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RANDOM RENORMALIZATION IN THE SEMICLASSICAL LONG-TIME LIMIT OF A PRECESSING SPIN

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Written to celebrate the 60th birthday of Joseph Ford, a wise friend

Discord between the semiclassical and long-time limits is illustrated by the trace of the propagator for a particle of spin $J^2 = \hbar^2 j(j+1)$ with Hamiltonian $J_z^2 \times$ constant. The trace can be expressed in terms of $S_j(t/j)$ where t is a dimensionless propagation time and $S_j(\tau) \equiv \sum_{m=1}^{j} \exp\{i\pi\tau m^2\}$. The long-time limit is $t \to \infty$ (j fixed), and the semiclassical limit is $j \to \infty$ (t fixed); in these limits the trace can be expressed as (different) finite sums generating patterns in the complex plane which although intricate are limited in complexity. But in the combined limit $j \to \infty$, $t \to \infty$ ($\tau \equiv t/j$ fixed) the pattern of $S_j(\tau)$ for typical τ is an infinite hierarchy of spirals with infinitely many scalings related by a chaotic renormalization map; the pattern is infinitely sensitive to the arithmetic of τ.

1. Introduction

The remarkably complicated behaviour of the sum

$$S_j(\tau) \equiv \sum_{m=1}^{j} \exp\{i\pi\tau m^2\} \tag{1}$$

as j increases with τ held fixed, has been discussed by several authors [1–4]. Here I will explain how this behaviour illuminates the well-known fact (see e.g. [5–7]) that for most quantum problems the semiclassical and long-time limits are discordant, in the sense that the result depends on the order in which they are taken. What will be shown is that all the complexity of $S_j(\tau)$ – including randomness – appears when the two limits are taken simultaneously, in a problem whose classical mechanics is trivial.

The classical problem concerns a spin $\boldsymbol{J}$ driven by a Hamiltonian $H(\boldsymbol{J})$, so that the evolution equation is

$$\dot{\boldsymbol{J}} = \nabla_{\boldsymbol{J}} H \times \boldsymbol{J}. \tag{2}$$

This conserves the length J of $\boldsymbol{J}$, and the sphere with radius J is the phase space for the motion, which is that of a system with one freedom whose canonical variables are the azimuthal angle ϕ (coordinate) about the z axis, and J_z (momentum). If H does not depend on time, it too is conserved; thus the spin moves around the energy contours on the J sphere, and the system is trivially integrable. Attention will be restricted to the simplest nonlinear Hamiltonian of this type, namely

$$H = J_z^2/2I, \tag{3}$$

where I is the (constant) moment of inertia of the spin (this Hamiltonian describes, for example, the 26000 years precession of the Earth's rotation axis in the Moon's gravity, or that of a crystal nuclear spin in an electric field). From (2), $\boldsymbol{J}$ precesses round circles of latitude, with frequency $\omega = J_z/I$.

All orbits are therefore periodic, and the shortest period, corresponding to $J_z = \pm J$, defines a characteristic time of the system:

$$T_{\min} = 2\pi I/J. \tag{4}$$

In quantum mechanics both J and J_z are quantized:

$$J^2 = \hbar^2 j(j+1) \quad (0 \le j \le \infty)$$

and (5)

$$J_z = m\hbar \quad (-j \le m \le j).$$

With the Hamiltonian (3) the energy eigenvalues are $E_m = m^2\hbar^2/2I$, and in the basis of m-states the operator determining the evolution of a quantum state for time T is

$$\langle m|\mathscr{U}|m'\rangle = \exp\{-\mathrm{i}E_m T/\hbar\}\,\delta_{mm'} = \exp\{-i\hbar T m^2/2I\}\,\delta_{mm'}. \tag{6}$$

The simplest characterization of this propagator, and hence of the quantum evolution, is its trace, that is

$$\mathrm{Tr}\,\mathscr{U} = \sum_{m=-j}^{j} \exp\{-\mathrm{i}\hbar T m^2/2I\}. \tag{7}$$

