# Wave propagation and scattering

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# TWINKLING EXPONENTS IN THE CATASTROPHE THEORY OF RANDOM SHORT WAVES

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ABSTRACT

The difficulty in obtaining a theoretical description of the statistics of random short waves arises from the presence of ray caustics, which cause intensity moments to diverge. The divergences are determined by the result of a competition between singularities, which contribute exponents calculated using catastrophe theory.

## 1. INTRODUCTION

The applications and illustrations of catastrophe theory in physics are now many and varied, and have been recently and extensively reviewed in several books (Poston and Stewart 1978, Gilmore 1981) and articles (Stewart 1981). In this paper I will concentrate on one group of topics, centred on the origin and applications of scaling laws in the theory of short waves and explain how these form the basis of a theory of the statistics of the waves when they encounter randomness.

There are several reasons why it is appropriate to write about this subject. The first reason is that although the theory is subtle it nevertheless provides an explanation of a familiar phenomenon: the intense twinkling of starlight.

The second reason is that the theory embodies in a novel way two themes with which theoretical physics has been much preoccupied, namely scaling and nongaussian fluctuations. Scaling is an expression of the fact that some physical quantities can depend nonanalytically on others. This is most familiar in phase transitions (Stanley 1971) where thermodynamic properties are nonanalytic functions of temperature at critical points. In this paper the scaling laws will concern the short-wave limit, in which the wavenumber $k(= 2\pi/\text{wavelength})$ becomes infinite. Nongaussian fluctuations occur in random variables or functions governed by processes with a high degree of mutual correlation, so that the central limit theorem cannot be applied. Familiar examples are the stable distributions of probability theory (Jona-Lasinio 1975), and certain fractal noises (Mandelbrot 1982). My example is unfamiliar in this context and involves large fluctuations for which statistical quantities (moments of wave intensity) may scale in ways dominated by competition amongst singularities (i.e. the catastrophes).

The third reason is that the theory makes essential use of concepts characteristic of catastrophe theory. This essential use is to be contrasted with the use 'in principle' which has been so common when catastrophe theory is invoked outside physics (and which has given rise to controversy - see Sussmann and Zahler (1975), Zahler and Sussmann (1977) and subsequent correspondence in Nature 270 (1977) 381-384, 658). The characteristic concepts in question are firstly, the structural stability of certain universality classes of singularity, and secondly the hierarchy of normal forms representing the singularities in each class.

The fourth reason is that this group of subjects is one of the less well known applications of catastrophe theory. The subjects are still in their infancy, and it is likely that much remains to be discovered by imaginative investigators.

The plan of this paper is as follows. Section 2 contains a brief account of the catastrophe theory of waves. Section 3 is devoted to the scaling laws expressing the nonanalyticity of short waves. These are essential preliminaries to section 4, where at last randomness is incorporated into the asymptotics, thus introducing the central idea of singularity-dominated strong fluctuations by deriving scaling laws for moments of wave intensity.

## 2. SHORT WAVES AS DIFFRACTION CATASTROPHES

Consider a monochromatic wave with wave number $k$, represented by a scalar wavefunction $\psi(C;k)$, where $C$ is an abbreviated notation for any quantities $C_1, C_2 \ldots$ on which the waves depend, such as time, position coordinates, or parameters describing diffracting objects or refracting media. In the language of catastrophe theory, $C$ are control parameters. $\psi$ will be assumed to satisfy a linear wave equation with boundary conditions. This framework is very broad, including optics, quantum mechanics, acoustics, elasticity and small-amplitude water waves.

The short-wave limit is $k\to\infty$. It is nontrivial because $\psi$ is a nonanalytic function of $1/k$, with an essential singularity at $1/k=0$. In optics, for example, it is not possible to express monochromatic electromagnetic wave fields as Taylor series in the wavelength with geometrical optics as the leading term. Thus the connection between wave and ray optics (and, similarly, between quantum and classical mechanics) is much more complicated than the connection between special relativity and Newtonian mechanics, which is simply a matter of expanding in powers of $v/c$ (relative velocity of coordinate frames divided by speed of light).

Large-$k$ asymptotics must give the correct mathematical description of three physically obvious facts. Firstly, $\psi(C;k)$ must be constructed in terms of the trajectories of the corresponding Hamiltonian problem (for example rays of light). Secondly, on the caustic or focal set, that is on the envelope of the rays representing the wave, $\psi$ must rise to high values, diverging as $k\to\infty$. And thirdly, the scale of diffraction fringes in C space must vanish as $k\to\infty$.

A crucial element in formulating asymptotics in accordance with these three criteria is the recognition that a wave corresponds not to a trajectory but to a family of trajectories; one is reminded of Dirac's (1951) remark: "presumably the family has some deep significance in nature, not yet properly understood". Different trajectories may pass through different points C, and more than one trajectory may pass through a given point C. To label the trajectories in a family we employ variables $s= s_1,s_2\ldots$. In the terminology of catastrophe theory, s are state variables; they may represent, for example, points at which trajectories intersect an initial wavefront, or directions of trajectories at an instant of time. The different trajectories through a point will be denoted by $s^\mu(C)$ $(\mu=1,2\ldots)$.

The trajectories $s^\mu(C)$ are determined by ray dynamics. These are governed by equations derived from a Hamiltonian function (whose operator generalization generates the wave equation satisfied by $\psi(C;k)$). Here it will be necessary only to invoke the fact that ray dynamics may also be derived from a variational principle, which may be expressed as follows: there exists an optical distance function (or action function) $\phi(s;C)$, whose stationary values $s^\mu(C)$ are the rays. Thus

$$\frac{\partial\phi}{\partial s_i}(s;C)=0 \quad \forall\, i \quad \text{if} \quad s=s^\mu(C). \tag{1}$$

An elementary example of this way of formulating ray dynamics is the evolution of rays from a curved wavefront W in a plane filled with homogeneous isotropic medium (figure 1).

