LETTER TO THE EDITOR

Classical non-adiabatic angles

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Received 18 December 1987

Abstract. If a family of tori in phase space is driven by a time-dependent Hamiltonian flow in such a way as to return after some time to the original family, there generally results a shift in the angle variables. One realisation of this process is in the cyclic adiabatic change of a classical Hamiltonian, and the angle change has previously been shown to separate naturally into a dynamical part and a geometrical part. Here the same geometrical angle change is extracted when the return is achieved non-adiabatically, and the 'dynamical' remainder calculated. Two examples are given: the precession of a spin and the rotation of phase-space ellipses.

It is known [1, 2] that the cyclic adiabatic change of an integrable Hamiltonian induces in the angle variable(s) a change $\Delta \theta$ which separates naturally into the obvious dynamical change $\Delta \theta_d$ (the time integral of the frequency), and an additional geometric change $\Delta \theta_g$. This is a classical analogue of the geometric quantum phase [3] arising naturally in the adiabatic cyclic change of a quantum Hamiltonian. As has recently been pointed out by Aharonov and Anandan [4], the same geometric part can be extracted from the phase change that occurs in a general, non-adiabatic, cyclic evolution of a quantum state, to leave a quite simple 'dynamical' remainder. Our purpose is to show that $\Delta \theta_g$ can be similarly extracted from the general, non-adiabatic, cyclic change of an action torus, with a simple remainder.

For simplicity we analyse a system with one freedom and later generalise to more. Consider an action-angle coordinate system on the phase plane, i.e. $I(q, p; X)$, $\theta(q, p; X)$ where $X = (X_1, X_2, \ldots)$ is a set of parameters with which the coordinate system can be changed. The action contours are loops (one-dimensional tori) with area $2\pi I$, and the angle is the canonically conjugate variable (whose uniform distribution is defined by the density $\delta(I - I(q, p; X))$).

The purpose of setting up this variable coordinate system is that we are now to imagine a flow in the phase space generated by a Hamiltonian $H(q, p, t)$ which causes an initial family of closed curves (tori), marked in the flow, to be carried through a cycle so as to return after time $T$ (figure 1). At all times $0 < t < T$ there is a parameter $X(t)$ for which the curves coincide with the action contours of $I(q, p; X(t))$. This process defines a classical cyclic evolution; it is not necessary that $H$ change slowly, or cyclically, or that the marked initial curves coincide with its contours.

Since by Liouville's theorem the area of a curve cannot change as it is transported, the action coordinate for any carried phase point is constant, $\dot{I} = 0$, and the cyclic change means $X(T) = X(0)$. In contrast, the angle variable (of a carried phase point) will generally vary in this process, and, in particular, when an initial curve has returned after time $T$ the individual points will be shifted by an angle (the same for all points on that curve) which we now determine.
Following [1] we write the rate of change of angle of a phase point as the sum of contributions from its motion in phase space and from the changing coordinates $I, \theta$:

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial I} + X \frac{\partial X}{\partial \theta}$$  \hspace{1cm} (1)

where

$$\mathcal{H}(\theta, I, t) = H(q(\theta, I; X(t)), p(\theta, I; X(t)), t)$$  \hspace{1cm} (2)

and $\frac{\partial X}{\partial \theta}$ is the rate at which the angle at fixed $q, p$ changes with parameters. Integrating (1) we obtain $\Delta \theta$, which does not depend on $\theta$, as a sum of two terms that individually do depend on $\theta$. These dependences can be eliminated by averaging round each contour of constant action; we denote this averaging by

$$\langle \ldots \rangle = \int dq \int dp \delta(I - I(q, p; X)) \ldots = \frac{1}{2\pi} \int_0^{2\pi} d\theta \ldots .$$  \hspace{1cm} (3)

Thus we obtain

$$\Delta \theta = \Delta \theta_d + \Delta \theta_s$$  \hspace{1cm} (4)

where

$$\Delta \theta_d = \int_0^T dt \frac{\partial \mathcal{H}}{\partial I}$$  \hspace{1cm} (5)
and
\[ \Delta \theta_g = \oint dX (\partial_X \theta) = \oint \langle d\theta \rangle \]

(6)

where \( d\theta \) is the angle variable derivative 1-form in parameter space.

