ADIABATIC PHASE SHIFTS FOR NEUTRONS AND PHOTONS

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1. INTRODUCTION

My purpose is to present some new results and make some explanatory remarks about the recently-discovered geometrical phase (Berry 1984) as applied to simple quantal and optical situations. At its most abstract, this phase is a continuation property of the eigenvectors $|n_X\rangle$ of a complex Hermitian matrix $\hat{H}(X)$ depending on (at least two) real parameters $X=\{X_1,X_2\ldots\}$. Let X be taken round a circuit C in parameter space, and let $|n_X\rangle$ (assumed nondegenerate) be continued according to the natural transport law

$$\langle n_X|dn_X\rangle=0\,. \tag{1}$$

Then $|n_X\rangle$ is not a single-valued function of X but acquires round C a phase $\gamma_n(C)$ given by the flux through C of a 2-form V, that is

$$\gamma_n(C)=-\iint_S V_n(X) \tag{2}$$

where S is any surface spanning C. V is given by

$$V_n(X)=\mathrm{Im}\,\langle dn(X)|\wedge|dn(X)\rangle \tag{3}$$

where $|n(X)\rangle$ is any choice of eigenvector which is single-valued over S (of course no such choice satisfies (1)) and $|dn\rangle$ is the change in eigenvector resulting from a parameter-space displacement dX. Simon (1983) explained how $\gamma_n(C)$ is an example of anholonomy.

The usefulness of this mathematics stems from the fact that the continuation rule (1) is enforced by several important equations of physics in the limit when X changes slowly in time or space. For example, the time-dependent Schrodinger equation in the adiabatic limit of slowly-changed Hamiltonian parameters $X(t)$ (e.g. external forces) gives (1) (Berry 1984), so that wavefunctions $|\Psi_n(t)\rangle$ clinging to the eigenstate $|n_X\rangle$ will acquire round C a geometric phase as well as the more familiar dynamical phase

$$\int_0^T dt\, E_n(X(t))/\hbar \tag{4}$$

where T is the (long) time taken to make the circuit and $E_n(X)$ is the eigenvalue corresponding to parameters X.

In section 2 I discuss two aspects of $\gamma_n(C)$ for beams of neutrons in magnetic fields. First, if the fields change slowly in space a Stern-Gerlach beam deflection seems inevitable, but the associated extra phase shift can be made negligible in comparison with $\gamma_n(C)$. Second, a classical particle spinning in a constant field is an example of an integrable system, for which there exists a classical analogue of the geometrical phase, namely shifts $\Delta\theta$ in the canonical angle variables, discovered by Hannay (1985) and related to $\gamma_n(C)$ by Berry (1985); for this simple special case $\Delta\theta$ will be shown to be identical with a shift in the 'gyrophase' discovered independently by Littlejohn (1985).

In section 3 I show that the classical optics of slowly-varying dielectric media with both birefringence and gyrotropy can generate geometrical phase shifts; this way of making photons turn is analogous to magnetic fields for making neutrons turn.

Both the neutron and photon examples depend on operators $\hat{H}(X)$ which can be represented as 2x2 matrices. For such $\hat{H}$, the phase 2-form may be denoted $V_\pm(X)$ where $\pm$ refers to the eigenstate with the higher/lower eigenvalue. There is a useful general formula for $V_\pm(X)$, derived in Appendix A using the fact that any 2x2 Hermitian matrix can be written in the form

$$\hat{H}(X) = \begin{pmatrix} A_0 + A_z & A_x - iA_y \\ A_x + iA_y & A_0 - A_z \end{pmatrix} = A(X)\,\hat{1} + \underline{A}(X)\cdot\hat{\underline{\sigma}} \tag{5}$$

where $\underline{A} \equiv (A_x(X), A_y(X), A_z(X))$ and $\hat{\underline{\sigma}}$ is the vector of three Pauli matrices. The formula is

$$V_\pm(X) = \pm \frac{(A_x dA_y \wedge dA_z + A_y dA_z \wedge dA_x + A_z dA_x \wedge dA_y)}{2(A_x^2 + A_y^2 + A_z^2)^{3/2}} \tag{6}$$