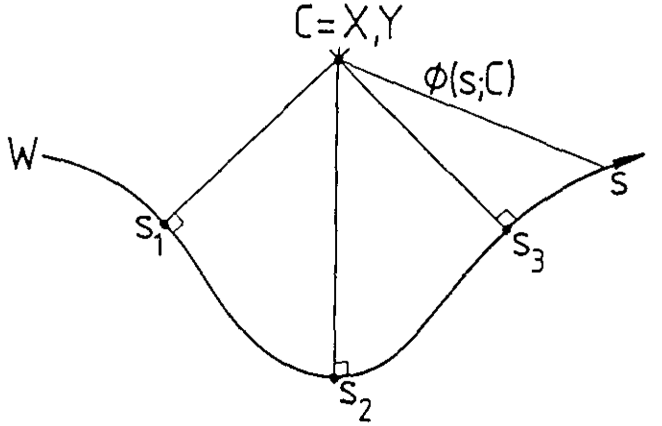


Fig. 1. Optical distance $\phi(s;C)$ from s on a wavefront W to control point C; also shown are three trajectories, for which $\phi$ is stationary.

C corresponds to position X,Y in the plane, s is a coordinate on the wavefront, and $\phi$ is the distance to C from s on W. It is clear that (1) simply expresses the condition that in these circumstances rays are straight lines normal to W.

In catastrophe terminology, (1) states that rays are determined by a gradient map (figure 2) from s to C.

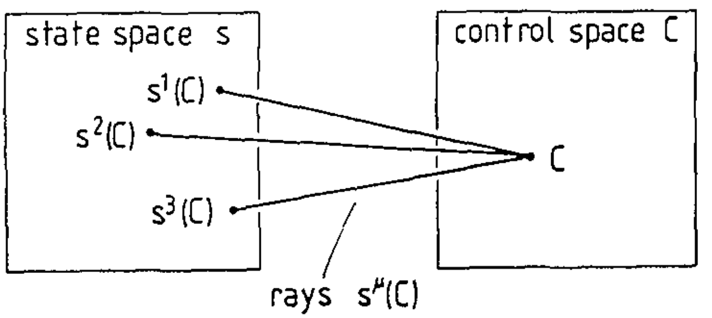


Fig. 2. Multivalued gradient mapping from state space to control space, induced by rays $s^\mu(C)$.

The caustics are envelopes of the ray family described by $\phi(s,C)$ and are defined as singularities in C space of the gradient map (1), that is hypersurfaces across which the number of rays suddenly changes. The condition for this is that $\phi$ is stationary to higher order, so that in addition to (1) the equation

$$\det \left\{ \frac{\partial^2 \phi}{\partial s_i \partial s_j} \right\} = 0 \qquad (2)$$

must hold. This is illustrated for our simple example in figure 3, which shows the family of rays normal to W. At points such as A, three rays pass through each point; at points such as B, one ray passes through each point. The separator set is the caustic, in this case a cusped curve whose points satisfy the focal condition, which follows from (2), of lying on the locus of centres of curvature of W.

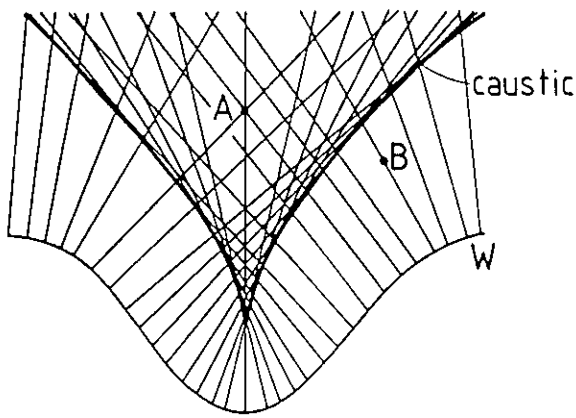


Fig. 3. Cusped caustic formed as envelope of trajectories (normals) from wave-front W; three trajectories reach A, one reaches B.

Caustics organize the multivaluedness of the ray family. In the space sXC the different solutions $s^{\mu}(C)$ join to form a smooth surface, called the critical manifold, whose foldings over the control space C correspond to the rays. This is illustrated for our example by figure 4.

The reason for introducing the function $\phi(s;C)$ is that as well as generating rays and caustics it also generates wave-functions $\psi(C;k)$ in the short-wave limit $k\to\infty$. A substantial body of asymptotic analysis leads to the following integral representation:

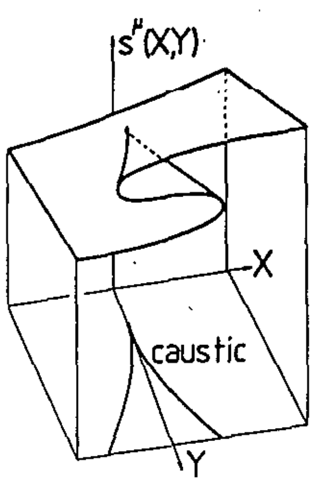


Fig. 4. Caustic of figure 3 formed by projection of critical manifold in s,C space.

$$\psi(C;k) = \left(\frac{k}{2\pi}\right)^{n/2} \int d^n s \; a(s;C)\, e^{ik\phi(s;C)} \qquad (3)$$

In this equation, n is the number of state variables and a is a smooth function of s and C dependent on the ray family. The integral (3) generalizes a number of mode expansions, Kirchhoff diffraction integrals and Fourier-transform representations (whose variety is a consequence of arbitrariness in interpreting s), which all become equivalent in the short-wave limit. The derivation of such appoximations and their correction terms was put on a firm basis by Maslov, in work reviewed by Kravtsov (1968), Duistermaat (1974) and in the book by Maslov and Fedoriuk (1981). Intuitive presentations have been given by Berry (1976, 1981) and Berry and Upstill (1980). Physicists will find (3) reminiscent of Feynman-path integrals, which are exact representations of $\psi$ as a 'super-superposition' in which s is infinite-dimensional and $\phi$ a functional; this point of view is well presented by Shulman 1981.

