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INCOMMENSURABILITY IN AN EXACTLY-SOLUBLE QUANTAL AND CLASSICAL MODEL FOR A KICKED ROTATOR

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The Hamiltonian $H = 2\pi\alpha p + V(\theta)\Sigma_{-\infty}^{\infty}\delta(t-n)(0 \leqslant \theta < 2\pi)$ is solved exactly, classically and quantally; the solutions depend strongly on α. There is no classical chaos and the phase cylinder p, θ is filled with invariant curves, which are finite loops around the cylinder if α is sufficiently irrational and are translates of the infinitely long p axis if α is rational. Quantal quasi-energy states correspond exactly to these invariant curves: localized in p and extended in θ if α is sufficiently irrational, and extended in p and localized in θ if α is rational. For a classical or quantal initial pure-momentum state, the energy at time $t = n$ grows as n^2 if α is rational (resonance) and remains bounded if α is sufficiently irrational (non-resonance). If α is very nearly rational (marginal resonance), the energy may grow as n^{λ} where λ is expressed in terms of exponents describing the irrationality of α and the continuity class of $V(\theta)$. If the value of α is uncertain, ensemble-averaging over α shows that the energy grows ultimately as n, i.e. diffusively, as though under random impulses.

1. Introduction

A model recently devised by Grempel, Fishman and Prange [1] combines two features of considerable current interest. These are firstly, the dependence of physical properties on the rationality or irrationality of a parameter, and secondly the connection between classical mechanics and quantum mechanics. In addition, their model has the attractive feature of being exactly soluble.

In the model, classical or quantal particles move freely with constant speed $2\pi\alpha$ on the ring $0 \leqslant \theta < 2\pi$ between kicks at unit intervals of time. Thus α is the rotation number between kicks. The magnitude of each kick depends on the positions of the particles through the periodic potential $V(\theta)$. These properties are embodied in the Hamiltonian

$$H = 2\pi\alpha p + V(\theta) \sum_{n=-\infty}^{\infty} \delta(t-n)$$
$$0 \leqslant \theta < 2\pi,\ -\infty < p < +\infty\,. \qquad (1)$$

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Grempel et al. [1] studied the quantum mechanics of this model, and showed how the nature of the quasi-energy spectrum and the localization of the quasi-energy states depend on the rationality or irrationality of the parameter α.

My purpose here is fourfold: firstly, to show (section 2) that the classical trajectories generated by (1) can be determined analytically and are confined via the existence of constants of motion to invariant curves in the phase plane of the mapping between 'kicks' (i.e. there is no chaos); secondly, to show (section 3) that exact quantal eigenstates are associated with the invariant curves by 'torus quantization' [2], and that the evolution of the kinetic energy p^2 is exactly the same in quantum mechanics as in classical mechanics; thirdly, to show (section 4) that the (quantal or classical) kinetic energy remains bounded (non-resonance) if finite classical invariant curves exist (α sufficiently irrational) and increases with time (resonance) if such invariant curves do not exist, the increase being quadratic if α is rational; and fourthly (section 5), to show that if α lies in a set of zero measure very close to rationals the energy

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increases nonquadratically (marginal resonance) in a way depending on exponents related to the analyticity of $V(\theta)$ and the degree of irrationality of α.

It is important to distinguish (1) from the much richer Hamiltonian obtained by replacing p by p^2. Unlike (1), this Hamiltonian is classically non-integrable and exhibits increasing chaos as $V(\theta)$ gets stronger. Moreover, the important discovery was made by Casati, Chirikov, Izraelev and Ford [3] that with this Hamiltonian, unlike (1) the kinetic energy grows more slowly in quantum mechanics than in classical mechanics. This phenomenon has given rise to some discussion and speculation [4–6]; because (1) does not exhibit the phenomenon, the present paper should not be regarded as a contribution to its understanding.

2. Classical invariant curves and evolution

Consider times $t = n$ immediately following the nth kick, and let $\theta(n) \equiv \theta_n$, $p(n) \equiv p_n$. Then Hamilton's equations give, for the mapping in the θ, p phase cylinder between successive kicks,

$$\begin{aligned} \theta_{n+1} &= \theta_n + 2\pi\alpha\,, \\ p_{n+1} &= p_n - V'(\theta_{n+1})\,. \end{aligned} \tag{2}$$

Without significant loss of generality, the periodic function $V(\theta)$ will be regarded as even and expanded as

$$V(\theta) = \sum_{m=1}^{\infty} a_m \cos m\theta\,. \tag{3}$$

Potentials with infinitely many non-zero a_m will be particularly interesting.

