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Uniform asymptotic smoothing of Stokes's discontinuities

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Across a Stokes line, where one exponential in an asymptotic expansion maximally dominates another, the multiplier of the small exponential changes rapidly. If the expansion is truncated near its least term the change is not discontinuous but smooth and moreover universal in form. In terms of the singulant F – the difference between the larger and smaller exponents, and real on the Stokes line – the change in the multiplier is the error function

$$\pi^{-\frac{1}{2}}\int_{-\infty}^{\sigma} \mathrm{d}t \exp(-t^2) \quad \text{where} \quad \sigma = \operatorname{Im} F/(2 \operatorname{Re} F)^{\frac{1}{2}}.$$

The derivation requires control of exponentially small terms in the dominant series; this is achieved with Dingle's method of Borel summation of late terms, starting with the least term. In numerical illustrations the multiplier is extracted from Dawson's integral (erfi) and the Airy function of the second kind (Bi): the small exponential emerges in the predicted universal manner from the dominant one, which can be 10^{10} times larger.

1. INTRODUCTION

Stokes's phenomenon concerns the behaviour of small exponentials while hidden by large ones (Stokes 1864, 1871, 1889, 1902). Such exponentials occur commonly in the asymptotic approximation of functions $y(X;k)$ defined by integrals or differential equations and dependent on a large parameter k and variables $X = (X_1, X_2, \ldots)$. In the simplest case there are just two exponentials, and the lowest-order approximation incorporating both can be written

$$y(X;k) \approx M_+(X;k)\exp\{k\phi_+(X)\} + \mathrm{i}S(X;k)M_-(X;k)\exp\{k\phi_-(X)\} \qquad (\operatorname{Re}\phi_+(X) > \operatorname{Re}\phi_-(X)), \quad (1)$$

where the dominant and subdominant contributions are labelled + and −, and the factor i is inserted for later convenience. The prefactors $M_\pm$ vary slowly with X and their k-dependences are simple powers. Attention will here focus on the *Stokes multiplier function* $S(X;k)$ weighting the subdominant exponential; this varies rapidly when X is near the *Stokes line* of y, where

$$\operatorname{Im}[\phi_+(X) - \phi_-(X)] = 0. \quad (2)$$

On the Stokes line (a set of codimension 1) it can be said that the dominance of + over − is maximal.

The need to retain the subdominant term, even though it is numerically insignificant in (1) as $k \to \infty$, and the need for Stokes's multiplier, both spring from a common cause; maintaining the validity of (1) when X crosses *anti*-Stokes lines (far from Stokes's lines) on which $\mathrm{Re}\,[\phi_+ - \phi_-] = 0$ and the exponential previously called $-$ becomes the dominant one. (Readers are warned that some authors employ the term Stokes line to denote what we here call an anti-Stokes line, and vice versa.)

A simple illustrative example, to which we shall later return, due to Stokes (1864) and well described by Dingle (1973, hereinafter called I) is the complex error function

$$\left.\begin{aligned} y(X;k) &= \int_{-\mathrm{i}\infty}^{Z} \mathrm{d}t \exp(kt^2), \\ (Z &= X_1 + \mathrm{i}X_2). \end{aligned}\right\} \tag{3}$$

Near the positive real Z axis the dominant contribution to y as $k \to \infty$ comes from the end-point of integration $t = Z$;

$$y \sim (2kZ)^{-1} \exp(kZ^2) \quad (Z \text{ positive real}). \tag{4}$$

Thus $\phi_+ = Z^2$ and $M_+ = (2kZ)^{-1}$. Near the positive imaginary axis this would predict that y is exponentially small, which is false because the integral is then dominated by the stationary point at $t = 0$, giving

$$y \sim \mathrm{i}(\pi/k)^{\frac{1}{2}} \quad (Z \text{ positive imaginary}). \tag{5}$$