For large k the integrand in (3) is a rapidly-oscillating function of s and direct evaluation of the integral is impractical. It is natural to attempt an approximate evaluation by the method of stationary phase, which consists in expanding to second order about its stationary points, which are precisely the rays $s^{\mu}(C)$ defined by (1). This gives

$$\psi(C;k) \approx \sum_{\mu} \left[ \frac{a(s;C)\exp\{ik\phi(s;C)+i\alpha_{\mu}\pi/4\}}{\left|\det \frac{\partial^2\phi(s;C)}{\partial s_i \partial s_j}\right|^{\frac{1}{2}}} \right]_{s=s^{\mu}(C)} \tag{4}$$

where a is an amplitude and $\alpha_{\mu}$ is the signature of the matrix $\partial^2\phi/\partial s_i \partial s_j$. Thus $\psi$ appears as a superposition in which each ray contributes a wave whose phase is precisely the stationary value of the optical distance function.

Although (4) gives a useful description of many interference effects, it suffers the fatal defect of failing just where we shall be most interested in $\psi$, namely on caustics. This is clear from (2), which implies that the approximation (4) diverges to infinity as C moves onto a caustic because of the vanishing of the denominators of its terms. Now when $k = \infty$ this divergence must surely occur: it simply expresses the infinite concentration of rays at a caustic. But we are interested in large-k asymptotics rather than the trajectory limit, and so seek to determine exactly how the divergence occurs as k increases. Thus near caustics (4) is too crude an approximation to the integral (3).

It is at this point that catastrophe theory enters to provide two remarkable simplifications of the problem for the important case (which occurs 'almost always') when the caustic has the property of structural stability. This means that a small smooth deformation of $\phi$ (i.e. a diffeomorphism such as would be produced by changing the nature or positions of diffracting objects) will cause a smooth deformation of the caustic. In figure 3, for example, the cusp will remain a cusp under small changes in the form of W. The central theorem of catastrophe theory (Poston and Stewart 1978, Gilmore 1981) is that the world of caustics is partitioned into universality classes. Any two caustics in the same class can be transformed into each other by diffeomorphism of their $\phi$'s. It is these universality classes (or in mathematical terminology, equivalence classes) that constitute the catastrophes. The classification of catastrophes, begun by Thom (1975), has been carried much further by Arnol'd (1975). Figure 5 shows the forms of the caustics for the catastrophes whose codimension K (essential number of control parameters C) satisfies $K \leq 3$.

The first way in which this classification enables the diffraction integral (3) to be simplified follows from replacing all $\phi(s;C)$ in a given universality class (labelled j) by one normal form $\Phi_j(s;C)$. The transformation of $\phi$ into $\Phi_j$ is achieved by diffeomorphism of s and C. From this point of view, catastrophe theory is the classification of normal forms; table I is a list of normal forms for the first few catastrophes (Arnol'd 1975 lists many more). Making the same transformation in (3), and replacing the factor a and the transformation Jacobian by unity (because these are smooth functions and near a caustic the contributing s values lie close together), we obtain, instead of the infinitely many integrals corresponding to all possible $\phi$, the following finite set of <u>diffraction catastrophes</u>

$$\Psi_j(C;k) = \left(\frac{k}{2\pi}\right)^{n/2} \int d^n s \; e^{ik\Phi_j(s;C)} \quad . \tag{5}$$

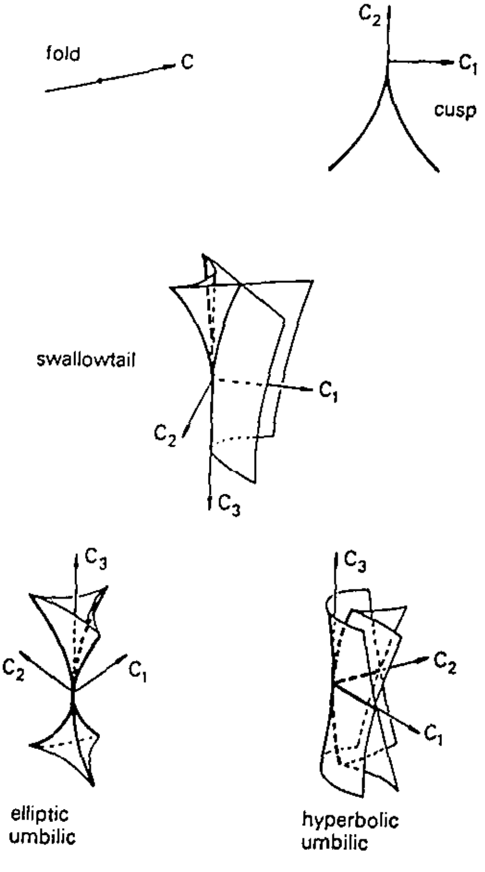


Fig. 5. The elementary catastrophes of codimension $K \leqslant 3$.