By design, then, the angle shift has been divided into a geometric part \( \Delta \theta_g \)—the same as that arising naturally in the adiabatic change of a Hamiltonian \([1, 2]\) and involving the parameter-space 1-form \( d\theta \)—and a remaining 'dynamical' part \( \Delta \theta_d \)—involving not the instantaneous frequency as in the adiabatic case but its average \( \langle \partial \mathcal{H} / \partial I \rangle \) round the action contour. Thus (4) is the classical analogue of Aharonov and Anandan's division \([4]\) of a non-adiabatic quantum phase change into a geometric part occurring naturally in adiabatic change \([3]\) and a remaining dynamical part.

Useful formulae for \( \Delta \theta_g \) will now be obtained by introducing the parameter-dependent generating function of the canonical transformation from \( q, p \) to \( \theta, I \):

\[
S(q, I; X) = \int_q^{q'} dq' p(q', I; X)
\]

(7)

We note that this allows (1) to be reinterpreted \([2]\) as a Hamilton equation in action-angle variables: the changing \( X \) introduces a time dependence which contributes to the transformed Hamiltonian a term \( \partial S / \partial t \), whose \( I \) derivative can be shown to equal the extra term \( \dot{X} \partial_X \theta \) in (1) (the proof proceeds by reducing both quantities to \( S_{IX} - S_{II}S_{Xq} / S_{Iq} \)).

Expressing \( S \) in action-angle variables by

\[
\mathcal{S}(\theta, I; X) = S(q(\theta, I; X), I; X)
\]

we have \( d \mathcal{S} = dS + pdq \) and hence in (6)

\[
\langle d\theta \rangle = \langle d(\partial S / \partial I) \rangle = d(\langle \partial \mathcal{S} / \partial I \rangle) - \frac{\partial}{\partial I} \langle pdq \rangle = -\frac{\partial}{\partial I} \langle pdq \rangle
\]

(9)

where \( dq \) is the coordinate displacement of a torus point with fixed \( \theta, I \) accompanying an infinitesimal parameter change. (The torus average \( \langle \partial \mathcal{S} / \partial I \rangle \) vanishes because \( \partial \mathcal{S} / \partial I \) is periodic in \( \theta \).)

Thus

\[
\Delta \theta_g = -\frac{\partial}{\partial I} \oint \langle pdq \rangle = -\frac{\partial}{\partial I} \left( \oint pdq \right) = -\frac{\partial}{\partial I} \langle A(\theta, I) \rangle
\]

(10)

where \( A(\theta, I) \) is the phase-space area swept out during the circuit (i.e. over time \( T \)) by the point labelled \( \theta \) on the torus \( I \) (figure 2). The torus average \( \langle A(\theta, I) \rangle \) is independent of the \( X \)-dependent choice of origin of \( \theta \). An alternative expression is obtained by writing the first circuit integral in (10) as the flux, through the parameter-space circuit, of the 2-form \( -\partial((dp \wedge dq))/\partial I \) (cf \([2]\)).

If the system has \( N \) freedoms, there are \( N \) actions \( I = \{I_i\} \), \( N \) angles \( \theta = \{\theta_i\} \) and hence \( N \) angle shifts \( \Delta \theta = \{\Delta \theta_i\} \) \((1 \leq I \leq N)\). The \( l \)th dynamical and geometric shifts are given by (5) and (10) with \( \partial I \) replaced by \( \partial I_l \) and \( A(\theta, I) \) replaced by the symplectic area

\[
A(\theta, I) = \sum_{l=1}^{N} \oint p_i(\theta, I; X) \, dq_i(\theta, I; X).
\]

(11)
The form (10) for the geometric angle implies a concise expression for the semi-classical quantum phase obeying the relation [2] \( \Delta \theta_g = -\hbar \partial \gamma / \partial I \). This evidently yields

\[
\gamma_g = (A(\theta, I))/\hbar
\]

(12)
a formula which could be rederived \textit{ab initio} from the non-adiabatic quantum mechanics of Aharonov and Anandan [4] by using the semiclassical wavefunctions associated with moving tori (see, for example, [5]).

