial anticipations of the dislocation concept. In 1836, Whewell [7] realised that the study of tides was greatly simplified by considering them as harmonic waves with a period of about 12 hours, whose wavefronts ('co-tidal lines') mark the loci of high water at different times. He discovered that the pattern of co-tidal lines has singularities ('amphidromic points') where the tide is always 'high' but has zero amplitude; one such singularity is shown on fig. 3. Of course these are wavefront dislocations, in this case points because the waves propagate in two dimensions. Then, in 1931, Dirac [8], while studying the effect of a magnetic field on the phase ϕ of an electron's wave function ψ, realised that there could exist circuits around which ϕ changed by a multiple of 2π , enclosing line zeros of ψ . However, he was more concerned with the ends of these lines, which are singularities not just of ψ but of the magnetic field, corresponding to magnetic monopoles. Next, in 1970, Riess [9] considered zeros ('nodal hypersurfaces') of quantum bound many-body systems such as molecules, and related the zeros to magnetic properties of the system. Finally, in work contemporary with ours, Hirschfelder and his collaborators [10] independently discovered dislocations in quantum mechanics, and called them 'quantised vortex lines', thus emphasising the vector field gradϕ rather than ϕ itself. All these authors confined themselves to monochromatic waves, and hence to stationary dislocations.

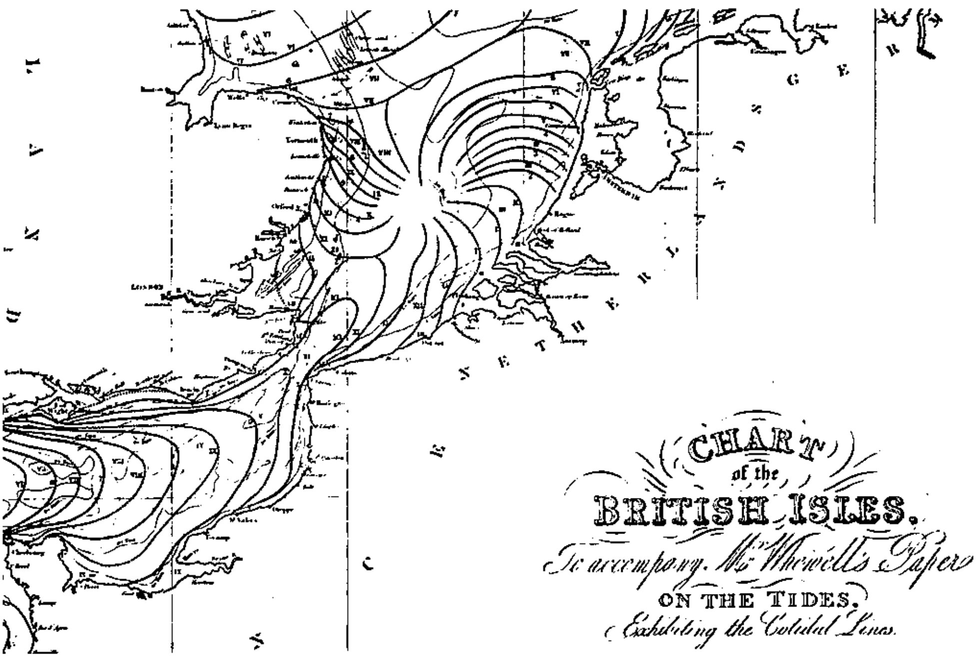


Fig. 3. Whewell's chart of co-tidal lines (wavefronts) in the North Sea [7] , showing a dislocation point.