The polynomial exponents in the integrands represent the irreducible topological complexity of the collisions of stationary points $s^{\mu}(C)$ of $\Phi_j$ (i.e. of caustics) as the parameters C vary. Diffraction catastrophes can be very complicated functions of C. Consider, for example the cusp. From table I, the quartic polynomial $\Phi_{cusp}$ (which on using (1) can easily be seen to correspond to the cubic critical manifold of figure 4) gives

$$\Psi_{cusp}(C_1,C_2) = (k/2\pi)^{1/2} \int_{-\infty}^{\infty} ds\, \exp\{ik(s^4/4+C_2s^2/2+C_1\ s)\} \qquad (6)$$

Table I

| Name | Symbol | K | $\phi(s:C)$ |
|---|---|---|---|
| fold | $A_2$ | 1 | $s^3/3+Cs$ |
| cusp | $A_3$ | 2 | $s^4/4+C_2s^2/2+C_1s$ |
| swallowtail | $A_4$ | 3 | $s^5/5+C_3s^3/3+C_2s^2/2+C_1s$ |
| elliptic umbilic | $D_4^-$ | 3 | $s_1^3-3s_1s_2^2-C_3(s_1^2+s_2^2)-C_2s_2-C_1s_1$ |
| hyperbolic umbilic | $D_4^+$ | 3 | $s_1^3+s_2^3-C_3s_1s_2-C_2s_2-C_1s_1$ |
| butterfly | $A_5$ | 4 | $s^6/6+C_4s^4/4+C_3s^3/3+C_2s^2/2+C_1s_1$ |
| parabolic umbilic | $D_5$ | 4 | $s_1^4+s_1s_2^2+C_4s_2^2+C_3s_1^2+C_2s_2+C_1s_1$ |

Standard polynomials $\Phi$ for the elementary catastrophes with codimension $K \leq 4$

Photographs of the intensity pattern of $|\Psi_{cusp}|^2$, and a computer simulation based on computing contours of (6), are shown in figure 6. This function was first studied by Pearcey (1946). The simplest diffraction catastrophe - the fold - was studied by Airy (1838). The more complicated elliptic umbilic diffraction catastrophe was studied in detail by Berry, Nye and Wright (1979). The computation of the oscillatory integrals (5) is growing into a small industry (see, for example, Connor and Farrelly 1981, Connor and Curtis 1982, and Upstill et al 1982).

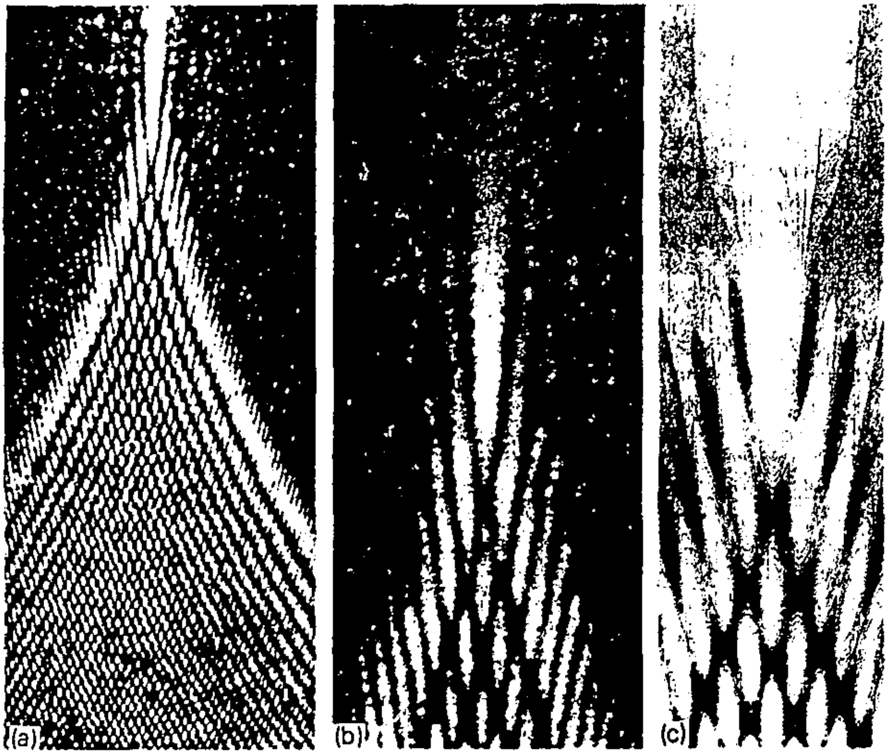


Fig. 6. Cusp diffraction catastrophe; a) experiment; b) magnification of a); c) computer simulation made by shading contours of $|\Psi_{cusp}(C_1,C_2)|^2$ calculated from (6).

## 3. SCALING

The second way in which catastrophe theory can simplify the asymptotic diffraction integrals follows from the fact that the normal forms $\Phi_j(s,C)$ are quasi-homogeneous polynomials in the variables s with coefficients linear in the parameters C. This has the consequence that the asymptotic parameter k can be scaled out of (5), so that the diffraction catastrophe $\Psi_j$ for any value of k can be expressed in terms of $\Psi_j$ for any other value (e.g. k=1). The explicit scaling law is (apart from possible logarithmic modifications to be mentioned later)

$$\Psi_j(C_i;k)=k^{\beta_j}\Psi_j(C_i k^{\sigma_{ij}};1), \tag{7}$$

where the separate control parameters have been denoted by $C_i$.

Table II lists the values of the exponents $\beta_j$ and $\sigma_{ij}$, and also the additional exponent

$$\gamma_j \equiv \sum_{i=1}^{K} \sigma_{ij} \tag{8}$$

for catastrophes with codimension $K \leq 4$. The method for obtaining the exponents is simply to first rescale s to eliminate k from the C-independent terms in the exponent of (5) and to then rescale the $C_i$ to eliminate k from the other terms. It is easy to confirm that when applied to (6) this procedure yields the correct cusp exponents in table II.