2. NEUTRONS

Nonrelativistic neutrons in a variable magnetic field $\underline{B}(\underline{r},t)$ in vacuo, with magnetic moment $-\mu$ (which is a positive quantity), satisfy

$$\left\{ -\frac{\hbar^2}{2M}\nabla^2 + \mu\underline{B}(\underline{r},t)\cdot\hat{\underline{\sigma}} \right\} \Psi(\underline{r},t) = i\hbar\frac{\partial \Psi(\underline{r},t)}{\partial t} \tag{7}$$

where Ψ is a two-component spinor. If Ψ is initially in an eigenstate of $\underline{B}\cdot\hat{\underline{\sigma}}$ for constant $\underline{B}$, and then $\underline{B}$ is caused by external means to vary slowly in space and/or time round a cycle, that is a closed loop C in $\underline{B}$ space, then there will be a geometrical contribution $\gamma_\pm(C)$ to the phase of Ψ. This is easily calculated using (6) by realising that the parameters X can here be chosen to be the components of $\underline{B}$ =

$\{B_x, B_y, B_z\}$, and moreover the vector $\underset{\sim}{A}$ in (5) is simply $\underset{\sim}{B}$. The 2-form $V(X)$ now becomes a vector $\underset{\sim}{V}(\underset{\sim}{B})$ in $\underset{\sim}{B}$ space, namely the half--strength monopole

$$\underset{\sim}{V}_{\pm} = \frac{\pm \underset{\sim}{B}}{2B^3} . \tag{8}$$

From (2), $\gamma_{\pm}(C)$ is simply given by the flux of this vector through C, namely

$$\gamma_{\pm}(C) = \mp \tfrac{1}{2}\Omega(C) \tag{9}$$

where Ω is the solid angle subtended by C at the origin of $\underset{\sim}{B}$ space.

If the variation in $\underset{\sim}{B}$ is purely temporal, that is $\underset{\sim}{B} = \underset{\sim}{B}(t)$, the space and time dependences of Ψ separate and the theory for $\gamma_{\pm}(C)$ is that given by Berry (1984). One way to vary $\underset{\sim}{B}$ is round a cone of semiangle θ, that is

$$\left.\begin{aligned} \underset{\sim}{B} &= B\left\{\sin\theta\cos\phi(t), \sin\theta\sin\phi(t), \cos\theta\right\} \\ &\text{with} \quad \phi(0) = 0, \quad \phi(T) = 2\pi \end{aligned}\right\}. \tag{10}$$

Then (9) gives

$$\gamma_{\pm}(C) = \mp\pi\left(1-\cos\theta\right) \tag{11}$$

which for $\theta = \pi/2$ is the familiar sign change for one complete planar rotation of a spinor. In the special case of uniform rotation, that is $\phi = 2\pi t/T$ in (10), Schrödinger's equation was solved exactly by Rabi (1937), and from this solution it is not hard to confirm (11) for the geometical phase in the adiabatic limit $T\to\infty$.

In practice it might be easier to vary $\underset{\sim}{B}$ spatially rather than temporally, and let a monochromatic beam of x-polarized neutrons with energy E, travelling in the z direction, pass through length L of a helical magnetic field, given by (10) with $\phi = \phi(z)$ where $\phi(0) = 0$ and $\phi(L) = 2\pi$. Again (7) can be solved exactly when $\underset{\sim}{B}$ screws uniformly, that is $\phi = 2\pi z/L$, and again (11) is confirmed in the adiabatic limit $L\to\infty$; the case $\theta = \pi/2$ ($\underset{\sim}{B}$ always in the xy plane) was worked out by Eder and Zeilinger (1976).

There are however difficulties with this simple helical field. Although solenoidal $(\nabla\cdot\underset{\sim}{B} = 0)$ it is not irrotational (i.e. $\nabla\wedge\underset{\sim}{B} \neq 0$) and so would imply a time-varying electic field. A helical magnetic field which does satisfy $\nabla\cdot\underset{\sim}{B} = 0$ and $\nabla\wedge\underset{\sim}{B} = 0$ and which therefore could be part of a static electromagnetic field is

$$\underset{\sim}{B}(\underset{\sim}{r}) = B\left\{\sin\theta\cos qz\cosh qx, \sin\theta\sin qz\cosh qy, \cos\theta - \sin\theta\left(\sin qz\sinh qx - \cos qz\sinh qy\right)\right\} \tag{12}$$

where $q \equiv 2\pi/L$. This is a good approximation to the simple helical field((10) with $\phi = qz$) if the neutron beam is close to the z axis, that is $|x| \ll L$, $|y| \ll L$

But the price paid for making $\underline{B}$ curl-free, as in (12), is that now the magnitude $|\underline{B}(\underline{r})|$ of the field is non-uniform. This means that the neutrons are passing through a medium of variable refractive index

$$n_{\pm}(\underline{r}) = \left(1 \mp \mu|\underline{B}(\underline{r})|/E\right)^{1/2} \tag{13}$$

and so will be deviated. This is the Stern-Gerlach effect, which can alternatively be regarded as arising from the classical force $+\nabla(\underline{\mu}\cdot\underline{B}) = \mp\mu\nabla|\underline{B}|$. It is important to confirm that the extra beam phase (action/$\hbar$) arising from this cause can be neglected in comparison with the geometrical phase (11) which is the quantity of principal interest, and this we now do.