It is important to distinguish wavefront dislocations from the singularities of 'geometrical-optics wavefronts' [3] . The latter are singularities of isosurfaces of solutions of the Hamilton-Jacobi equation and differ from wavefront dislocations in several ways. Firstly, the geometrical-optics wavefronts can cross (because several rays can pass through a point), whereas the true wavefronts ϕ = constant cannot cross (except on a dislocation). Secondly, the singularities of geometrical-optics wavefronts are ray caustics, which are typically surfaces in space (section 3), whereas wavefront dislocations are lines in space. And finally, while the solutions of the Hamilton-Jacobi equation are useful ingredients in short-wave approximations to ψ , the geometry of the geometrical-optics wavefronts does not itself possess direct physical significance; by contrast, the true wavefronts are directly detectable as crests and troughs of ψ.

3. DIFFRACTION CATASTROPHES. Question: which wave morphology is complementary (in the sense employed by Bohr in discussing the philosophy of quantum mechanics) to the wavefront dislocation? Answer: the caustic. Reason: at a dislocation, the wave's phase is singular and its amplitude vanishes. Observation of a dislocation requires discrimination of the wave's structure on wavelength scales - circumstances under which caustics are blurred by diffraction. At a caustic, on the other hand, the wave's geometrical-optics amplitude is infinite. The observation of a caustic takes place on scales large compared with the wavelength - circumstances under which phase details, and hence dislocations, are hard to discern. In other words, caustics are the singularities of ray theory, and dislocations are the singularities of wave theory.

Caustics are envelopes of ray families. Stable caustics are classified as catastrophes, that is as stable singularities of gradient maps. The reason for this ([11 - 14]), in outline, is that by Fermat's (or Hamilton's) principle rays are given by the extrema of optical distance (or action) and hence correspond to gradient maps, while caustics, on which neighbouring rays touch, are the loci of coalescence of the extrema and hence correspond to singularities of the maps.

Each catastrophe gives an equivalence class, consisting of caustics related by diffeomorphism (this is the sense in which the caustics are stable). The classification is in terms of codimension, at least for the simplest cases [3] . The catastrophe labelled i is defined by a generating polynomial $G_i\ (S_j;\ C_k)$ depending on 'state variables' S_j and 'control parameters' C_k. In ray theory, C_k could represent spacetime coordinates X, S_j could represent coordinates on a given initial wavefront, and G_i could represent optical path or action. The caustic inhabits control space, and is defined by the locus of C_k for which

$$\frac{\partial G_i}{\partial S_j} = 0 \quad \text{and} \quad \det\left\{\frac{\partial^2 G_i}{\partial S_j\, \partial S_{j'}}\right\} = 0 . \tag{6}$$

In three-dimensional space, the stable caustics are smooth 'fold' surfaces (codimension one), 'cusp' lines (codimension two), and 'swallowtail', 'elliptic umbilic' and 'hyperbolic umbilic' points (codimension three). These forms can be seen in light focused by irregular glass (of the type used for bathroom windows) or by irregular water-droplet 'lenses' [11, 15, 16] .

Catastrophes play an important role in the physics of waves [27], analogous to that played by atoms in the physics of matter. It is helpful to think of them as 'elemental atomic forms', occupying a 'mesoscale' between 'macroscopic' caustic networks, formed by combinations of catastrophes, and 'microscopic' interference patterns decorating individual caustic singularities. On this analogy, wavefront dislocations are the elementary particles of wave physics.

'Macroscopic' caustic networks are a familiar sight as dancing lines of sunlight focused onto swimming-pool floors after refraction by the wavy water surface. The bright lines often meet in junctions, apparently violating catastrophe theory, according to which the only stable caustics in two dimensions are fold lines and cusp points. But the junctions are illusions, resulting from the fact that the patterns are imperfectly resolved. Figs. 4a and 4b show two commonly-observed caustic networks, and figs. 4c and 4d show their resolutions into detail suggested by catastrophe theory and confirmed by experiment [17, 18]. The study of caustic networks is still in its infancy; apart from some rules related to the indices of umbilic points [19, 14], no general results are known.