Table II

| catastrophe | $\beta$ | $\sigma_i$ | $\gamma$ |
|---|---|---|---|
| fold | 1/6 | $\sigma_1=2/3$ | 2/3 |
| cusp | 1/4 | $\sigma_1=3/4, \sigma_2=1/2$ | 5/4 |
| swallowtail | 3/10 | $\sigma_1=4/5, \sigma_2=3/5, \sigma_3=2/5$ | 9/5 |
| elliptic umbilic | 1/3 | $\sigma_1=2/3, \sigma_2=2/3, \sigma_3=1/3$ | 5/3 |
| hyperbolic umbilic | 1/3 | $\sigma_1=2/3, \sigma_2=2/3, \sigma_3=1/3$ | 5/3 |
| butterfly | 1/3 | $\sigma_1=5/6, \sigma_2=2/3, \sigma_3=1/2, \sigma_4=1/3$ | 7/3 |
| parabolic umbilic | 3/8 | $\sigma_1=5/8, \sigma_2=3/4, \sigma_3=1/2, \sigma_4=1/4$ | 17/8 |

Exponents governing scaling of wave amplitudes and fringe spacing as $k \to \infty$

In mathematical terms, the scaling (7) is a precise expression of the nonanalyticity of wave functions near caustics, as $k \to \infty$. In physical terms, the exponent $\beta_j$ describes the short-wave divergence of wave intensity $|\Psi_j|^2$ at the caustic singularity ($C_i=0$): the intensity scales as $k^{2\beta_j}$. The exponents $\sigma_{ij}$ describe the shrinking of the diffraction fringes in the $C_i$ control-space direction: the fringe spacings scale as $k^{-\sigma_{ij}}$. The exponent $\gamma_j$ describes the shrinking of the K-dimensional hypervolume of the main diffraction maximum: this scales as $k^{-\gamma_j}$. $\beta_j$ is the 'singularity index' introduced by Arnol'd (1975) and computed for a large number of cases by Varchenko (1976); $\sigma_{ij}$ and $\gamma_j$ (the 'fringe index') were introduced by Berry (1977).

To illustrate with a quantum-mechanical example the way in which the scaling laws can quickly lead to interesting physics, consider an isotropic source emitting particles with mass m and speed v in a gravitational field with acceleration g (figure 7).

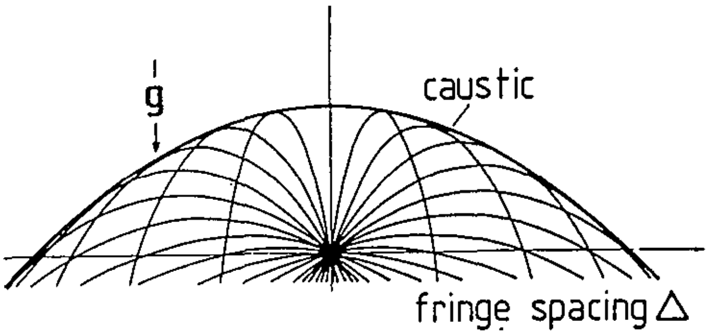


Fig. 7. Section of paraboloidal caustic formed by particles issuing from a source under uniform gravity.

The corresponding parabolic Newtonian trajectories envelop a caustic which is a fold catastrophe in the form of a paraboloid (this is the 'bounding paraboloid' of elementary gunnery). Within the paraboloid, two rays pass through each point, and their interference gives rise to diffraction fringes in space. What is the semiclassical separation Δ (figure 7) of the bright fringes nearest the caustic? The answer can be obtained by realizing that Δ may depend on m,v,g and Planck's constant $\hbar$. Since $\hbar$ corresponds to $k^{-1}$ in the scaling laws, table II gives $\Delta \sim \hbar^{2/3}$ for the fold. Elementary dimensional analysis now leads to

$$\Delta = a(\hbar^2/m^2 g)^{1/3}, \tag{9}$$

where a is dimensionless. This does not involve v and so the spacing of these quantal fringes is, curiously, unaffected by altering the de Broglie wavelength h/mv of the particles (this is a case where the semiclassical limit is not quite the same as the short-wave limit). A more refined analysis (Berry 1982a) gives, for the constant, a=3.53897, which for neutrons in the earth's gravity gives Δ=0.026mm - an almost-macroscopic quantum effect.

Chillingworth and Romero-Fuster (1983) have proved an interesting relation between $\gamma_j$ and $\beta_j$, namely

$$\gamma_j = (K_j+1)(1-\beta_j) - 1 \tag{10}$$

where $K_j$ is the codimension of the j'th catastrophe. For the catastrophes in table II the relation can be checked directly. For some higher catastrophes, the $\gamma_j$ and $\beta_j$ as calculated by Berry (1977) do not satisfy (10) and this is connected with the phenomenon of modality which must now be briefly discussed.

A remarkable feature of Thom's early classification of catastrophes was that the list of universality classes of given codimension was finite. If the number of state variables s-the corank - is unity, there is indeed only one singularity for each value of K. But if the corank is two then a new phenomenon appears when K=7: the finiteness of classification breaks down, and universality classes are parameterized by one or more continuous moduli, denoted a. Singularities with different a are not equivalent under diffeomorphism, although they may be topologically equivalent.

If the moduli a are treated as additional control parameters and their indices $\sigma_{ij}$ are computed and included in the computation of $\gamma_j$ from equation (8), then the relation (10) is satisfied. But this procedure although mathematically impeccable, is physically unsatisfactory, because the $\sigma_{ij}$ for moduli are always found to be negative or zero, so that as $k\to\infty$ diffraction fringes do not shrink along modal directions in parameter space.