There are in fact four contributions to the beam phase χ . In terms of the neutron speed v in field-free space, these are:

$$\begin{aligned}
\chi_1 &= MvL/\hbar \quad \text{(kinetic phase)}\\
\chi_2 &= \mu BL/v\hbar \quad \text{(refractive phase; cf.13)}\\
\chi_3 &= \gamma_{\pm}(C) \quad \text{(geometrical phase; equation (11))}\\
\chi_4 &= \mu^2 B^2 L \sin^2 2\theta / 2Mv^3 = \chi_2^2 \sin^2 2\theta / 2\chi_1 \quad \text{(deviative phase, from Stern-Gerlach effect)}
\end{aligned} \tag{14}$$

(The expression for the deviative phase χ_4 is obtained by solving the classical equations for the neutron's motion with the Stern-Gerlach force generated by the field (12), to lowest order in the adiabatic parameter q, and calculating the consequent correction to $\int \underline{p}\cdot d\underline{r}/\hbar$ along the beam path.)

Now $\chi_1 \ggg 1$ for all realistic path lengths and speeds. Moreover $\chi_1 \gg \chi_2$ because the refractive index $n_{\pm}$ in (13) is very close to unity (for thermal neutrons in a field of $10T$, $\mu B/E \sim 10^{-5}$). In order for the quantum adiabatic theory to be applicable, we must have

$$\text{time to cycle the } \underline{B} \text{ field} = L/v \gg \text{ period of transition radiation between } |+\rangle \text{ and } |-\rangle = 2\pi\hbar/\mu B \tag{15}$$

which immediately implies $\chi_2 \gg 1$. The geometrical phase χ_3 is of order unity. And for the deviative phase χ_4 which is our main concern, (15) gives the inequality

$$\chi_4 \gg \sin^2 2\theta / 2\chi_1 \tag{16}$$

Because $\chi_1 \ggg 1$ this can easily be satisfied with $\chi_4 \ll 1$ (for a $0{\cdot}1$m beam path with neutrons with $v = 200\,\mathrm{ms}^{-1}$ in a field of $0{\cdot}01$T, $\chi_4 \sim 10^{-5}$, $\chi_2 \sim 500$ and $\chi_1 \sim 10^8$), confirming that the inevitable beam deviation resulting from the requirement $\nabla \wedge \underline{B} = 0$ would not spoil any experiment to detect $\gamma_{\pm}(C)$.

There remains the problem of detecting the geometrical phase $\gamma_{\pm}(C)$. In the experiment I proposed earlier (Berry 1984) this would be achieved by interference: the neutron beam would be split into two beams, each passed through a length L of magnetic field, one of which is helical (e.g. 12) and one uniform; the beams are then recombined.

Then $\gamma_{\pm}(C)$ should be revealed as a shift of interference fringes with variation of B, v or L (χ_1, χ_2 and χ_3 depend differently on these quantities and so in principle they could be separated).

Now we turn to the semiclassical interpretation of the geometrical phase for neutrons, which is based on regarding them as precessing classical spins. Only the magnetic part of the Hamiltonian (7) is involved, and this will be written in terms of the spin angular momentum operator as

$$\hat{H} = \omega(X(t))\,\underline{b}(X(t)) \cdot \hat{\underline{S}} \tag{17}$$

where

$$\hat{\underline{S}} \equiv \hbar\hat{\underline{\sigma}}/2, \quad \underline{b} \equiv \underline{B}/|\underline{B}|, \quad \omega \equiv 2\mu|\underline{B}|/\hbar \tag{18}$$

and where the parameters X have been reinstated. The quantal equation of motion for the expectation value $\underline{S}(t)$ of $\hat{\underline{S}}$ is exactly the same as the classical spin precession equation, namely

$$\frac{d\underline{S}}{dt} = \omega\,\underline{b}\wedge\underline{S} \tag{19}$$

For slow change of the unit magnetic field vector $\underline{b}$, this describes precession of $\underline{S}$ about $\underline{b}$ with instantaneous angular velocity ω and adiabatic invariant

$$I = \underline{S}\cdot\underline{b} \tag{20}$$

(for neutrons, $I = \pm\hbar/2$). Canonically conjugate to I is the angle variable ϕ, defined (fig.1a) with respect to perpendicular axes labelled by unit vectors $\underline{e}_1$, $\underline{e}_2$ in the plane perpendicular to $\underline{b}$ (i.e. $\underline{e}_1\wedge\underline{e}_2 = \underline{b}$).