For irrational α, the conserved quantity is identified by using the trigonometric identity

$$\sin m\theta_{n+1} = \frac{\cos[m(\theta_n + \pi\alpha)] - \cos[m(\theta_{n+1} + \pi\alpha)]}{2\sin m\pi\alpha}\,, \tag{4}$$

which enables the second mapping equation to be written in the form

$$\begin{aligned} p_{n+1} &+ \frac{1}{2}\sum_{m=1}^{\infty} \frac{m a_m \cos[m(\theta_{n+1} + \pi\alpha)]}{\sin m\pi\alpha} \\ &= p_n + \frac{1}{2}\sum_{m=1}^{\infty} \frac{m a_m \cos[m(\theta_n + \pi\alpha)]}{\sin m\pi\alpha}\,. \end{aligned} \tag{5}$$

The has meaning whenever the series converge, which in section 4 will be shown to occur when α is sufficiently irrational; then (5) expresses a conservation law.

Because all angles θ are eventually explored if α is irrational, iterates of (2) will fill an invariant curve encircling the cylinder and labelled by the value K of the constant of motion (5). There are thus no periodic orbits. The equation of the curve is

$$p_K(\theta) = K - \frac{1}{2}\sum_{n=1}^{\infty} \frac{m a_m \cos m(\theta + \alpha\pi)}{\sin m\pi\alpha}\,. \tag{6}$$

Fig. 1 shows six examples of invariant curves computed by iterating the mapping (2); they will be discussed at the end of section 4, after the mathematical theory establishing their existence has been presented. The phase cylinder is filled by this one-parameter family of invariant curves, and so

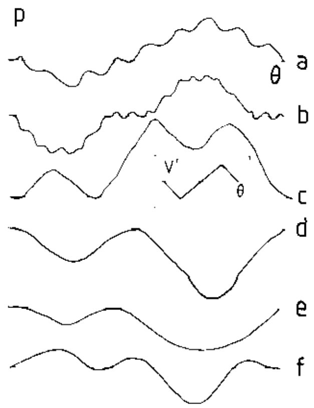

Fig. 1. Invariant curves on the p, θ cylinder. For (a), (b) and (c) the impulse $V'(\theta)$ is the sawtooth shown in the inset; for (d), (e) and (f) the potential is $V(\theta) = \exp\{-\cos\theta\}$. For (a) and (d), the incommensurability parameter is $\alpha = 1/\sqrt{5}$; for (b) and (e), $\alpha = 1/e$; for (c) and (f), $\alpha = 1/\pi$. The θ range is 0 to 2π; the p scale is arbitrary.

the system is integrable. Moreover it is a particularly simple system because the different invariant curves are related by translation parallel to the p-direction and are therefore similar.

When α is rational, invariant curves of the type (6) do not exist. The trajectories hop between a finite number of angles. The phase cylinder is still filled with invariant curves, each of which consists of a finite number (q if $\alpha = p/q$) of infinite lines parallel to the q-axis. The different invariant curves are now related by parallel translation in θ, and are labelled by a 'constant of motion' which is simply an initial angle $\theta_0 (0 \leqslant \theta_0 < 2\pi/q$ if $\alpha = p/q)$. In this rational case, orbits do not fill their invariant curves but can hop to $p = \pm\infty$ if $\theta_0 \neq 0$ and hop periodically if $\theta_0 = 0$.

The explicit evolution of the initial state θ_0, p_0 is easily obtained from (2) and (3) as

$$\theta_n = \theta_0 + 2\pi\alpha n\,,$$
$$p_n = p_0\,,$$
$$p_n = p_0 + \sum_{m=1}^{\infty} \frac{m a_m \sin m(\theta_0 + \pi\alpha(1+n)) \sin nm\pi\alpha}{\sin m\pi\alpha}\,. \tag{7}$$

For any initial momentum p_0, the evolution of the kinetic energy p_n^2 depends on the initial angle θ_0. Averaging over these angles gives the evolution of the mean kinetic energy of an ensemble of particles with p_0, uniformly distributed in θ_0, namely

$$\langle p_n^2 \rangle_{\theta_0} \equiv \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta_0 p_n^2(\theta_0)$$
$$= p_0^2 + \frac{1}{4} \sum_{m=1}^{\infty} \frac{m^2 a_m^2 \sin^2 nm\pi\alpha}{\sin^2 m\pi\alpha}\,. \tag{8}$$

The behaviour of this sum for large n will be shown in sections 4 and 5 to depend delicately on the rationality of α.