An example is the singularity $Z_{11}$, whose normal form is

$$\Phi\ (s_1 s_2; a, C_1 \ldots C_9) = s_1^3 s_2 + s_2^5 + a s_1 s_2^4 + C_1 s_1 + C_2 s_1^2 + C_3 s_1 s_2$$
$$+ C_4 s_1 s_2^2 + C_5 s_1 s_2^3 + C_6 s_2 + C_7 s_2^2 + C_8 s_2^3 + C_9 s_s^4 . \qquad (11)$$

Removing k in (5) from the first two terms by rescaling $s_1, s_2$ gives $\beta=8/15$. The rescaling $C_1$----$C_9$ to eliminate k from the last nine terms gives a series of positive $\sigma_i$ whose sum is $=21/5=63/15$, which with K=9 does not satisfy (10). If now the modulus a is rescaled, the corresponding index $\sigma_a=-1/15$, so that diffraction fringes expand along the a direction as $k\to\infty$. When this is included to give $\gamma=62/15$, and K increased to 10, then (10) is satisfied.

Another example is $X_9$, whose normal form may be written

$$\Phi\ (s_1,s_2;a,C_1\ldots C_7)=s_1^4+s_2^4+as_1^2s_2^2+C_1s_1+C_2s_2+C_3s_1^2+C_4s_2^2$$
$$+C_5s_1s_2+C_6s_1s_2^2+C_7s_1^2s_2. \qquad (12)$$

Rescaling the first two terms gives $\beta=1/2$, and rescaling the last seven terms gives $\gamma=7/2$ which with K=7 fails to satisfy (10). If now a is rescaled, its index is $\sigma_a=0$, so that diffraction fringes neither expand nor shrink in the a-direction as $k\to\infty$. Of course this zero index cannot alter the value of $\gamma$, but if a is counted as a control parameter, thus increasing K to 8, then (10) is again satisfied.

These arguments show that although (10) can be confidently employed to relate $\beta_j$ and $\gamma_j$ for nonmodal catastrophes, it should not be relied on for modal ones if $\gamma_j$ is taken to represent the index governing fringe shrinking. This is an indication of the fact that in diffraction physics moduli a have a significance different from that of control parameters C. In some cases, modality requires (7) to be modified by logarithmic factors, as discussed by Varchenko (1976). I emphasise, however, that in the great majority of applications of shortwave scaling the catastrophes involved are nonmodal, so that both (7) and (10) may be employed without regard to the complications introduced by modality.

Diffraction catastrophe scaling laws and exponents are strongly reminiscent of those occurring in the theory of phase transitions (Stanley 1971, Fisher 1967), with $k\to\infty$ corresponding to $T\to T_c$. There too one starts with integrals (for partition functions rather than wave functions) whose quadratic approximation yields the wrong behaviour close to singularities (thermodynamic critical points rather than caustics). Moreover there exist exponent-equalities analogous to (10), and variables may be relevant (analogous to control parameters) or irrelevant or marginal (analogous to moduli).

But there are differences between the two sorts of scaling. Diffraction exponents are always rational numbers, whilst thermodynamic exponents need not be. Diffraction integrals can be reduced to low-dimensional integrals, and scaling accomplished in a finite number of steps, whilst partition functions involve infinite dimensions and it is their recursive transformation, via the renormalization group (Wilson 1975 -

see also Pfeuty and Toulouse 1977), which generates the scaling laws.

## 4. TWINKLING OF RANDOM SHORT WAVES

When refracted by atmospheric turbulence, the steady light from a star acquires a fluctuating intensity that we perceive as twinkling. When reflected or refracted by irregular undulations on a water surface, sunlight forms fantastic patterns of moving bright lines on the sides of boats, under bridges and on the bottoms of swimming pools. Because of the unpredictability of the air turbulence or the undulating water, one may speak of the fluctuating light intensity as a random function and seek to comprehend its statistics. The wavelength of light is small in comparison with the smallest cells of atmospheric turbulence or the smallest ripples on water, so that these problems lie in the domain of random short waves.

Wave propagation in random media has given rise to an extensive literature (as illustrated by the other papers in this volume, and reviewed by Uscinski 1977 and in a more general context by Ziman 1979) as has wave reflection from irregular surfaces (Beckmann and Spizzichino 1963, Bass and Fuks 1979). In spite of intense study, these problems have proved remarkably resistant to analytical solution. This is the case even for the simple phase screen model Mercier 1962, Bramley & Young 1967, Salpeter 1967) in which a random spatial phase modulation is imparted to the wavefronts of an initially plane wave which then propagates freely; the phase fluctuations are converted by diffraction into intensity fluctuations which are the object of study. Until recently the only tractable case was that of weak scattering, which could be treated by perturbation techniques such as the Born approximation.

Now, however, a clear picture is emerging of the short-wave limit, i.e. the limit $k\to\infty$, based on the realization that random waves must be dominated by random caustics, near which wave functions $\psi(C;k)$ take the form of one of the diffraction catastrophes discussed in sections 2 and 3. The caustics give rise to intense nongaussian fluctuations in the intensity $|\psi|^2$. In the ray limit $k=\infty$, the caustics are singularities of $\psi$ (not softened by diffraction), and the intensity fluctuations are infinitely strong. It is important to realise that these singularity-dominated strong fluctuations are produced by natural focusing of the rays when the fluctuations in atmospheric refractive index, or in the height of the water surface, are themselves gentle (or even Gaussian-distributed).

The large fluctuations are described by the <u>moments</u> $I_m$ of the probability distribution of the wave intensity, defined by

$$I_m \equiv \langle |\psi|^{2m} \rangle. \qquad (13)$$

where $\langle\ \rangle$ denotes averaging over an ensemble of random media or undulating surfaces. $I_1$ is the average wave intensity and is not large when $k\to\infty$ because the caustic singularities are integrable. $I_m \geq 2$ diverge as $k\to\infty$, behaviour to be contrasted with that of a Gaussian random wave (Re$\psi$ and Im$\psi$ independent random variables with zero mean), for which $I_m = m!$ and which is independent of k. Nongaussian fluctuations in the twinkling light from Sirius were measured by Jakeman, Pike and Pusey (1976). In precatastrophic studies of the second moment, Shishov (1971) and Buckley (1971), established that $I_2 \sim \ln k$ as $k\to\infty$. In what follows I shall outline the leading k-asymptotics of $I_m$ for general m; a fuller treatment is given in the original paper (Berry 1977), and an important development has been made by Hannay (1982, 1983, 1985).