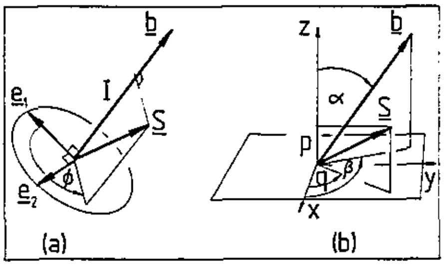

figure 1

When $\underline{b}(X)$ is taken round a circuit C in X space in a (long) time T, the total change in ϕ is not obtained simply by integrating the instantaneous angular velocity $\omega(X)$, but contains an extra shift $\Delta\phi(C)$, i.e.

$$\phi(T) - \phi(0) = \int_0^T dt\, \omega(X(t)) + \Delta\phi(C). \tag{21}$$

This $\Delta\phi$ is an example of the canonical angle shifts discovered by Hannay (1985) to be a general phenomenon in the Hamiltonian mechanics of slowly-changed systems whch are instantaneously integrable. In the present case the motion of $\underline{S}$ on the sphere with (conserved) radius $S \equiv |\underline{S}|$, generated by (19), can be obtained by regarding (17) as a classical Hamiltonian whose canonical coordinate q is the azimuth angle of $\underline{S}$ relative to fixed Cartesian axes xyz (fig.1b) and whose canonical momentum p is S_z. If relative to these axes $\underline{b}(X)$ has polar angles $\alpha(X)$ and $\beta(X)$ then the Hamiltonian may be written explicitly as

$$H(q,p;X) = \omega(X)\left[p\cos\alpha(X) + (S^2-p^2)^{1/2}\cos(q-\beta(X))\sin\alpha(X)\right] \tag{22}$$

where X depends slowly on time. (It may be confirmed that Hamilton's equations reproduce (19)).

It would seem that the specification of $\Delta\phi(C)$ by (21) is incomplete, because the unit vectors $\underline{e}_1$, $\underline{e}_2$, relative to which ϕ is defined, have themselves not been specified. The remarkable fact is that such specification is not necessary: Hannay (1985) showed that <u>any</u> X-dependent choice of origin of ϕ may be chosen (provided this is single-valued over a domain in X space containing C) and $\Delta\phi(C)$ is independent of the choice.

To calculate $\Delta\phi(C)$ I use the formalism in Berry (1985), where the angle shift is the flux of a 2-form written as

$$\Delta\phi(C) = -\iint_S \frac{\partial W(X,I)}{\partial I} \tag{23}$$

W is expressed in terms of parameter-space displacements of the phase-space variables p and q, considered to depend on X and the adiabatic (action-angle) variables I and ϕ:

$$W(X,I) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\ dp(I,\phi;X) \wedge dq(I,\phi;X). \tag{24}$$

To get dp and dq we first write the spin vector in the basis $\underline{b}$, $\underline{e}_1$, $\underline{e}_2$ as

$$\underline{S} = I\underline{b} + \cos\phi\,(S^2-I^2)^{1/2}\underline{e}_1 + \sin\phi\,(S^2-I^2)^{1/2}\underline{e}_2 \tag{25}$$

Now $dp = dS_z$, and

$$\begin{aligned} dS_y &= d\left((S^2-p^2)^{1/2}\sin q\right) = (S^2-p^2)^{1/2}\cos q\, dq + \text{term in } dp \\ &= S_x\, dq + \text{term in } dp. \end{aligned} \tag{26}$$

Thus

$$dp \wedge dq = dS_z \wedge dS_y / S_x \tag{27}$$

where

$$dS_z = I\,db_z + \cos\phi\,(S^2-I^2)^{1/2} de_{1z} + \sin\phi\,(S^2-I^2)^{1/2} de_{2z} \tag{28}$$

and similarly for dS_y.