If we measure time in units of $T_{\min}$ (eq. (4)), that is in terms of

$$t \equiv T/T_{\min}, \tag{8}$$

and substitute the quantum number j for $\hbar$ from (5), $\mathrm{Tr}\,\mathscr{U}$ can be written as

$$\mathrm{Tr}\,\mathscr{U} = 1 + 2S_j^*(\tau), \tag{9}$$

which has the same form as (1) with

$$\tau \equiv t/\sqrt{j(j+1)} \approx t/j. \tag{10}$$

In principle $\mathrm{Tr}\,\mathscr{U}$ could be measured. One way is to start with a beam of spin-j particles (for example nuclei with high spin), all in the same (arbitrary) state $|m_0\rangle$, and divide it into $2j+1$ identical beams, still in the state $|m_0\rangle$. Then transform the mth beam from $|m_0\rangle$ to $|m\rangle$ (there exists a unitary operator – different for each $|m_0\rangle$ – which does this); this gives beams in states $|-j\rangle, |-j+1\rangle \dots |m\rangle \dots |j\rangle$. Let this collection of coherent beams propagate for time T in the Hamiltonian H, so that the mth beam acquires the phase factor in (6). Now transform each of the phase-shifted beams back into the original state $|m_0\rangle$ (another unitary operation) and recombine them. The amplitude of the recombined beam will be proportional to $\mathrm{Tr}\,\mathscr{U}$. (A different way to realize $S_j(\tau)$ is with optical diffraction gratings [1], and experiments of this type are now being carried out.)

The behaviour of (1) as $j \to \infty$ also appears [8] to control the quantum-classical discordance in the long-time behaviour of the kicked plane rotator (which, unlike a three-dimensional spin, can have any value m of angular momentum quantum number). In the same context, the statistics of the phases in (1) (that is $\pi\tau m^2 \bmod 2\pi$) has been investigated [9].

$S_j(\tau)$ (and also $\mathrm{Tr}\,\mathscr{U}$) is a complex number which it is helpful to regard as the sum of j unit vectors in the complex plane. Now we will discuss the geometry of the sum in three limits.

2. Long-time limit

This is $t \to \infty$ with $\hbar$ (and hence j) fixed. Therefore the number of unit vectors j in $S_j(\tau)$ is constant, and the parameter τ increases (eq. (10)). But $S_j(\tau)$ is periodic in τ, with period 2, so that $\mathrm{Tr}\,\mathscr{U}$ is periodic in T with recurrence time $2jT_{\min}$. Thus any quantum state recurs, in contrast with classical phase-space distributions which need not (each individual orbit is periodic, but different orbits have different periods). If j is large (semi-