3. Quantal eigenstates and evolution

In quantum mechanics the Hamiltonian (1) involves operators $\hat{p}$ and $\hat{\theta}$ and the 'quantum map' [7] relates the states $|\psi_n\rangle$ after successive kicks by means of a unitary evolution operator. With units such that $\hbar = 1$ (which in this problem does not obscure the quantum-classical correspondence), the evolution under (1) can be written explicitly as

$$|\psi_{n+1}\rangle = \mathrm{e}^{-\mathrm{i}V(\hat{\theta})}\,\mathrm{e}^{-\mathrm{i}2\pi\alpha\hat{p}}|\psi_n\rangle\,. \tag{9}$$

In the position representation the fact that the second unitary factor is a translation operator can be used to write the evolution of wave functions in the simple form

$$\langle\theta\,|\,\psi_{n+1}\rangle \equiv \psi_{n+1}(\theta) = \mathrm{e}^{-\mathrm{i}V(\theta)}\psi_n(\theta - 2\pi\alpha)\,. \tag{10}$$

The eigenstates labelled $\phi_s(\theta)$ of the unitary evolution are associated with quasi-energies ω_s and satisfy

$$\mathrm{e}^{-\mathrm{i}V(\theta)}\phi_s(\theta - 2\pi\alpha) = \mathrm{e}^{-\mathrm{i}\omega_s}\phi_s(\theta)\,. \tag{11}$$

Use of the Fourier expansion (3) and the identities (cf. (4))

$$\cos m\theta = \frac{\sin m(\theta + \pi\alpha) - \sin m(\theta - \pi\alpha)}{2\sin m\pi\alpha}\,,$$
$$\omega_s = \frac{\omega_s}{2\pi\alpha}(\theta + \pi\alpha) - \frac{\omega_s}{2\pi\alpha}(\theta - \pi\alpha)\,, \tag{12}$$

gives a 'conservation law' relating quasi-energy states at angles differing by $2\pi\alpha$, namely

$$\exp\left\{\frac{\mathrm{i}}{2}\sum_{m=1}^{\infty}\frac{a_m \sin m(\theta + \pi\alpha)}{\sin m\pi\alpha} - \frac{\omega_s}{2\pi\alpha}(\theta + \pi\alpha)\right\}\phi_s(\theta)$$
$$= \exp\left\{\frac{\mathrm{i}}{2}\sum_{m=1}^{\infty}\frac{a_m \sin m(\theta' + \pi\alpha)}{\sin m\pi\alpha} - \frac{\omega_s}{2\pi\alpha}(\theta' + \pi\alpha)\right\}\phi_s(\theta')\,, \tag{13}$$

where $\theta' = \theta - 2\pi\alpha$.

If α is irrational, repeated use of this relation connects the values of ϕ_s at any two angles and hence gives the explicit (normalized) eigenfunctions

$$\phi_s(\theta) = \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-\mathrm{i}}{2} \sum_{m=1}^{\infty} \frac{a_m \sin m(\theta + \pi\alpha)}{\sin m\pi\alpha} + \frac{\omega_s \theta}{2\pi\alpha}\right\}. \quad (14)$$

The spectrum of quasienergies ω_s is determined by the requirement that $\phi_s(\theta)$ be periodic, which gives

$$\omega_s = 2\pi s\alpha (\mathrm{mod}\ 2\pi) \quad (s \text{ integer}). \quad (15)$$

Thus states are labelled by a discrete quantum number s, and the spectrum is dense on the range $0 \leqslant \omega < 2\pi$.