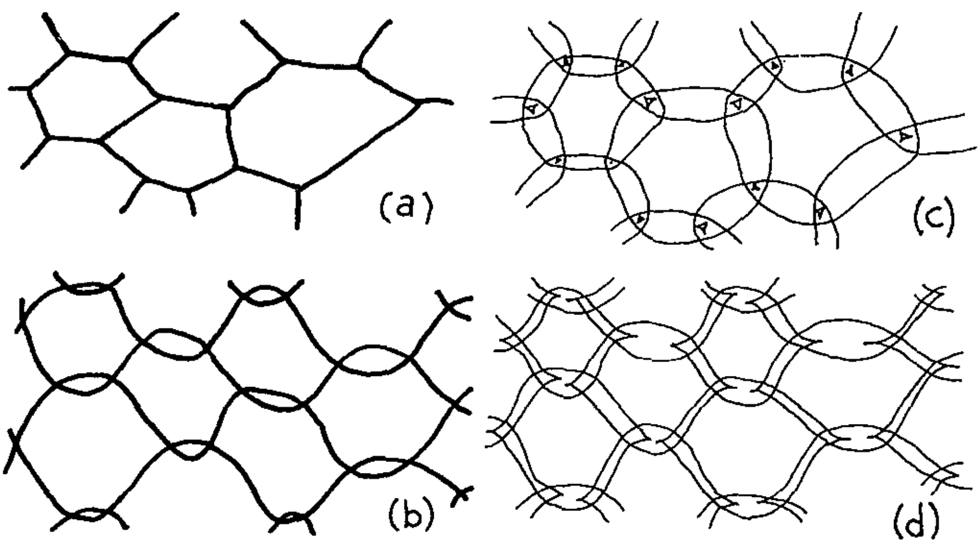


Fig. 4. Two sorts of 'swimming-pool' caustic network, seen with poor resolution (a,b) and good resolution (c,d).

On 'microscopic' scales, caustic singularities are softened by the effects of the finite wavelength λ. Asymptotic approximation of the solution of the wave equation [3, 13, 14, 20-22] yields the remarkable result that each catastrophe i gives rise to a canonical wave function $\psi_i(C_k;\lambda)$ describing diffraction near the caustic. The ψ_i take the form of oscillatory integrals, defined in terms of the generating polynomials G_i, as follows:

$$\psi_i(C_k;\lambda) = \frac{1}{\lambda^{n/2}}\int\cdots\int d^n S_i \, e^{\frac{2\pi i}{\lambda} G_j(S_j;C_k)} \qquad (7)$$