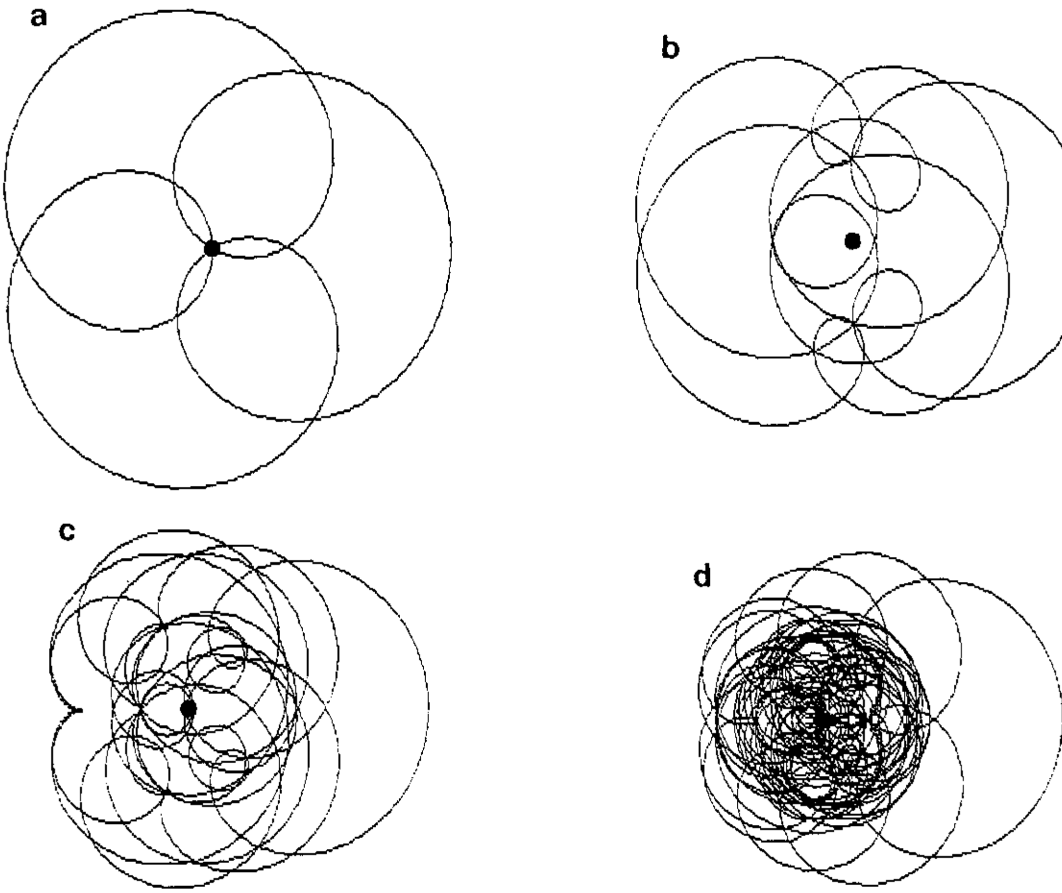


Fig. 1. Trace of the spin-j propagator over a quantum period, i.e. the complex number representing the sum $S_j(\tau)$ (eq. (1)) for $0 \le \tau \le 2$, for (a) $j=2$; (b) $j=3$; (c) $j=5$; (d) $j=10$. The rightmost points have $\tau=0$ and the points $\tau=1$ are indicated by dots.

classical conditions), the quantum period is large (for the Earth's rotation, $j \sim 10^{61}$, so the quantum recurrence time for precession is about $2jT_{\min} \sim 10^{66}$ years).

Evidently we need only study $\mathrm{Tr}\,\mathscr{U}$ over a quantum period, that is for $0 \le \tau \le 2$. When $\tau = 0$, all j vectors in S_j are in line, and $S_j(0)=j$. As τ increases, the vectors in (1) rotate at different speeds and the end-point $S_j(\tau)$ of their sum winds up. The windings continue until $\tau=1$, when $S_j(1)=0$ if j is even and 1 if j is odd. Thereafter the sum unwinds until $\tau=2$ when $S_j(2)=j$.

Some of these windings are shown in fig. 1. They are surprisingly intricate, even for $j=3$, and the intricacy increases rapidly with j (that is, semiclassically). Nevertheless, the complexity of each pattern is strictly limited by the fixed value of j.

3. Semiclassical limit

This is $j \to \infty$ (i.e. $\hbar \to 0$ with J constant) with t fixed. Therefore the number of unit vectors j in $S_j(\tau)$ increases, and by (10) the parameter τ decreases proportionately. The crudest approximation is then to regard m as a continuous variable, so that $S_j(t/j)$ becomes the convergent integral

$$S_j(t/j) \approx \int_0^\infty \mathrm{d}m \exp\{\mathrm{i}\pi t m^2/j\} = \tfrac{1}{2}\mathrm{e}^{\mathrm{i}\pi/4}(j/t)^{1/2}. \quad (11)$$