To obtain the semiclassical interpretation of these eigenstates it is necessary only to notice that (14) can be written in terms of the invariant curve (6) by

$$\phi_s(\theta) = \frac{1}{\sqrt{2\pi}} \exp\left\{\mathrm{i} \int_0^{\theta} p_s(\theta')\, \mathrm{d}\theta'\right\}. \quad (16)$$

Thus the state with quantum number s is associated with the invariant curve with constant of motion $K = s$. This is only to be expected in view of the general correspondence [7] between eigenstates of a quantum map and quantized invariant curves (if they exist) of the associated classical map (or, more generally still, between semiclassical eigenstates and quantized invariant tori [2]). The only surprise is that in this case the correspondence is exact rather than asymptotic: the Hamiltonian (1) gives rise to a 'correspondence identity' in the sense of Norcliffe and Percival [8], and Norcliffe, Percival and Roberts [9].

In momentum space, where p is integrally quantized, these states with irrational α have wave functions

$$\bar{\phi}_s(p) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \mathrm{d}\theta\, \mathrm{e}^{-\mathrm{i}p\theta} \phi_s(\theta). \quad (17)$$

It is not hard to show (for example by the method of stationary phase) that when the potential is strong (the semiclassical limit for this problem) $\bar{\phi}_s(p)$ decays faster than exponentially beyond the 'momentum caustics' at the maximum and minimum momentum values on the associated invariant curve. As explained by Grempel et al., this momentum localization [1] implies that the *local* quasi-energy spectrum is discrete, even though the full spectrum (15) is not.

If α is rational such that $m\alpha$ is an integer for some m for which the Fourier coefficient a_m is non-zero, finite invariant curves do not exist because the series (6) diverges. Then the construction leading to the eigenstates (16) fails. For this case Grempel et al. [1] show how (11), which now relates wave functions at a finite set of θ values, can be employed in a different way to construct the quasi-energy states. They show that the quasi-energy spectrum has a band structure (in contrast to that given by 15) and the states are extended in momentum (in contrast to those given by 16). This extension in momentum is consistent with the fact that the classical invariant curves for rational α are infinite lines parallel to the p-axis. Note however that the quasi-energy states *are* localized in angle, just like the classical invariant curves, and the (non-normalizable) eigenstates in angle representation are again exactly given by a semiclassical construction in terms of a finite series of delta-functions.

Now let us consider the evolution of an initial state $|\psi_0\rangle$ which is localized near momentum p_0 but is not a quasi-energy state. The kinetic energy, initially close to p_0^2, will change under the quantum map as $|\psi_0\rangle$ evolves to $|\psi_n\rangle$ according to (9) or (10). The evolution as $n \to \infty$ is independent of the precise nature of the initial state, and for analytical reasons a convenient choice is the pure momentum state

$$\psi_0(\theta) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}p_0\theta}. \quad (18)$$

After n iterations this becomes (from eq. (10))

$$\psi_n(\theta) = \frac{1}{\sqrt{2\pi}} \exp\left\{\mathrm{i}p_0(\theta - 2\pi n\alpha) - \mathrm{i} \sum_{j=1}^{n} V(\theta - (j-1)2\pi\alpha)\right\}. \quad (19)$$

The kinetic energy is the expectation value

$$\langle\psi_n|\hat{p}^2|\psi_n\rangle = -\int_0^{2\pi} \psi_n^*(\theta)\frac{d^2}{d\theta^2}\psi_n(\theta)\,d\theta$$
$$= p_0^2 + \frac{1}{2\pi}\int_0^{2\pi} d\theta\left[\sum_{j=1}^{n} V'(\theta-(j-1)2\pi\alpha)\right]^2. \tag{20}$$

On substituting the Fourier expansion (3), this can easily be evaluated and is found to be exactly equal to the right side of the purely classical formula (8), i.e.

$$\langle\psi_n|\hat{p}^2|\psi_n\rangle = \langle p_n^2\rangle_{\theta_0}. \tag{21}$$

This is another correspondence identity. It states that the quantal expectation of kinetic energy for an initial pure-momentum state evolves in exactly the same way as the average classical energy of an ensemble of trajectories with the same initial momentum and a uniform distribution in angle. This is reasonable in view of the uncertainty principle, according to which precise knowledge of momentum implies complete ignorance of angle.