In working out W from (24) it is convenient to choose the fixed axes xyz to coincide with the instantaneous position of the triad $\underline{b}$, $\underline{e}_1$, $\underline{e}_2$, i.e.

$$\underline{b} = \underline{e}_1 \wedge \underline{e}_2 = (0,0,1)\ ,\ \underline{e}_1 = (1,0,0)\ ,\ \underline{e}_2 = (0,1,0) \tag{29}$$

Thus

$$db_z = de_{2y} = 0 \quad , \quad db_y = -de_{2z} \tag{30}$$

and (25–28) give

$$dp \wedge dq = (\cos\phi\, de_{1z} + \sin\phi\, de_{2z}) \wedge \left(-I de_{2z} + \cos\phi (S^2 - I^2)^{1/2} de_{1y}\right) \Big/ \cos\phi$$

$$= -I de_{1z} \wedge de_{2z} + \cos\phi (S^2 - I^2)^{1/2} de_{1z} \wedge de_{1y} + \sin\phi (S^2 - I^2)^{1/2} de_{2z} \wedge de_{1y} \tag{31}$$

Now (24) gives

$$W = -I\, de_{1z} \wedge de_{2z} \tag{32}$$

which on reverting to arbitrary axes gives, for the 2-form in (23),

$$-\frac{\partial W}{\partial I}(X, I) = d\underline{e}_1(X) \wedge \cdot\, d\underline{e}_2(X). \tag{33}$$

If the parameters X are position coordinates of the space in which is varying, then d becomes the gradient operator and the right-handside of (33) can be written

$$\nabla \wedge (\underline{e}_1 \cdot \nabla\ \underline{e}_2) \tag{34}$$

Exactly this quantity appears in an elegant analysis by Littlejohn (1985) of the adiabatic spiralling of a charged particle around a curving magnetic field line, which is analogous to the spin precession problem I am considering here. Littlejohn calls the azimuthal angle variable ϕ the 'gyrophase', emphasizes the arbitrariness in its definition, and shows that the quantity (34) has 'gyrogauge invariance' that is invariance under arbitrary position-dependent of rotations of $\underline{e}_1$ and $\underline{e}_2$ in the plane perpendicular to $\underline{b}$.

If instead the parameters X are components of $\underline{B}$ itself, the 2-form (33) becomes the vector

$$-\frac{\partial \underline{W}}{\partial I}(\underline{B}, I) = \nabla_{\underline{B}} \wedge \left(\underline{e}_1(\underline{B}) \cdot \nabla_{\underline{B}}\, \underline{e}_2(\underline{B})\right). \tag{35}$$

One way to evaluate this (probably not the simplest) is to choose

$$\left.\begin{aligned} \underline{e}_1 &= \left(-B_x B_z\ ,\ -B_y B_z\ ,\ B_x^2 + B_z^2\right) \Big/ B\,(B_x^2 + B_y^2)^{1/2} \\ \underline{e}_2 &= (B_y\ ,\ -B_x) \Big/ (B_x^2 + B_y^2)^{1/2} \end{aligned}\right\} \tag{36}$$

Then direct calculation gives

$$-\frac{\partial \underline{W}(\underline{B}, I)}{\partial I} = \underline{B}/B^3 \tag{37}$$

and hence, by (23) the angle shift

$$\Delta\phi = \Omega(C). \tag{38}$$

(Hannay (1985) points out that this is the extra angle turned through by a spinning symmetrical top whose axis is forced round a closed circuit enclosing solid angle Ω.)

The result (38) also illustrates the semiclassical relation (Berry 1985) between $\Delta\phi$ and the geometrical phase γ for a state with quantum number n of a classically integrable system, namely

$$\Delta\phi(C) = -\partial\gamma_n(C)/\partial n, \tag{39}$$

because as shown by Berry (1984) the generalization of (9) for states with quantum number n for the spin component along $\underline{B}$ is $\gamma_n(C) = -n\Omega(C)$.

Finally, it is worth remarking that the quantum phase γ and the classical angle $\Delta\phi$ arise only after three levels of rotation: a <u>turn</u> through Ω of the axis about which the already <u>spinning</u> neutron is <u>precessing</u>.

2. PHOTONS

Photons can be made to turn by twisting an anisotropic medium in which they are propagating. Consider first a dielectric medium that is not twisting, and in which there are plane waves with electric displacement and field vectors

$$\underline{D}(\underline{r},t) = \underline{D}\exp\{i(\underline{k}\cdot\underline{r}-\omega t)\}\ ;\ \underline{E}(\underline{r},t) = \underline{E}\exp\{i(\underline{k}\cdot\underline{r}-\omega t)\}. \tag{40}$$

Anisotropy is embodied in the reciprocal dielectric tensor $\underline{\underline{\eta}}$ defined by

$$\underline{E} = \underline{\underline{\eta}}\cdot\underline{D}/\varepsilon_0 \tag{41}$$