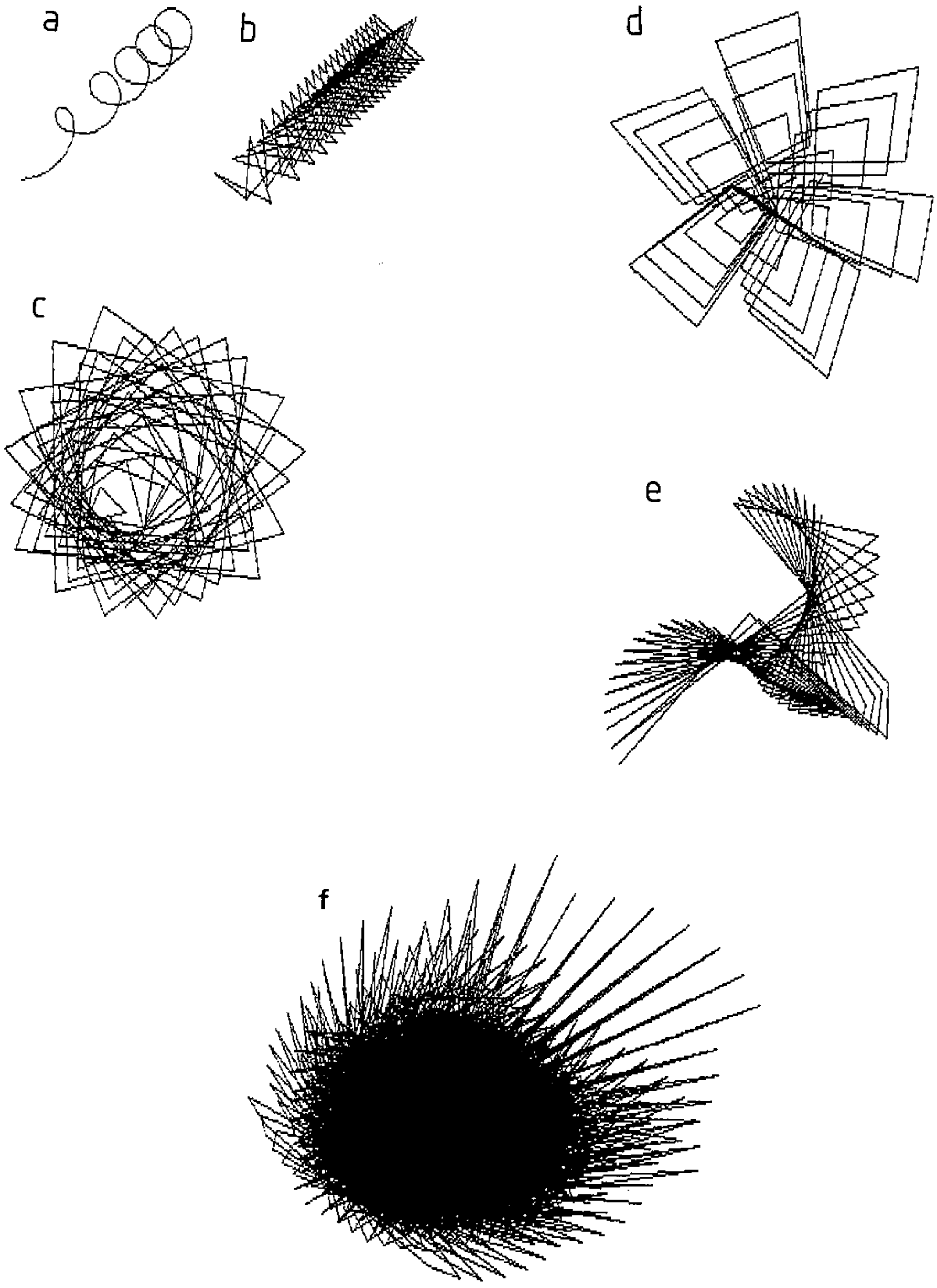