4. Resonant and nonresonant evolution

It is clear from (21) and (8) that the classical and quantal evolution of kinetic energy away from its initial value is governed by the sum

$$S(n,\alpha) \equiv \sum_{m=1}^{\infty} \frac{m^2 a_m^2 \sin^2 n\pi m\alpha}{\sin^2 m\pi\alpha}. \tag{22}$$

We wish to study the behaviour of S as $n\to\infty$ and determine when the energy remains bounded and when and how it may grow without limit.

First consider the case where α is rational, so that

$$\alpha = \frac{p}{q} \quad (p, q, \text{relatively prime integers}). \tag{23}$$

Then the terms for which m is an integer multiple of q have vanishing numerators and denominators and dominate the sum for large n, so that

$$S\left(n,\frac{p}{q}\right) \xrightarrow{n\to\infty} n^2q^2\sum_{s=1}^{\infty} s^2 a_{sq}^2. \tag{24}$$

The series always converges if $V(\theta)$ is a smooth function, so that the energy grows quadratically when α is rational. *This is resonant evolution.* It is clear from (6) that the series for the invariant curve diverges (for almost all θ) when α is rational, so that for resonant evolution bounded invariant curves do not exist – indeed we already know that in this case the invariant curves are infinite lines parallel to the p axis. (If the potential and the rational value of α are such that all a_{sq} vanish (for example if the potential has a finite Fourier expansion with $a_m = 0$ for $m > N$, and $q > N$) then the right-hand side of (24) vanishes and the evolution is not resonant.)

Now consider the case where α is irrational. The denominators in the series (22) are never zero, and in the numerator the phase of $\sin^2$ never recurs so that it is tempting to replace $\sin^2$ by its average value of $\frac{1}{2}$ as $n\to\infty$. This would give

$$S(n,\alpha) \xrightarrow{n\to\infty} \frac{1}{2}\sum_{m=1}^{\infty}\frac{m^2a_m^2}{\sin^2 m\pi\alpha} \equiv S_\infty, \tag{25}$$

provided the series converges, and suggests that the energy is bounded by a finite constant value, i.e. *nonresonant* evolution. To justify this procedure it is necessary to examine the convergence of the series S_∞. First, however, we remark that if it converges, then so does the series (6) for the invariant curve – in other words, nonresonant evolution is associated with the existence of bounded invariant curves (the exact connection between (6) and (25) is obtained in terms of the average of $p_K^2(\theta)$ over θ).

If $V(\theta)$ has a finite Fourier expansion, S_∞ is obviously finite for any irrational α and the evolution is certainly non-resonant. If not, the convergence of the series is delicate, because the denominators may get arbitrarily close to zero as m increases and $m\alpha$ may get increasingly close to integers, and this source of divergence may or may not be compensated by the decrease in the a_m. To estimate the series, it is sufficient to concentrate on the worst terms, which depend on good rational approximations to α. The best rational approximations are the successive convergents α_k to the continued fraction for α [10], defined by

$$\alpha \equiv \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots}},$$

$$\alpha_k \equiv \frac{p_k}{q_k} = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots \cfrac{1}{b_k}}} \tag{26}$$

The worst terms in (25) are those for which m is an integer multiple of one of the q_k, and then, in terms of the deviations

$$\epsilon_k(\alpha) \equiv \alpha - \alpha_k\,, \tag{27}$$

an estimate for S_∞ is

$$S_\infty \sim \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\frac{1}{\epsilon_k^2(\alpha)}\sum_{s=1}^{\infty} a_{sq_k}^2 \sim \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\frac{a_{q_k}^2}{\epsilon_k^2(\alpha)}\,; \tag{28}$$

the replacement of the second series by its first term being justified because the existence of the potential $V(\theta)$ implies the convergence of the Fourier sequence a_m^2.

However smooth the potential is, that is however quickly the a_m decrease, there will always be α so close to rationals that the smallness of ϵ_k causes S_∞ to diverge. Such cases will be discussed in the next section. My aim here is to show that the series converges *for almost all* α. The argument involves the way ϵ_k decreases as k and hence the denominators q_k increase. For any α whatever, the decrease cannot be slower than $\mathcal{O}(q_m^{-2})$. This follows trivially from the relations [10]

$$|\epsilon_k| \approx \left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right| = \frac{1}{q_{k+1}q_k} < \frac{1}{q_k^2}\,. \tag{29}$$