What catastrophe theory provides is the understanding of the nature of the divergence of $I_m$ as $k\to\infty$. A crucial step is realising that the ensemble (of phase screens, or atmospheres, for example), over which the average in (13) is taken, can be <u>smoothly parameterised</u> by a large number of variables which can be considered as extra control parameters C. Each choice of C thus gives an atmosphere, or an irregular surface. For example, the deviation of an irregular surface from a plane, or the variations in refractive index of air, may be described by Gaussian random functions, which are superpositions of infinitely many sinusoids whose phases are the extra controls (giving a control space in the form of an infinite-dimensional torus). Averaging consists of integrating over these C with smooth probability density P(C) of realisations of members of the ensemble, so that

$$I_m = \int dC P(C) \ |\psi(C;k)|^{2m}. \qquad (14)$$

Now, for large k this enormously augmented control space is dominated by caustics on which $\psi$ is large; because of the high dimensionality of C, catastrophes of very high codimension can occur. The integral (14) is dominated by the caustics, and it is natural to assess the separate contributions of each universality class of singularity, that is to discover the k-dependence of the contribution $I_{mj}$ of the j'th catastrophe to the m'th moment. The assumption here is that contributions of different catastophes to (14) can be separated.

To estimate $I_{mj}$ the diffraction catastrophe scaling laws are employed as follows. The localized control-space regions of high intensity corresponding to the j'th catastrophe give contributions whose 'strength' is $|\psi|^{2m} \sim k^{2m\beta_j}$ and whose 'extent' is $k^{-\gamma_j}$ where $\gamma_j$ and $\beta_j$ are the exponents of section 3. Thus

$$I_{nj} \sim k^{2m\beta_j - \gamma_j} \qquad (15)$$

This estimate is confirmed by a careful scaling of the integral (14).

Thus each catastrophe contributes a power-law divergence to the n'th moment, provided the exponent $2m\beta_j - \gamma_j$ is positive. Obviously $I_n$ is dominated by the catastrophe(s) for which this exponent is largest, so that the asymptotic behaviour is

$$I_m \to A_m k^{\nu_m} \quad \text{as } k \to \infty, \qquad (16)$$

where

$$\nu_m \equiv \max_j (2m\beta_j - \gamma_j) \qquad (17)$$

will be called the twinkling exponents. In the competition to dominate $I_m$, which catastrophe wins? This is fully discussed by Berry (1977). The main result is that the codimension K(m) of the winning catastrophe increases with m: higher moments are dominated by higher catastrophes. This is physically reasonable, because high moments of $|\psi|^2$ are dominated by large rare fluctuations at the light detector, and these correspond to the close passage of high-codimension diffraction catastrophes.

The value of the twinkling exponent $\nu_m$ depends on which catastrophes are permitted to enter the competition, and this in turn depends on physical circumstances. For waves propagating in two space dimensions, wavefronts are one-dimensional and so only catastrophes with corank unity (i.e. one state variable s- cf.section 2) can compete. These are the cuspoid singularities, and (17) gives, for the twinkling exponents,

$$\nu_m = \max_K \frac{K(2m-K-3)}{2(K+2)} \tag{18}$$

The first few exponents are listed in table III. The value $\nu_2=0$ reflects the fact, already mentioned, that $I_2$ grows as $\ell nk$ rather than as a power. As $n\to\infty$, $\nu_m\to m$ and the codimension $K_{max}(m)$ of the dominating catastrophe grows as $K_{max}(m)\to 2\sqrt{m}$.

Table III

| m | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| K | 1 | 1 | 2 | 2 | 3 | 3 | 3 and 4 | 4 | 4 | 4 and 5 | 5 | 5 |
| $\nu_m$ | 0 | $\frac{1}{3}$ | $\frac{3}{4}$ | $\frac{5}{4}$ | $\frac{9}{5}$ | $\frac{12}{5}$ | 3 | $\frac{11}{3}$ | $\frac{13}{3}$ | 5 | $\frac{40}{7}$ | $\frac{45}{7}$ |

Twinkling exponents $\nu_m$ and codimension K for catastrophes of corank unity winning competition to dominate intensity moments $I_m$.

For waves in three space dimensions, wavefronts are two-dimensional and so catastrophes with corank unity and two may compete. The difficulty now is that many corank-two singularities exhibit modality (discussed near the end of section 3) and the classification of modal catastrophes is incomplete. Nevertheless, a study of the completely-classified singularities with $K \leq 11$ shows that for given m the quantity $2m\beta_j-\gamma_j$ increases with K and then decreases, so that there is still one (or sometimes two) dominating catastrophe(s). The resulting twinkling exponents and the dominating catastrophe(s) are listed in table IV. It is clear that the corank-two singularities soon dominate those of corank unity.

An interesting unsolved problem for wave propagation in three space dimensions is: how does $\nu_m$ behave as $m\to\infty$? The answer would require knowledge of the asymptotics of $\beta_j$ and $\gamma_j$ as $K\to\infty$.

For coranks unity and two, the existence of a maximum in $2m\beta_j-\gamma_j$ for finite K, on which the whole concept of dominating singularity depends, came as a pleasant surprise. It is not at all clear whether this feature would persist for singularities with corank $\geq 3$, that is for wave propagation in spaces of higher dimensionality. Perhaps there is a critical dimensionality, above which $2m\beta_j-\gamma_j$ does not have a maximum. Progress on this problem is frustrated by lack of systematic classification of high-corank catastrophes.