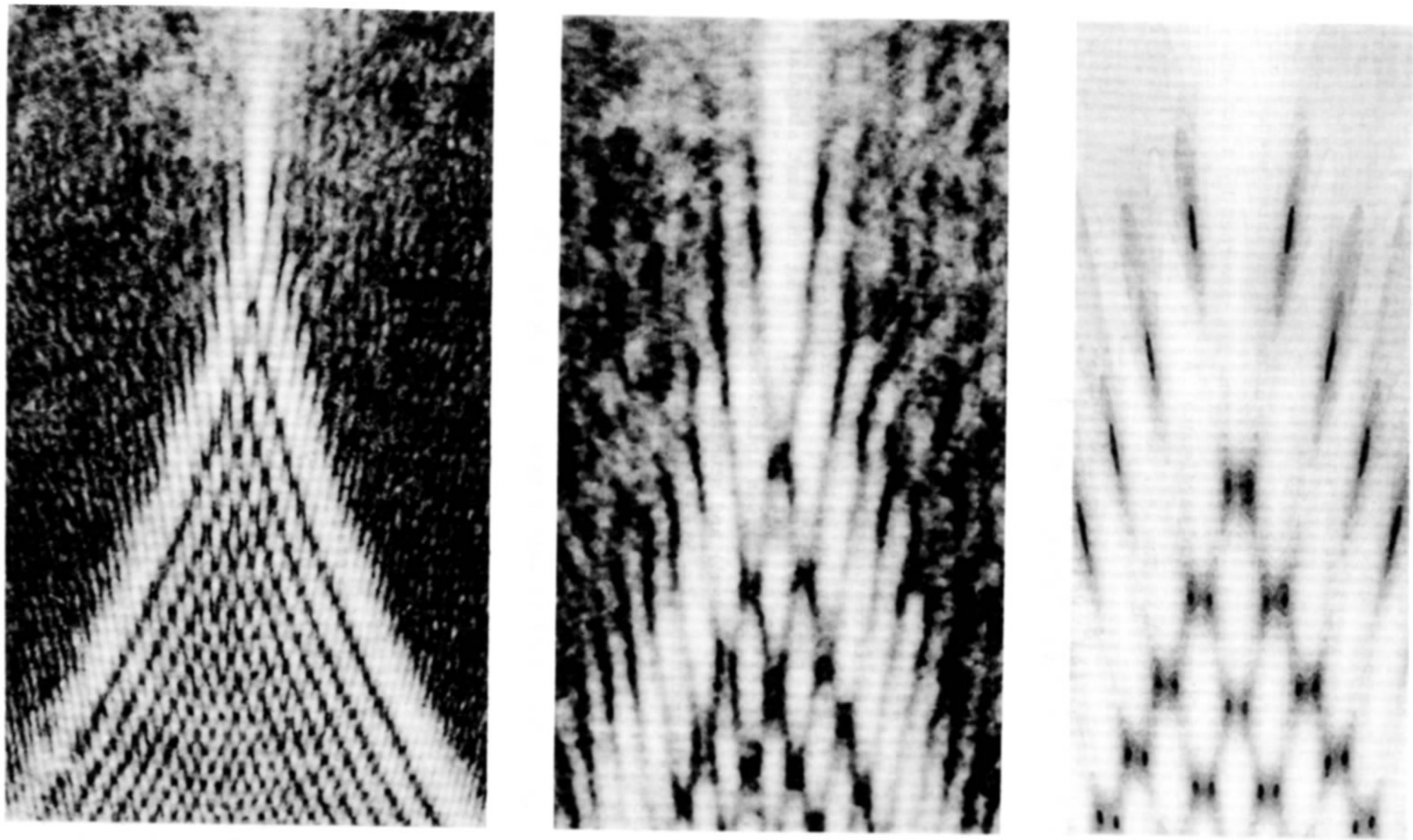
(n is the corank of the catastrophe, that is the number of state variables s_j.) Thus catastrophe theory classifies stable short-wave diffraction patterns as well as stable caustics. The 'diffraction catastrophes' (7) constitute a new hierarchy of standard functions; our present state of knowledge concerning them can be summarised under three headings.

(i) Computations. In 1838, Airy [23] computed the fold diffraction catastrophe $\psi_{fold}(C_1;\lambda)$ (which in modern notation [24] is proportional to the Airy function Ai). In 1946, Pearcey [25] computed the cusp diffraction catastrophe $\psi_{cusp}(C_1, C_2;\lambda)$. The close agreement between theory and experiment is obvious from fig. 5. Notice in particular the wavefront dislocation points (figs. 5b, c), which appear on these intensity pictures as elongated dark regions where $|\psi_{cusp}| = 0$, occurring in pairs within the arms of the caustic and singly in two rows flanking the caustic. The ability of diffraction catastrophes to give an accurate description of these delicate features is remarkable.

The only higher diffraction catastrophe so far studied in detail is the elliptic umbilic [26]. Its wave function $\psi_E(C_1, C_2, C_3;\lambda)$ describes an elaborate diffraction pattern in space, whose maxima are stacked in a regular array like atoms in a distorted crystal, and whose dislocation lines form either hexagonally puckered rings or 'hairpins'. We found that, just as for the cusp, experiment and theory agree down to the finest details.