It is a nontrivial result of measure-theoretic continued-fraction theory [10] that $|\epsilon_k|$ in fact decreases as $1/q_k^2 \ln q_k$ for almost all α. Therefore

$$S_\infty \sim \sum_{k=1}^{\infty} q_k^4 a_{q_k}^2 \ln^2 q_k \sim \int_0^\infty \mathrm{d}q\, \frac{q_k^4 a_q^2 \ln^2 q}{\mathrm{d}q/\mathrm{d}k}\,. \tag{30}$$

To estimate the derivative it is necessary to know that denominators increase exponentially for almost all α, i.e.

$$q_k \sim \mathrm{e}^{\delta k}, \quad \text{i.e. } \mathrm{d}q/\mathrm{d}k \sim \delta q\,, \tag{31}$$

where $\delta = \pi^2/12 \ln 2$ [10]. Thus

$$S_\infty \sim \int_0^\infty \mathrm{d}q\, q^3 a_q^2 \ln^2 q\,, \tag{32}$$

which converges if the function $V'(\theta)$ (i.e. the impulse at θ) is continuous (it need not be differentiable).

It has been shown, therefore, that the evolution of the energy is nonresonant, and the map (2) has invariant curves, for almost all irrational α. Some of these curves are shown in fig. 1. Curves (a), (b) and (c) are for the illustrated sawtooth impulse, which is continuous but not differentiable. Curves (d), (e) and (f) are for the analytic potential $\exp(-\cos\theta)$. Both potentials possess infinitely many Fourier components. For each potential, the curves are drawn for $\alpha = 1/\sqrt{5}$, $1/e$ and $1/\pi$.

5. Marginal resonance

For irrational α so close to rationals that the series (25) for S_∞ diverges, the asymptotic evolution is still governed by $S(n, \alpha)$ as given by (22) but now the convergence of the sum depends crucially on the $\sin^2$ terms in the numerator, which may not be approximated by $\frac{1}{2}$. As before, the important terms are obtained by successively truncating the continued fraction (26) for α, so that (cf. (28))

$$S(n,\alpha) \xrightarrow{n\to\infty} \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{a_{qk}^2}{\epsilon_k^2(\alpha)}\sin^2\{n\pi\epsilon_k(\alpha)q_k\}\,. \tag{33}$$

The asymptotic n-dependence is determined by the competition between the diminishing Fourier components a_m and deviations ϵ_k, and the classes of irrational α which product marginal resonance are different for potentials of different continuity classes. I shall consider two cases.

The first continuity class of potential $V(\theta)$ consists of those whose lowest discontinuous derivative (occuring at isolated angles θ) is the νth. Then from

$$a_m = \mathrm{Re}\,\frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\theta\, \mathrm{e}^{\mathrm{i}m\theta}V(\theta) \tag{34}$$

it follows (for example by integrating by parts) that

$$|a_m| \sim \frac{1}{m^{\nu+1}}, \quad \text{as } m\to\infty\,, \tag{35}$$

where ν will be called the analyticity exponent and exceeds unity for smooth potentials, i.e. those for which the impulse is a continuous function of θ. The appropriate class of irrationals is those for which the deviations ϵ_k decrease as

$$\epsilon_k \simeq \frac{1}{q_k^{\tau+1}}, \quad \text{as } k\to\infty\,, \tag{36}$$

when τ will be called the incommensurability exponent, and exceeds unity for the special classes of irrational α now being considered. Then (cf. (30))

$$S(n,\alpha) \xrightarrow{n\to\infty} \int_1^{\infty} \frac{\mathrm{d}q}{\mathrm{d}q/\mathrm{d}k}\, q^{2(\tau-\nu)}\sin^2\!\left(\frac{\pi n}{q^{\tau}}\right). \tag{37}$$

These α are so close to rationals that the denominators q_k increase faster than exponentially. A rough estimate (based on the recursion formula relating successive q_k) gives

$$q_k \sim (q_1)^{\tau^{k-1}}, \tag{38}$$

so that

$$\frac{\mathrm{d}q}{\mathrm{d}k} \sim q\ln q\ln\tau\,. \tag{39}$$

The logarithmic factor will be ignored because it does not affect the leading-order power-law asymptotics of $S(n, \alpha)$, and (37) becomes

$$S(n,\alpha) \xrightarrow{n\to\infty} \int_1^{\infty} \frac{\mathrm{d}q}{q}\, q^{2(\tau-\nu)}\sin^2\frac{\pi n}{q^{\tau}}\,. \tag{40}$$

This is a convergent integral whose dominant behaviour can be determined as

$$S(n,\alpha) \longrightarrow \text{constant} \quad \text{if } \tau<\nu\,, \qquad \searrow n^{2(1-\nu/\tau)}, \quad \text{if } \tau>\nu\,, \tag{41}$$

apart from possible logarithmic corrections.