Table IV

| m | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Symbol | A | $A_2$ and $D_4$ | $D_4$ | $D_4$ and $E_6$ | $E_6$ and X | X | X | X | X and $W_{12}$ | $W_{12}$ | $W_{12}$ | $W_{13}$ |
| K | 1 | 1 and 3 | 3 | 3 and 5 | 5 and $\geq 7$ | $\geq 7$ | $\geq 7$ | $\geq 7$ | $\geq 7$ and 10 | 10 | 10 | 11 |
| $\nu_m$ | 0 | $\frac{1}{3}$ | 1 | $\frac{5}{3}$ | $\frac{5}{2}$ | $\frac{7}{2}$ | $\frac{9}{2}$ | $\frac{11}{2}$ | $\frac{13}{2}$ | $\frac{38}{5}$ | $\frac{87}{10}$ | $\frac{157}{16}$ |

Twinkling exponents $\nu_m$, codimension K and symbolic notation for catastrophes of corank unity or two winning competition to dominate intensity moments $I_m$.

The central result (16) of this theory is that intensity moments diverge as $k\rightarrow\infty$, with twinkling exponents $\nu_m$ which are universal for a given class of competing catastrophes. This universality means that $\nu_m$ are independent of the statistics of the random medium or undulating surface. But the coefficients $A_m$ in (16) are not universal and do depend on the nature of the randomness. Hannay (1982, 1983, 1985), in a powerful analysis, has calculated the $A_m$ for the corank-unity catastrophes involved in diffraction from a one-dimensional random phase screen. He finds that for certain exceptional moments $I_m$ (where 16m-7 is the square of an odd number, i.e. m = m = 2,3,5,5.5,8....) namely those at which the dominating catastrophe changes (see table III) the power-law divergence is multiplied by a factor $\ell nk$. For the particular case where the phase screen has Gaussian statistics and m=2, he obtains results in agreement with the earlier calculations of Shishov (1971) and Buckley (1971).

In optics, the twinkling exponents predicted by (16) can be tested by measuring the moments using light of different wavelengths, because

$$\nu_m = \lim_{k\rightarrow\infty} \frac{d\,(\ell n I_m)}{d\,(\ell n\ k)}\ . \tag{19}$$

Such a test has been carried out by Walker, Berry and Upstill (1983) with just two wavelengths of laser light refracted by randomly rippling water. The experiment was very difficult because the high moments depended on rare events and so their values took a long time to stabilise. Figure 8 shows the measured exponents compared with the predictions given by various classes of catastrophes. The best fit is given by cusps with $\gamma_j$ calculated using only one of the control directions (across the cusps), and is consistent with the visual observation that in the experiments the detector plane was dominated by lines where it was almost touched by the cusped edges of caustic surfaces in space. This question of a 'partial asymptotics', for cases where caustics are so asymmetrically deformed (e.g. elongated by paraxiality) that not all control parameters contribute to the $\gamma_j$ involved in scaling the intensity moments, is very subtle and needs further study.

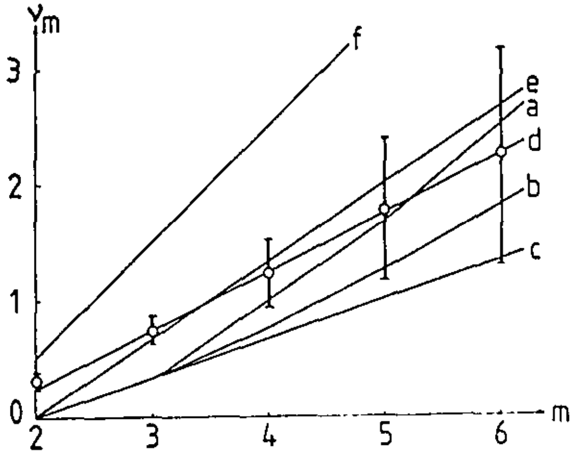


Fig. 8. Twinkling exponents $\nu_m$ (after Walker, Berry and Upstill 1983). Circles and error bars: experimental; curve a: unrestricted competition amongst catastrophes of coranks unity and two (table IV); curve b: competition restricted to corank-unity (cuspoid) catastrophes (table III); curve c: exponents from fold catastrophes; curve d: exponents from nongeneric transverse sections of cusp catastrophes; curve e: exponents from nongeneric transverse sections of elliptic and hyperbolic umbilic catastrophes; curve f: exponents from nongeneric transverse sections of $X_9$ catastrophes.

Finally, I wish to point out that there is an important class of problems involving waves and randomness for which the shortwave asymptotics is not characterised by singularity-dominated strong fluctuations. This occurs whenever the randomness has a self-similar, i.e. fractal structure (Mandelbrot 1982) extending to infinitesimal length scales. Then the limit $k\to\infty$ probes ever finer scales and the randomness never appears smooth on a wavelength scale. Caustics do not form and instead it is expected and found that the wave statistics as $k\to\infty$ will depend on a fractal dimension D describing the self-similarity of the randomness.

Two such cases have been studied so far, both concerning the phase-screen model for waves propagating in two space dimensions. Such a phase screen is characterised by the deviation h(x) it produces it initially rectilinear wavefronts normal to the incident beam. In the first case ('diffractals'), Berry (1979) and Berry and Blackwell (1981) studied the propagation of monochromatic waves and quasimonochromatic pulses when the wavefront and thus the graph of the function h(x) is a D-dimensional fractal curve (for example one of the Weierstrass-Mandelbrot functions studied by Berry and Lewis 1980); such curves are continuous but not differentiable, so that rays (normals) do not exist and a fortiori caustics do not exist. In the second case ('subfractals'), Jakeman (1982ab) studied monochromatic waves evolving from wavefronts for which h(x) was smooth but its derivative dh/dx is a D-fractal curve; thus rays exist but caustics do not (because the curvature does not exist).

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