(ii) Projection identities. Because the G_i are polynomials, the diffraction catastrophes ψ_i satisfy partial differential equations; these are very complicated. However, they also satisfy a series of astonishing nonlinear identities [28]. These relate the intensity $|\psi_i|^2$ to an integral over the wave function ψ_i corresponding to the same catastrophe (or a particular hypersection of it) or to a less singular catastrophe. Here are three examples, written for simplicity for the case $\lambda = 2\pi$:

$$\left.\begin{aligned}
|\psi_{fold}(C_1)|^2 &= \sqrt{\frac{2}{\pi}}\, 2^{-1/3}\int_{-\infty}^{\infty} du\, \psi_{fold}\{2^{2/3}(u^2+C_1)\} \\
|\psi_{cusp}(C_1, C_2)|^2 &= \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} du\, |6u|^{-1/3}\psi_{fold}\{2\,\mathrm{sgn}\,u(6|u|)^{-1/3}(u^3+C_2u+C_1)\} \\
|\psi_E(C_1, C_2, C_3)|^2 &= \frac{2^{1/3}}{\pi}\int_{-\infty}^{\infty} du_1\int_{-\infty}^{\infty} du_2\, \psi_E\{2^{2/3}(C_1+2C_3u_1+3(u_2^2-u_1^2)), \\
&\qquad 2^{2/3}(C_2+2C_3u_2+6u_1u_2), 0\}
\end{aligned}\right\} \qquad (8)$$



(a) (b) (c)

Fig. 5(a). Observation of diffraction near a cusp (courtesy of J.F. Nye); (b) enlargement of (a); (c) simulation of (b), obtained by shading a computed contour map of $\psi_{\text{cusp}}(C_1, C_2)$ (courtesy of F.J. Wright).

The first of these was already known [29, 30] as a relation between Airy functions.

These identities have a quantum-mechanical interpretation [28] in terms of the projections from phase space to control space of Wigner functions [30] associated with classical Lagrangian manifolds [21].

(iii) Wavelength scaling. As λ tends to zero, the wave intensity on a caustic becomes infinitely large and the scale of detail in interference fringes becomes infinitely small. These phenomena are governed by a scaling law giving the wave $\psi_i(C_k; \lambda)$ for wavelength λ in terms of the wave for some other wavelength λ_o. The law, derivable from (7), is

$$\psi_i \{ C_k ; \lambda \} = \left(\frac{\lambda_o}{\lambda}\right)^{\beta_i} \psi_i \left\{ \left(\frac{\lambda_o}{\lambda}\right)^{\sigma_{ki}} C_k ; \lambda_o \right\} , \tag{9}$$

where the exponents β_i and σ_{ki} are different for each catastrophe and depend on the precise form of the polynomial G_i. Therefore ψ_i need be computed only for a single wavelength. All scaling information about diffraction catastrophes is contained in the exponents, whose number is $K_i + 1$, where K_i is the codimension of the catastrophe i.

β_i governs the divergence of $|\psi_i|$ as $\lambda \to 0$, at the most singular point $C_k = 0$ in control space. It was introduced by Arnol'd [3] (who called it the 'singularity index') and studied in detail by Varchenko [31]. β_i increases with codimension K (and faster for umbilics than cuspoids). This is reasonable, since the larger K is, the greater is the number of rays coalescing to touch the caustic at its most singular point.

σ_{ki} govern the shrinking fringe spacings in different control directions C_k, as $\lambda \to 0$; these numbers were introduced by Berry [32]. The sum

$$\gamma_i \equiv \sum_{k=1}^{K} \sigma_{ki} \tag{10}$$

governs the behaviour of the K-dimensional hypervolume of the main diffraction maximum in control space; this scales as λ^{γ_i}. Romero [33] showed that γ_i is invariant under diffeomorphism. γ_i, which may be called the 'fringe index', also increases with K, showing that the intense region near the singularity, where all rays interfere constructively, shrinks very rapidly as $\lambda \to 0$ for the higher catastrophes. This shrinking is anisotropic in control space (as indicated by the fact that for a given catastrophe the σ_{ki} are not all equal), which explains why diffraction catastrophes formed by objects large in comparison with λ are observed [26] to be greatly elongated in certain directions.