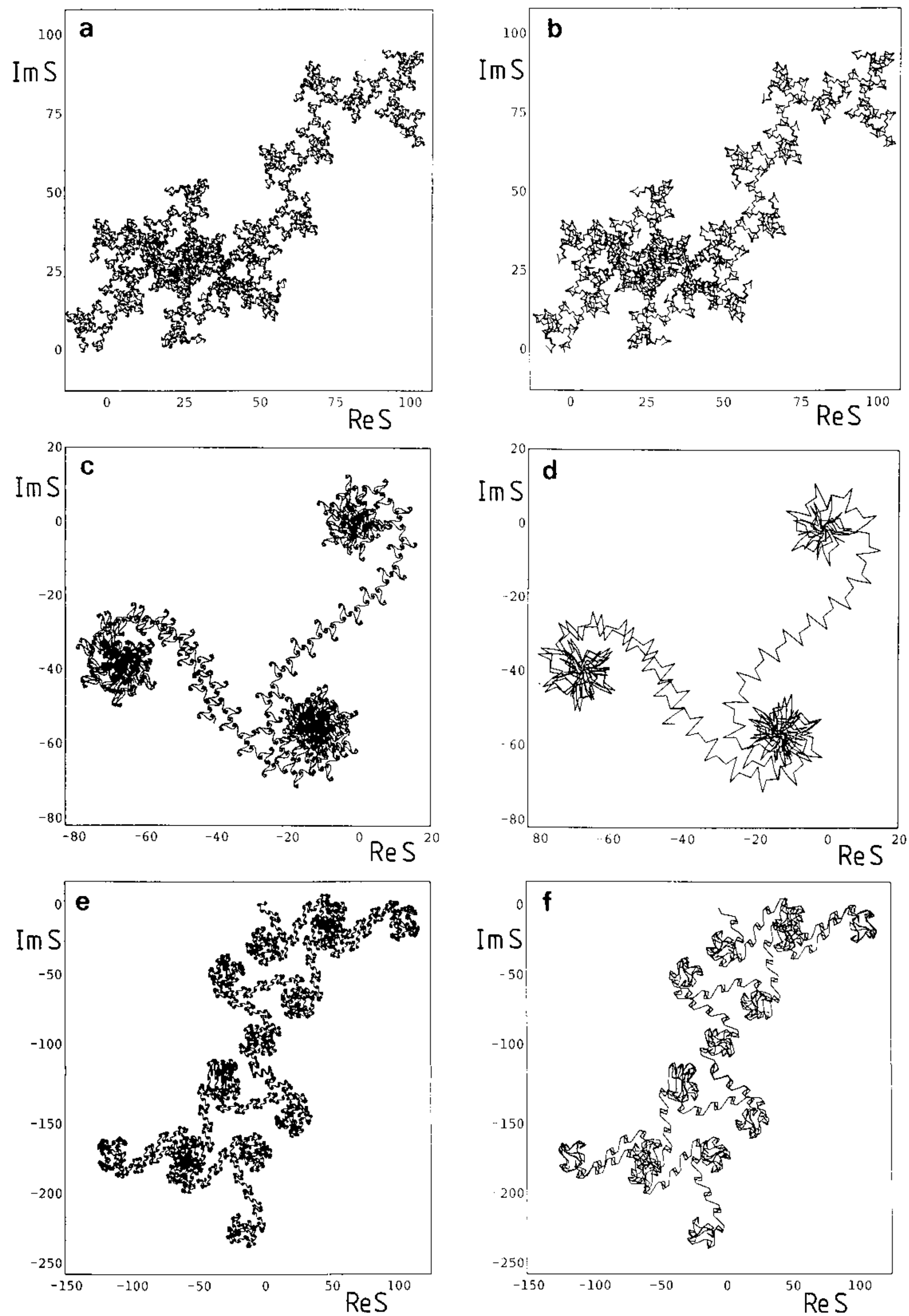