Maxwell's equations then show (see e.g. Landau, Lifshitz and Pitaevskii 1984) that $\underline{D}$ is perpendicular to the wavector $\underline{k}$ and satisfies

$$\underline{\underline{\eta}}_T(\underline{k})\cdot\underline{D} - \underline{D}/n^2 = 0 \tag{42}$$

where $\underline{\underline{\eta}}_T(\underline{k})$ is the 2x2 tensor made of the components of $\underline{\underline{\eta}}$ in the plane transverse to $\underline{k}$, and n is the refractive index

$$n \equiv ck/\omega \tag{43}$$

Thus the propagating plane waves $\underline{D}$ in each direction $\underline{k}$ are the two eigenvectors of $\underline{\underline{\eta}}_T(\underline{k})$, corresponding in general to elliptic polarization, and their refractive indices are the eigenvalues.

In a general nonabsorbing medioum, $\underline{\underline{\eta}}_T$ is Hermitian. This suggests that a light beam could be made to acquire a geometrical phase shift γ by adiabatically twisting the medium by one complete turn along the beam path. For γ to be nonzero, the anisotropy must at least combine uniaxial birefringence and gyrotropy, for which the dielectric law is

$$\underline{E} = \left(\underline{D}/n_o^2 + \underline{b}\cdot\underline{D}\,\underline{b} - i\,\underline{g}\wedge\underline{D} \right)/\varepsilon_o \tag{44}$$

When $\underline{g} = 0$ the medium has pure birefringence, determined by $\underline{b}$, and the eigenmodes are linearly polarized with refractive indices differing most if $\underline{b} \perp \underline{k}$. When $\underline{b} = 0$ the medium has pure gyrotropy, determined by $\underline{g}$, and the eigenmodes are circularly polarized with refractive indices differing most if $\underline{g} \parallel \underline{k}$, Physical ways of realizing the optic vectors $\underline{b}$ and $\underline{g}$ will be described later. If the wave vector is directed along the z axis, comparison of (44) with (41) gives

$$\underline{\eta}_T = \begin{pmatrix} \frac{1}{n_o^2} + b_x^2 & b_x b_y + i g_z \\ b_x b_y - i g_z & \frac{1}{n_o^2} + b_y^2 \end{pmatrix} \tag{45}$$

There are many ways to make $\underline{b}$ and $\underline{g}$ vary along the beam path (i.e. with z) so as to twist once. The simplest choice is to make $\underline{b}$ and $\underline{g}$ parallel and turn them about the z axis. Let the single optic axis thus defined by $\underline{b}$ and $\underline{g}$ have polar angles θ and ϕ relative to the z axis. Then (45) becomes

$$\underline{\eta}_T = \begin{pmatrix} \frac{1}{n_o^2} + \frac{b^2}{2}\sin^2\theta + \frac{b^2}{2}\sin^2\theta\cos 2\phi & \frac{b^2}{2}\sin^2\theta\sin 2\phi + ig\cos\theta \\ \frac{b^2}{2}\sin^2\theta\sin 2\phi - ig\cos\theta & \frac{1}{n_o^2} + \frac{b^2}{2}\sin^2\theta - \frac{b^2}{2}\sin^2\theta\cos 2\phi \end{pmatrix} \tag{46}$$

where $b \equiv |\underline{b}|$ and $g \equiv |\underline{g}|$.

The matrix has exactly the form (5), so that (6) gives the phase 2-form. If θ and ϕ are regarded as the parameters, a short calculation gives

$$\left.\begin{aligned} V_\pm(\theta,\phi) &= \mp \frac{2\sigma^2\sin^3\theta\,(1+\cos^2\theta)}{(\sigma^2\sin^4\theta + 4\cos^2\theta)^{3/2}}\, d\theta \wedge d\phi \\ \text{with} \quad \sigma &\equiv b^2/g \end{aligned}\right\} \tag{47}$$

Now let the optic axis be turned so as to sweep out a cone, by keeping θ constant and increasing ϕ by 2π. This circuit will result in a geometrical phase $\gamma(\theta)$, which is obtained from (2), by integrating over θ and ϕ, as

$$\gamma(\theta) = \pm 2\pi\left(1 - \frac{2\cos\theta}{(\sigma^2\sin^4\theta + 4\cos^2\theta)^{1/2}}\right) \tag{48}$$