Our first example is the precession of a spin \( J_r \) (with unit direction \( r \)) according to the law

\[
\dot{r} = \omega \wedge r.
\]

(13)
The phase space is a sphere of radius \( J \), and the flow is a rigid rotation with instantaneous angular velocity \( \omega \). This is a Hamiltonian system whose canonical variables \( q, p \) are azimuthal polar angle relative to a fixed direction \( \hat{z} \) (coordinate) and \( J_z \) (momentum); the Hamiltonian is

\[
H = \omega(t) \cdot J = J\omega(t) \cdot r.
\]

(14)
The action contours are chosen to be circles of colatitude \( \alpha \) (imagined as painted on the sphere) with direction \( a \) (called polar) as axis (figure 3). We define the action \( I \)
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1/27 times the area of the antipolar spherical cap bounded by the contour, i.e.

$$I = J(1 + \cos \alpha) = J(1 + a \cdot r).$$

(15)

Let $\omega(t)$ be such as to take $a$ on a closed circuit, thereby fulfilling the conditions of our general analysis. If in addition $\omega \cdot a = \text{constant} = \omega \cos \chi$, then (13) can be shown to model the free motion of a spinning top (the sphere) whose axle $a$ is forcibly cycled. (Two special cases are: $\omega$ parallel to $a$ and changed slowly (adiabatic); and $\omega$ = constant (simple precession).)

From (5) the dynamical angle shift is

$$\Delta \theta_d = J \frac{\partial}{\partial I} \int_0^T dt \omega \cdot r = J \frac{\partial}{\partial I} \int_0^T dt \omega \cdot a \cdot r \cdot a = \omega T \frac{\partial}{\partial I} (I - J) \cos \chi = \omega T \cos \chi.$$

(16)

The geometric angle shift is the solid angle $\Omega$ swept out by the axis $a$. This was anticipated by a physical argument [1] and derived elsewhere [6,7]. Here we obtain it from the formula (10) with the area $A(\theta, I)$ built up from individually torus-averaged elements, that is

$$\Delta \theta_g = -\int \frac{\partial}{\partial I} (dA).$$

(17)

For $dA$ it is sufficient to consider the triangle $r, r + b \wedge r d\theta_b, r + c \wedge r d\theta_c$, where $b \, d\theta_b$ and $c \, d\theta_c$ are two infinitesimal rigid rotations of a painted action circle. The torus-averaged area ($J$ times solid angle) of the triangle (correctly signed) is

$$(dA) = \frac{1}{2} d\theta_b d\theta_c \int_{\text{sphere}} d^2 r (b \wedge r) \wedge (c \wedge r) \cdot (-r) \delta(a \cdot r + 1 - I/J)/2\pi$$

$$= -\frac{1}{2} d\theta_b d\theta_c [-a \cdot (b \wedge c)(I-J)]$$

$$= -\frac{1}{2} d\theta_b d\theta_c (b \wedge a) \wedge (c \wedge a) \cdot a(I-J).$$

(18)

Thus $-\partial(dA)/\partial I = d\Omega$ and $\Delta \theta_g = \Omega$, as claimed.

For simple precession ($\omega = \text{constant} = \omega \hat{z}$), $T = 2\pi/\omega$ gives a cyclic evolution and $\Omega = 2\pi(1 - \cos \chi)$, so $\Delta \theta = 2\pi$, reflecting the fact that the tori have been rigidly rotated about $a$, leaving points in their original positions. The quantum version of this particular case is a slight generalisation of one considered by Aharonov and Anandan [4]. We have $J = \hbar(j(j+1))^{1/2}$ ($2j$ integer) and, for an arbitrary initial state, the following evolution generated by (14):

$$|\psi(t)\rangle = \sum_{m=-j}^j a_m \exp(-im\omega t)|m\rangle$$

(19)