Fig. 2. Semiclassical spin propagator, i.e. the complex number representing $S_j(t/j)$ for $1 \leq j \leq 100$, for (a) $t = 0.1$; (b) $t = 0.8$; (c) $t = 1.7$; (d) $t = 2.1$; (e) $t = 3.01$. In (f), $1 \leq j \leq 10000$ and $t = 4\pi$.


a
b
c
d
e
f
Im S
Re S

When combined with (9) this gives the same result as the simplest semiclassical trace formula, which is the phase-space integral

$$\mathrm{Tr}\,\mathscr{U} \approx \frac{1}{2\pi\hbar}\int_0^{2\pi}\mathrm{d}\phi\int_{-J}^{J}\mathrm{d}J_z \exp\{-\mathrm{i}H(J)T/\hbar\}. \tag{12}$$

The prediction from (11) is that the point representing $S_j(t/j)$ recedes along the 45° line as $j \to \infty$. This is, however, only a rough approximation, because the recession is accompanied by fluctuations that are more complicated for larger t. One way to see that these fluctuations must occur is to note that the phase difference between successive terms comprising a given sum (that is, for given j and t) is small only whilst $m < j/t$. The sum of the remaining terms $j/t < m \leq j$ cannot legitimately be approximated by an integral; the final terms $m = j - n$ ($n \ll j$) differ in phase by the constant $2\pi t$, so that the point representing $S_j(t/j)$ ends by cycling polygonally round the point given by (11) rather than converging onto it.

As fig. 2 shows, these effects combine to produce pictures that are surprisingly varied and complicated, especially for larger t. Nevertheless, the complexity is limited. This can be understood in terms of a more comprehensive semiclassical approximation in which corrections to (11) are obtained (see [1]) by Poisson transformation of (1):

$$\mathrm{Tr}\,\mathscr{U} \approx \mathrm{e}^{\mathrm{i}\pi/4}(j/t)^{1/2} \times\left[1 + 2\sum_{n=1}^{\mathrm{Int}(t)} \exp\{\mathrm{i}\pi j n^2/t\}\right]. \tag{13}$$

These corrections can be interpreted [1] as contributions from the classical closed orbits with period T. Semiclassically (i.e. as $j \to \infty$) the terms oscillate rapidly as t varies, and the jumps as t passes each integer n – corresponding to the entry into the sum of the nth closed orbit – have width $\Delta t \approx (n/j)^{1/2}$. But for fixed t the number ($\mathrm{Int}(t)$) of closed orbits is finite (and independent of j), guaranteeing that, as claimed, the complexity of $\mathrm{Tr}\,\mathscr{U}$ is limited – this time by the fixed value of $\mathrm{Int}(t)$.

4. Renormalization limit

This is the combined limit $t \to \infty$ and $j \to \infty$ with $\tau = t/j$ fixed (i.e. T a fixed fraction of the quantum recurrence time $2jT_{\min}$) and $0 \leq \tau \leq 1$. Now the number of terms in both the original series (1) and the closed-orbit expansion (13) increases limitlessly, and we can expect complicated behaviour. And fig. 3 illustrates that the behaviour is indeed complicated: as j increases, the point in the complex plane representing $S_j(\tau)$ traces out a pattern of curlicues [10] forming a hierarchy with ever-increasing scales. The details of the curlicues depend delicately on τ.

It is natural to seek to understand the hierarchy with the technique of renormalization, that is repeatedly transforming $S_j(\tau)$ to a sum of similar form. Of several renormalization schemes, the most efficient is based on the transformation (derived in [1] and accurate in the limit $j \to \infty$)

$$S_j(\tau_0) \approx \frac{\exp\{\mathrm{i}\pi F'(\tau_0)/4\}}{[F(\tau_0)]^{1/2}} \times K^{1+\mathrm{Int}(1/\tau_0)} S_{jF(\tau_0)}(\tau_1(\tau_0)), \tag{14}$$

whose ingredients will now be explained. $F(\tau_0)$ is the truncation factor (fig. 4) by which the number of terms in the transformed sum is reduced relative to that in the original sum, namely

$$F(\tau) = \begin{cases} \tau & \text{if } \tau < 1/2, \\ (1-\tau)(1 + \{1/(1-\tau)\} \bmod 1) & \\ \quad \text{if } \tau > 1/2. \end{cases} \tag{15}$$

Fig. 3. Semiclassical long-time spin propagator, i.e. the complex number representing $S_j(\tau)$, for some typical values of τ: (a) $\tau = 2^{-1/3}$ ($1 \leq j \leq 8000$); (b) renormalization of (a) using (14) (3000 vectors); (c) $\tau = (2000)^{-1/3}$ ($1 \leq j \leq 8000$); (d) renormalization of (c) (634 vectors); (e) $\tau = (20000)^{-1/3}$ ($1 \leq j \leq 40000$); (f) renormalization of (e) (1474 vectors) (these pictures are kindly supplied by J. Goldberg).