Details of the computation of these exponents (and an account of difficulties encountered for higher catastrophes) are given by Berry [32] in connection with the statistics of twinkling starlight - a problem involving the whole hierarchy of catastrophes. Simplified treatments can be found in [14, 34].

4. DIFFRACTALS. Question: how is it possible to prevent the emergence of diffraction catastrophes as $\lambda \longrightarrow 0$? Answer: by making waves encounter fractal objects. Reason: fractals [4] possess a hierarchy of structure which extends to infinitely small length scales. Therefore as $\lambda \longrightarrow 0$ waves explore ever-finer details of the objects, which never appear smooth; this means that rays do not exist, so the geometrical-optics limit is never encountered. For this new regime in wave physics I employ the term 'diffractal' [5]. Examples of diffractals are radio waves reflected by landscapes or refracted by turbulence, light in fluids near the critical point, and modes of vibration of trees and sponges.

Mandelbrot [4] shows that the simplest characterisation of a fractal's geometry is its Hausdorff-Besicovitch dimension D, which need not be an integer. And for the three diffractal problems so far studied D indeed plays a central role in the (non-catastrophic) scaling laws governing the wavelength-dependence of a variety of wave properties. Here I shall give a very brief progress report on these problems, two of which involve scattering while the other involves confined modes.

No exact solution is yet available for any case where a wave strikes an impenetrable fractal object or propagates in a fractally inhomogeneous medium. But it seems a reasonable approximation to assume that such a structure will impose a fractal deformation on wavefronts encountering it. This suggests the following canonical mathematical problem. A plane wave with wavelength λ travels along the positive z axis. The wavefront $z = 0$ is deformed into a scale-free random fractal curve with dimension $D (1 \leqq D < 2)$ and equation $z = h(x)$, and thereafter the wave $\psi(x,z)$ propagates freely towards $z = +\infty$, governed by the wave equation. The problem is to describe how the phase fluctuations at $z = 0$ develop as z increases into fluctuations of the intensity $|\psi|^2$. These intensity fluctuations are embodied in the 'second moment'

$$I_2 \equiv \frac{\langle |\psi(x,z)|^4 \rangle}{\langle |\psi(x,z)|^2 \rangle^2}, \tag{11}$$

where $\langle \ \rangle$ denotes averaging over the ensemble of $h(x)$.

A detailed discussion of this problem is given in [5] and summarised in [34] . It turns out that I_2 depends on D and the single parameter

$$\zeta \equiv \frac{z}{\lambda}\left(\frac{L}{\lambda}\right)^{\frac{D-1}{2-D}} . \tag{12}$$

The length L in this expression is the fractal curve's 'topothesy', i.e. the x separation between points on the curve whose connecting chord makes a root-mean-square slope of 1 radian with the x direction.

As ζ increases from 0 to ∞ , I_2 varies from 1 (no intensity fluctuations) to 2 (Gaussian fluctuations). The form of this variation depends on D: for the 'rougher' fractals $1.5 \leq D < 2$, I_2 increases monotonically with ζ and approaches 2 from below as ζ^{-1} ; for the 'less rough' fractals $1 < D < 1.5$, I_2 rises to a maximum of finite height at a finite value of ζ and approaches 2 from above as $\zeta^{-2(D-1)}$. This contrasts sharply with the behaviour of I_2 as $\lambda \longrightarrow 0$ when the initial wavefront h(x) is smooth [32] ; then the maximum of I_2 is $O(\ln \lambda^{-1})$, giving rise to infinitely strong fluctuations in the geometrical-optics limit, which arise from the formation of caustics in the ray family (normals to the initial wavefront).