The results make sense. If $\tau < \nu$, α is sufficiently far from rational for the evolution to be nonresonant. If $\tau > \nu$ the evolution is marginally resonant with the energy growing in a way dependent on the analyticity and incommensurability exponents. In the extreme $\tau\to\infty$, the evolution is quadratic in n, i.e. resonant, as would be expected because the deviations (36) correspond to α too close to rational for marginality.

The second continuity class of potential $V(\theta)$ consists of those which are analytic. Then from (34) it follows (for example by using Cauchy's theorem on the complex cylinder $0 \leqslant \mathrm{Re}\,\theta < 2\pi$, $-\infty < \mathrm{Im}\,\theta < +\infty$) that

$$|a_m| \sim \mathrm{e}^{-\mu m}, \quad \text{as} \quad m\to\infty\,, \tag{42}$$

where μ is the analyticity exponent for this case and depends on Im θ for the singularity or stationary point of (34) nearest the real axis. (There exist potentials so smooth that $\mu = \infty$, i.e. the $|a_m|$ decrease faster than exponentially, but these will not be considered here.) The appropriate class of irrationals is now those for which the deviations ϵ_k decrease as

$$\epsilon_k \sim e^{-\sigma q_k}/q_k^2, \quad \text{as} \quad k \to \infty\,, \tag{43}$$

where σ is the incommensurability exponent for this case. Then (33) becomes

$$S(n,\alpha) \xrightarrow{n\to\infty} \int_0^\infty \frac{dq q^4}{dq/dk} \times e^{2(\sigma-\mu)q} \sin^2\left(\frac{\pi n\, e^{-\sigma q}}{q}\right). \tag{44}$$

These α are so exquisitely close to rationals that the law of increase of the denominators q_k cannot be written in any obvious way as a function of k; a rough estimate gives

$$q_k \sim \exp\{\exp\{\exp \ldots (k-1 \text{ iterations}) \times \{\exp q_1\}\} \ldots .\}\,. \tag{45}$$

Nevertheless, dq/dk is still proportional to q apart from logarithmic factors, because if $dq/dk \sim q^{1+\delta}$ then, for any positive δ, however small, $q(k)$ diverges for some finite k, and this corresponds to a rational α. Therefore (44) can be written

$$S(n,\alpha) \xrightarrow{n\to\infty} \int_0^\infty dq q^3\, e^{2(\sigma-\mu)q} \sin^2 \frac{\pi n\, e^{-\sigma q}}{q}\,. \tag{46}$$

This is a convergent integral which if $\sigma < \mu$ tends to a constant as $n \to \infty$; this is the nonresonant case. If $\sigma > \mu$ the dominant behaviour can be found by changing variables to $x \equiv \exp\{-\sigma q\}/q$ and again ignoring logarithmic factors, which gives

$$S(n,\alpha) \xrightarrow{n\to\infty} \int_0^\infty \frac{dx}{x} x^{(2/\sigma)(\mu-\sigma)} \sin^2(\pi n x)\,, \tag{47}$$

whose n-dependence can be extracted by scaling. The result is

$$S(n,\alpha) \begin{cases} \longrightarrow \text{constant}, & \text{if } \sigma < \mu\,, \\ \longrightarrow n^{2(1-\mu/\sigma)}, & \text{if } \delta > \mu\,, \end{cases} \tag{48}$$

apart from possible logarithmic factors.

Once again the results make sense, with the evolution exponent for marginal resonance tending to zero as $\sigma \to \mu$ (non-resonant evolution) and tending to two as $\sigma \to \infty$ (resonant evolution).