(Essentially the same phase is obtained by rotating other rigid connections between $\underline{b}$ and $\underline{g}$, for example $\underline{b}$ perpendicular to $\underline{g}$ and $\underline{k}$, or $\underline{b}$ parallel to the projection of $\underline{g}$ perpendicular to z.) Equation (48) is the central result of this section; it shows that as the cone opens from $\theta = 0$ to $\theta = \pi/2$ the associated gometrical phase increases from zero to 2π

An obvious way to detect $\gamma(\theta)$ is by the interference of two recombined beams that have been split and passed through two anisotropic media, one twisted and one not. A difficulty with this experiment, arising precisely from the anistropy and analogous to the Stern-Gerlach deflection of neutrons, is that the refractive index n depends on the propagation direction (relative to the optic axis), so that rays and waves are not parallel and a narrow beam (ray) will spiral in the twisting medium instead of travelling in the z direction. The difficulty can be avoided as follows. (46) and (42) give the two refractive indices as

$$\frac{1}{n_\pm^2} = \frac{1}{n_o^2} + \frac{g}{2}\left[\sigma \sin^2\theta \pm \left(\sigma^2 \sin^4\theta + 4\cos^2\theta\right)^{1/2}\right] \tag{49}$$

If we now choose the strengths b and g of the birefringence and gyrotropy such that

$$\sigma = b^2/g = 1, \tag{50}$$

then

$$\frac{1}{n_+^2} = \frac{1}{n_o^2} + g, \quad \frac{1}{n_-^2} = \frac{1}{n_o^2} - g\cos^2\theta \tag{51}$$

Thus for this special choice n_+ is independent of θ (spherical dispersion surface) and gives rise to an ordinary ray, propagating parallel to its wave $\underline{k}$ (i.e. along z) instead of spiralling. The corresponding phase is

$$\gamma_+(\theta) = \frac{2\pi(1-\cos\theta)^2}{1+\cos^2\theta} \tag{52}$$

Again this increases from 0 to 2π as θ grows from 0 to $\pi/2$; γ would be easiest detected when it equals π, which occurs when $\theta = \arccos(2-\sqrt{3}) = 74{\cdot}46^\circ$.

To realize the optic vectors $\underline{b}$ and $\underline{g}$ in practice, the most promising procedure would appear to be to impose strong external electric and magnetic fields $\underline{E}$ and $\underline{B}$ on an isotropic dielectric liquid (e.g. water). Then by the Kerr electro-optic effect $\underline{E}$ induces, parallel to itself, birefringence $\underline{b}$ with strength

$$b^2 = 4\pi B' E^2/k n_o^3 \tag{53}$$

where k is the vacuum wavenumber and B' the Kerr constant tabulated by Kaye and Laby (1973) (they call it B; for water, its value is $52\, fm\, V^{-2}$). And by the Faraday effect (optical rotation) $\underline{B}$ induces, parallel to itself, gyrotropy $\underline{g}$ with strength

$$g = 443\, rB/kn_o^3 \tag{54}$$

where r is the Verdet constant tabulated by Kaye and Laby (1973) ((54) is written in SI units; for water, $r = 0{\cdot}018$ min A^{-1} and gives the optical rotation angle in a distance L by $231{\cdot}5\, rBL$ rad).

$\underline{E}$ and $\underline{B}$ are parallel, with magnitudes which to avoid ray spiralling must be related by (50); this gives

$$E = \left(\frac{443 r B}{4\pi B'} \right)^{1/2} \tag{55}$$

For water,

$$E = 5 \times 10^6 \, B^{1/2} \, V m^{-1} \tag{56}$$

with B in Tesla. The fields cannot be too small, because of the adiabatic restriction, fundamental to the whole theory, that the distance L over which the medium (i.e. $\underline{E}$ and $\underline{B}$) twists must be such as to generate a refractive phase much larger than unity. When $\sigma = 1$ this is just the Faraday rotation over distance L, namely $231 \cdot 5 r B L$. For water with $B = 1T$ and $L = 1m$, the phase is $4 \cdot 2\,\mathrm{rad}$, which is (just) large enough, and then (55) gives $E = 5 \times 10^6 V m^{-1}$ which is not unattainably large.

Strictly speaking, the electric field $\underline{E}$ is not necessary: a sufficiently large $\underline{B}$ alone will produce a Cotton-Mouton effect, that is a birefringence $\underline{b}$, as well as the Faraday rotation $\underline{g}$, but in order to satisfy (50) the magnitude B must be at least of the order of several hundred Tesla. Another possibility is to employ a chiral smectic C liquid crystal, with no fields, as the anisotropic medium (de Gennes 1974). This is a layered arrangement of rod-like molecules tilted with respect to the layers' normal and rotated from layer to layer. The normal determines a single optic axis, with birefringence generated by the tilt and gyrotropy by the rotation. To produce a geometric phase, the optic axis must itself be twisted down the beam path. As for neutrons, the geometric phase requires a hierarchy of structure: in this case the molecules must be <u>tilted</u> about an axis relative to which they are <u>rotated</u> and which is itself <u>twisted</u>.