In the second scattering problem (unpublished), a quasi-monochromatic wave train is emitted isotropically from a source at distance z above a D-dimensional fractal landscape ($2 < D < 3$) with topothesy L; the echo $\psi(t,z)$ reflected from the landscape is received back at the source. For large z the echo has a long tail, arising from reflections by distant irregularities. The average intensity in this tail decays as

$$\langle|\psi(t,z)|^2\rangle \underset{\substack{t\to\infty\\ z\to\infty}}{\sim} \left(\frac{L}{\lambda}\right)^{2(D-2)} \frac{1}{z^{D-1}\, t^{4-D}} . \tag{13}$$

This algebraic dependence involving D contrasts strongly with reflection from a smooth surface, for which $\langle|\psi|^2\rangle \sim \exp\left(-O(t/z)\right)$ when z and t are large.

The bound diffractal problem concerns the asymptotic mode distribution in fractal resonators [35] . Consider a D-dimensional region R with Hausdorff measure M_D, whose boundary ∂R is d-dimensional with Hausdorff measure m_d, where D and d need not be integers. Modes ψ_n are defined by

$$\left(\Delta + k_n^2\right)\psi_n = 0 \quad \text{in } R \text{, and } \psi_n = 0 \text{ on } \partial R . \tag{14}$$

Let N(k) be the number of modes with eigenvalues $k_n < k$. Then I conjecture that

$$N(k) \underset{k\to\infty}{\sim} \frac{M_D\, k^D}{(D/2)!\,(4\pi)^{D/2}} - \frac{m_d\, k^d}{4(d/2)!\,(4\pi)^{d/2}} + \cdots \tag{15}$$

This formula generalises a well-known result [36] for the case where D is an integer and d = D - 1.

In [35] I list the far-reaching implications of the conjecture (15) and give a plausibility argument for its correctness. This is based on the scaling properties of fractals under magnification, together with the idea that modes with $k_n < k$ are insensitive to detail in R and ∂R on scales much smaller than k (the same argument suggests an interpretation of Δ in (14) when R is a fractal).

At this early stage of its development, diffractal theory poses a number of mathematical problems, which should be studied for their intrinsic interest and also because of the wide potential applicability of the subject. I conclude with a list:

(i) Study of the wavefront dislocations (section 1) in a scattered diffractal.

(ii) Search for exactly-soluble scattering diffractal problems. (Evolution of a wavefront in the form of a Weierstrass function [4] ? Scattering from a Cantor set of points on a line? Scattering from a triadic island in the form of Koch's snowflake curve [4] ?)

(iii) Rigorous examination of the conjecture (15).

(iv) Search for exactly-soluble fractal resonator problems. (Modes of a drum bounded by the Koch curve? Modes of a shivering Sierpinski sponge [4] ?)

5. CONCLUDING REMARKS. Space prevents more than reference to an important morphological problem in quantum mechanics that seems to cut across the categories of this paper. This is the nature of quantum states in the semiclassical limit (Planck's constant $\hbar \rightarrow 0$), for bound systems whose classical motion is nonintegrable (e.g. a nonsymmetric vibrating molecule); such classical motion exhibits stochasticity [38, 39] , i.e. a sensitive 'unpredictable' dependence on initial conditions. The problem was originally posed by Einstein [40] but neglected until recently [41] . Two promising approaches are to describe the quantum system using Wigner functions [30, 42] , thus retaining contact with the classical phase space, and to study a new class of model system constructed by quantising area-preserving maps of the plane [43] .

The morphologies described here are not restricted to waves satisfying a particular equation. Wavefront dislocations, diffraction catastrophes and diffractals can occur in any sort of wave, monochromatic or non-monochromatic, scalar or vector, in media that may be inhomogeneous, anisotropic or time-dependent (though for vector waves the map defining dislocations is more complicated). Nowadays much attention is concentrated on non-linear waves, but I hope I have shown by describing these three extreme regimes that linear wave theory still contains much to interest mathematicians.

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