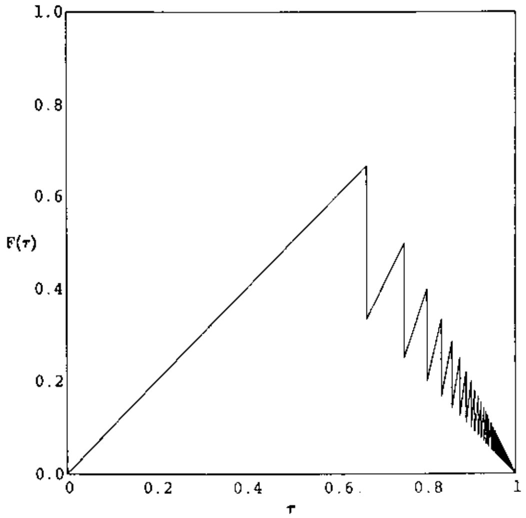


Fig. 4. Truncation factor $F(\tau)$ (eq. 15) in the renormalization (14) (from [1]).

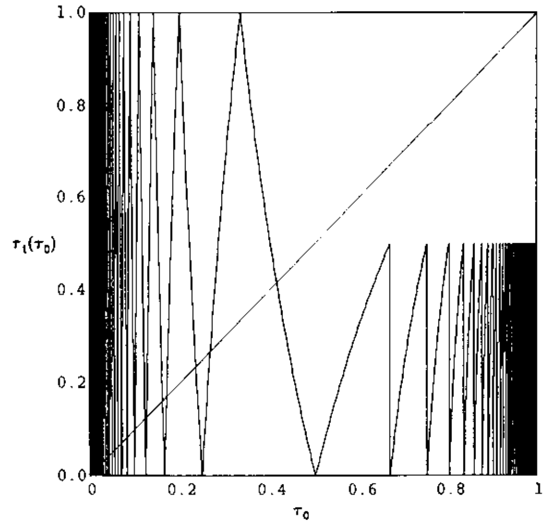


Fig. 5. Renormalization map $\tau_1(\tau_0)$ (eq. (16)) (from [1]). (Fixed points are intersections with the diagonal.)

The terms in the new sum are magnified by $F^{-1/2}$. The derivative $F'(\tau_0)$ appearing in the phase factor in (14) is an integer. The symbol K denotes the operation of complex conjugation applied to what follows it. Finally, the new sum involves a different τ, related to the old by the renormalization map (fig. 5)

$$\tau_1(\tau_0) = \begin{cases} (1/\tau_0) \bmod 1 \\ \quad \text{if Int}(1/\tau_0) \text{ is even,} \\ 1 - (1/\tau_0) \bmod 1 \\ \quad \text{if Int}(1/\tau_0) = 3, 5, 7, \ldots, \\ (\{1/(1-\tau_0)\} \bmod 1)/ \\ (1 + \{1/(1-\tau_0)\} \bmod 1) \\ \quad \text{if Int}(1/\tau_0) = 1. \end{cases} \tag{16}$$

Iteration of the transformation (14) simplifies the sum in a manner that depends on the arithmetic of τ_0. The simplest case is when τ_0 is rational. Then a finite number k of renormalizations with the map (16) leads to $\tau_k = 0$ or $\tau_k = 1$, and hence to a finite hierarchy [1] of curlicues with $S_j(\tau)$ either linearly increasing ("quantum resonance" [11]) or repeatedly retracing a finite pattern, respectively. The next simplest case is when τ_0 is a fixed point of the map (16). Then the asymptotic $S_j(\tau_0)$ has infinitely many scales related by self-similarity with the single scaling τ_0; the curlicues form a fractal (with dimension 2 [1]).