Finally, we compare the results (11) for neutrons and (48) for light. These show that for a complete planar turn (cone angle $\theta = \pi/2$) γ is π for neutrons and 2π for light. In view of the fact that light can be regarded as spin-one bosons, it is not surprising that there is no sign change for this rotation. On the other hand, results for both light and neutrons were obtained from 2x2 matrices: (45) for light, and $\mu \underline{B} \cdot \hat{\underline{\sigma}}$ in (7) for neutrons. But whereas the operator $\mu \underline{B} \cdot \hat{\underline{\sigma}}$ returns to its initial form only after a complete turn of $\underline{B}$ - and thereby generates a sign change of the eigenvectors - the matrix $\underline{\eta}_T$ in (45) (or (46)), and hence the anisotropic medium, returns to its intial form after a half-turn of the optic vector $\underline{b}$ ($\underline{g}$ is irrelevant in this case because $g_z = 0$) - and it is not surprising that the field vectors of linearly polarized light are reversed by a slowly-executed half-turn of the birefringent medium in which they are propagating.

APPENDIX A: PHASE 2-FORM FOR 2x2 MATRICES

In calculating $V_\pm(X)$ from (3) using the operator (5), we can ignore $A_0(X)$ without losing generality because this quantity does not affect the eigenstates $|\pm\rangle$. These states satisfy

$$\hat{H} \, |\pm\rangle = \pm E \, |\pm\rangle \tag{A.1}$$

where diagonalization trivially yields

$$E = \left(A_x^2 + A_y^2 + A_z^2 \right)^{1/2} \tag{A.2}$$

and we do not indicate explicitly the dependence on the parameters X. For V_+, (3) gives, using the completeness relation $|+\rangle\langle+| + |-\rangle\langle-| = 1$,

$$V_+ = \mathrm{Im}\, \langle d+|_\wedge |d+\rangle = \mathrm{Im}\langle d+|+\rangle_\wedge \langle+|d+\rangle + \mathrm{Im}\langle d+|-\rangle_\wedge \langle-|d+\rangle \tag{A.3}$$

The first term vanishes because $\langle+|d+\rangle$ is purely imaginary so that $\langle+|d+\rangle = -\langle d+|+\rangle$. For the second term we require $\langle-|d+\rangle$, for which (A.1) gives

$$\langle-|d+\rangle = \frac{\langle-|d\hat{H}|+\rangle}{2E} = \frac{d\underline{A}\cdot\langle-|\hat{\underline{\sigma}}|+\rangle}{2E} \tag{A.4}$$

Thus
$$V_+ = \mathrm{Im}\, \frac{d\underline{A}\cdot\langle+|\hat{\underline{\sigma}}|-\rangle_\wedge\langle-|\hat{\underline{\sigma}}|+\rangle\cdot d\underline{A}}{4E^2}$$

$$= \frac{dA_x \wedge dA_y}{4E^2}\,\mathrm{Im}\left\{\langle+|\hat{\sigma}_x|-\rangle\langle-|\hat{\sigma}_y|+\rangle - \langle+|\hat{\sigma}_y|-\rangle\langle-|\hat{\sigma}_x|+\rangle\right\}$$
$$+ \text{ cyclic permutations} \tag{A.5}$$

Using completeness we simplify the quantity in $\{\}$ to

$$\mathrm{Im}\{\} = \mathrm{Im}\langle+|[\hat{\sigma}_x, \hat{\sigma}_y]|+\rangle = 2\langle+|\hat{\sigma}_z|+\rangle \tag{A.6}$$

Now, the eigenvector $|+\rangle_\wedge$ defined by (A.1) and (5) is a spinor for which the expectation value of $\hat{\underline{\sigma}}$ is a unit vector directed along $\underline{A}$, so

$$\langle+|\hat{\sigma}_z|+\rangle = \frac{A_z}{(A_x^2 + A_y^2 + A_z^2)^{1/2}} \quad \text{etc.,} \tag{A.7}$$

Substituting this into (A.6) and thence (A.5), we obtain the desired formula (6). (It is trivial to check that $V_- = -V_+$, implying equal and opposite geometrical phase shifts for the two states.)

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