where $|m\rangle$ is the eigenstate with $\langle m|J_x|m\rangle = m\hbar$. This is also cyclic for $T = 2\pi/\omega$, with total phase shift $\gamma = 2\pi j$ (up to $2\pi$), and hence a geometric phase $\gamma_g$ given in terms
of the dynamical phase $\gamma_\text{d}$ by

$$\gamma_\text{d} = 2\pi j - \gamma_\text{d} = 2\pi j + \frac{1}{\hbar} \int_0^{2\pi/\Omega} \text{d}t \langle \psi|H|\psi \rangle$$

$$= 2\pi \left( j + \sum_{m=-j}^{j} m|a_m|^2 \right) = 2\pi (j + \langle \psi|J|\psi \rangle/\hbar).$$  \hspace{1cm} (20)

Corresponding to the torus $I$ is an eigenstate of the component of $J$ along $a$, with eigenvalue $I - j\hbar$ (= integer $\times \hbar/2$), so that the expectation value in (20) is $(I = j\hbar) \cos \chi$. Thus

$$\gamma_\text{d} = 2\pi [j + (I/\hbar - j) \cos \chi] \to 2\pi [J + (I - j) \cos \chi]/\hbar \text{ as } j \to \infty.$$  \hspace{1cm} (21)

It can be shown that $2\pi [J + (I - j) \cos \chi]$ is the torus average of the signed areas swept out by $\theta$ points during the cycle, i.e. $\langle A(\theta, I) \rangle$ in (10), so that the semiclassical relation (12) is confirmed.

In our second example the tori are rotating ellipses in phase space. The initial tori, with area $2\pi I$, can be written

$$I = (aq^2 + 2bqp + cp^2)/2(ac - b^2)^{1/2}$$  \hspace{1cm} (22)

with $a, b, c$ constant and satisfying $ac > b^2$. The Hamiltonian

$$H = \omega(p^2 + q^2)/2$$  \hspace{1cm} (23)

makes them rotate rigidly and non-adiabatically in $T = 2\pi/\omega$.

We find $\Delta \theta_\text{g}$ and $\Delta \theta_\text{d}$ by attaching values of $\theta$ to points moving rigidly with the tori (i.e. in circles). The transformation (7) gives, with (22),

$$q = (2Ic)^{1/2}(ac - b^2)^{-1/4} \cos \theta$$

$$p = (2I/c)^{1/2}[-b(ac - b)^{-1/4} \cos \theta + (ac - b^2)^{1/4} \sin \theta]$$  \hspace{1cm} (24)

whence averaging over $\theta$ gives

$$\langle \partial \mathcal{H}/\partial I \rangle = \frac{\omega}{2} \frac{\partial}{\partial I} \langle p^2 + q^2 \rangle = \frac{\omega(a + c)}{2(ac - b^2)^{1/2}}.$$  \hspace{1cm} (25)

Thus the dynamical angle shift (5) is

$$\Delta \theta_\text{d} = \pi(a + c)/(ac - b^2)^{1/2}.$$  \hspace{1cm} (26)

The geometrical shift (10) involves the areas $A(\theta, I)$ (figure 2), in this case circles whose torus average is

$$\langle A(\theta, I) \rangle = \pi \langle p^2 + q^2 \rangle = \pi(a + c)I/(ac - b^2)^{1/2}$$  \hspace{1cm} (27)

so that

$$\Delta \theta_\text{g} = -\pi(a + c)/(ac - b^2)^{1/2}.$$  \hspace{1cm} (28)

Note first that $\Delta \theta_\text{g}$ is of course the same as that calculated elsewhere [1] for an adiabatic rotation (and shown to be equal to $-\pi(a + a^{-1})$ where $a$ is the axis ratio of the ellipses) and second that $\Delta \theta_\text{d}$ and $\Delta \theta_\text{g}$ cancel exactly for this rigid rotation which, as with simple precession, leaves phase points back where they started.
Note added in proof. Anandan [8] has a similar argument to ours.

References