Not all marginal resonance involves a power-law growth in energy. For example if the impulse is a discontinuous function of θ($\nu = 1$ in (35)) and typical fractional numbers are considered ($\tau = 1$ in (36)), then (41) gives the exponent zero. In fact a delicate argument shows that the energy for this case grows as $\ln^2 n$; for quadratic irrationals it grows as $\ln n$.

6. Concluding remarks

The Hamiltonian (1) has the special property that there exists an exact correspondence between classical and quantal behaviour in both the commensurate case (rational α) and the incommensurate case (irrational α).

In the incommensurate case the energy remains bounded (nonresonant evolution) and as already known [1] the quasi-energy states are localized in momentum. It is unconvincing to regard this as localization by disorder (Anderson localization), even though an ingenious technique [1] (which amounts to taking the logarithm of the unitary operator in (9)) transforms the quantal Hamiltonian (1) to a tight-binding form with hopping between momentum sites. The reason is that the classical problem is integrable: the phase plane is covered with finite invariant curves and there are no chaotic orbits.

In the commensurate case the energy increases quadratically (resonant evolution) and [1] the quasi-energy states are extended in momentum although localized in angle. Each classical invariant curve now consists of a finite number of translates of the p-axis.

In the marginal case, where α is very close to a rational, the energy again increases, but in non-quadratic ways (eqs. (41) and (48)) which depend on the irrationality of α and the continuity of the potential. This marginally resonant asymptotics is probably accompanied by large fluctuations [11] (which I conjecture to be associated with nearly periodic classical orbits (7) which exist for particular values of θ_0). I do not know the nature of the quasi-energy spectrum and the degree of localization of the eigenstates for the marginal values of α.

The sensitivity of the evolution to the rationality or irrationality of α naturally raises the question of how the energy evolves in the presence of uncertainty about the value of α. The answer is obtained by averaging the energy, as given by (8), over the interval of uncertainty in α. The extreme case is when α is completely unknown, and then it is not difficult to obtain the following exact result:

$$\int_0^1 \mathrm{d}\alpha \langle p^2_{n\theta_0} \rangle = p_0^2 + \frac{n}{2\pi} \int_0^{2\pi} [V'(\theta)]^2 \,\mathrm{d}\theta \,. \tag{49}$$

This describes diffusion under the action of periodic impulses given by the mapping (2); the impulses act randomly when the energy is averaged over α. Two aspects of this result are interesting; firstly, the average is not dominated by the non-resonant energy saturation which occurs for sufficiently irrational α, that is for all α except a set of zero measure; and secondly, diffusion occurs only after averaging over an ensemble of Hamiltonians, and not merely after averaging over initial conditions θ_0.

If α is specified more closely, e.g. to within an accuracy $\Delta\alpha$, then averaging over this range also gives an ultimately diffusive energy growth, although the coefficient of n is smaller than in (49) and diminishes with $\Delta\alpha$. The terms in the series (8) which give rise to this asymptotic diffusion are those for which $m > (\Delta\alpha)^{-1}$. If $\Delta\alpha$ is small enough, m will be large enough for the nondiffusive evolutions discussed in the main body of this paper to appear as an intermediate regime before diffusion ultimately sets in for very large n. For marginal resonance the relevant m-values are so enormous (cf. (38) and (45)) that the largest value of $\Delta\alpha$ which allows this type of evolution is so small that its 'observation' would probably require special programming techniques. (This remark does not apply to the situation mentioned at the end of section 5, in which the impulse $V'(\theta)$ is discontinuous and $\langle p^2 \rangle$ grows logarithmically for most α; in this case the nonexistence of invariant curves is easy to illustrate on a computer, and iteration of (2) generates beautiful transitory 'fractal curtains' which readers will find it instructive to produce for themselves.)

Dr. J.H. Hannay has pointed out to me that the evolution of momentum for the model discussed in this paper is governed by the solution of a problem which surely occurs in other contexts, namely the sum of n values of a periodic function sampled at regular intervals which may or may not be commensurate with its period. He also remarks that the momentum evolution under the Hamiltonian (1) is exactly reproduced by the spatially uniform kicking of a Newtonian free particle, i.e. by

$$H = \tfrac{1}{2}p^2 + q \sum_{n=-\infty}^{\infty} V'(\alpha t)\delta(t-n) \,, \tag{50}$$

where V has period unity with the phase plane stroboscopically lit at unit times (this system is exactly soluble quantally as well as classically).

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