Most interesting, however, is the case of typical τ_0, for which renormalization depends on the ergodic properties of (16). Because this map is everywhere hyperbolic it is both ergodic and

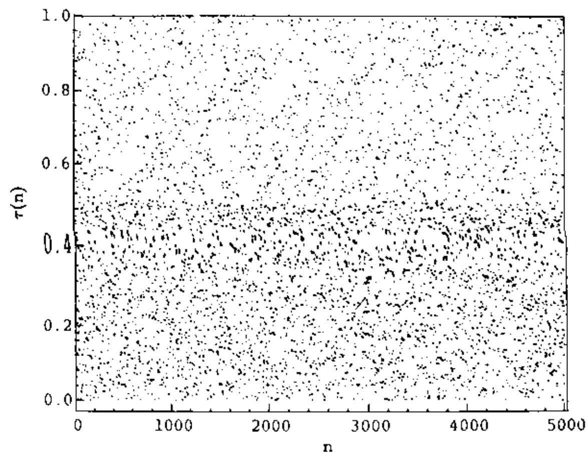


Fig. 6. Ergodicity and chaos in 5000 iterations of the renormalization map (from [1]).

chaotic, as illustrated in fig. 6. Therefore the fractal curlicues for typical τ_0 possess not only infinitely many scales but also infinitely many scalings forming a random cascade of renormalizations. The scalings are distributed over the range $0 < \tau < 1$ according to the invariant measure of the map, which is a simple function of τ [1].

Each renormalization gives a smaller sum with bigger terms. Thus renormalization is a coarsening transformation, removing the finest curlicues at each stage (fig. 3). Repeated renormalization must eventually produce a sum with just one term, no matter how large j is. For almost all τ_0 the number of renormalizations that produces this result can be shown [1] to be 0.653 ln j. (The fact that a sum with j terms can be comprehended with ln j operations must reflect the null algorithmic complexity [12] of both quantum and classical evolutions for this spin problem.) Of course a sum with one term is trivial, so that this "renormalization to the finish" is a powerful tool [1, 4] for estimating the size of the original sum as the product of the accumulated magnifications; the result is that $|S_j(\tau)| \approx j^{1/2}$ for almost all τ.

5. Discussion

The semiclassical and long-time limits can be regarded as approaches to the origin of the plane

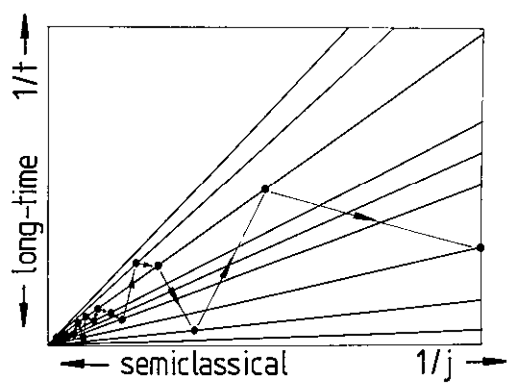


Fig. 7. The plane of asymptotic parameters j^{-1}, t^{-1}, with dots illustrating the history of the random renormalization of the propagator, starting near the origin which is a powerful singularity.

with coordinates j^{-1}, t^{-1} (fig. 7), along the j^{-1} and t^{-1} axes respectively. In both these extreme cases the limiting propagator can be expressed as a finite sum. But these two limits are special; more general is the renormalization limit, in which the origin is approached along a line with slope τ. Then the chaos in the renormalization reveals the origin to be a dragon's lair, so singular that the propagator is infinitely sensitive to the slope of approach. A generic point close to the origin renormalizes outwards along directions that change randomly.

One might call this behaviour "quantum chaos", but that would merely add provocation without illumination.

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