This would suggest the asymptotics

$$y \sim (2kZ)^{-1} \exp(kz^2) + \mathrm{i}(\pi/k)^{\frac{1}{2}} \tag{6}$$

involving $\phi_- = 0$ and $M_- = (\pi/k)^{\frac{1}{2}}$. Thus the Stokes line (2) is the real axis $X_2 = 0$, and the anti-Stokes lines, where dominance is exchanged, are the diagonals $X_1 = \pm X_2$. But (6) fails near the *negative* imaginary axis because it would predict $y \sim \mathrm{i}(\pi/k)^{\frac{1}{2}}$ whereas it is obvious from (3) that y is exponentially small and given by the continuation of (4). To encompass the three regions discussed, we must write

$$y \sim (2kZ)^{-1} \exp(kZ^2) + \mathrm{i}S(Z;k)\,(\pi/k)^{\frac{1}{2}}, \tag{7}$$

incorporating Stokes's multiplier S which must change from 0 to 1 between the anti-Stokes lines $X_2 = -X_1$ and $X_2 = +X_1$. This change in S is Stokes's phenomenon.

The conventional view (Stokes 1864) is powerfully (and unconventionally) argued by Dingle in I. It asserts that the change in S is discontinuous and localized at the Stokes line: on one side, S takes a value, S_- say; on the other, $S = S_- + 1$; on the line itself, $S = S_- + \frac{1}{2}$. For the example (3) the intuition behind this view is illustrated by figure 1, which shows how the steepest-descent contours of the integral ($\mathrm{Im}\, t^2 = \mathrm{Im}\, Z^2$) change discontinuously across the Stokes line, suddenly bringing in the subdominant contribution from the stationary point at $t = 0$

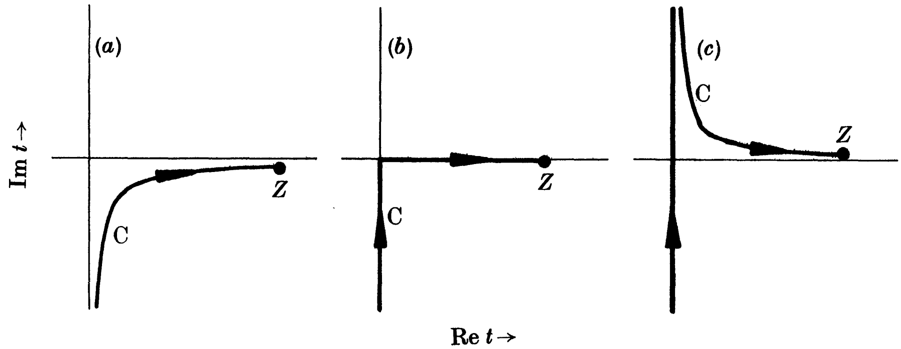

FIGURE 1. Steepest-descent contours of (3) from $t = -i\infty$ to $t = Z = X_1 + iX_2$, for three values of Z near the Stokes line $X_2 = 0$.

($S_- = 0$ in this case). It is worth repeating Stokes's description of the asymptotic emergence of his discontinuity. As a Stokes line is crossed,

> ...the inferior term enters as it were into a mist, is hidden for a little from view, and comes out with its coefficient changed. The range during which the inferior term remains in a mist decreases indefinitely as [the asymptotic parameter] increases indefinitely. (Stokes 1902)

From the context it is clear that Stokes is referring to asymptotic series interpreted by truncation near their least term. My aim here is to dispel Stokes's mist and show that his discontinuity is an artefact of poor resolution: with the appropriate magnification, S changes smoothly. Moreover, with the appropriate variable to describe the crossing of Stokes's line the change in the function $S(X;k)$ is *universal*, that is, the same for all problems in a wide class.

Obtaining this result requires control of the magnitude of the dominant exponential contribution with error small compared to the size of the subdominant exponential. Such control will be achieved by analysing the dominant asymptotic series (in descending powers of k) of which only the first term is included in (1). The series is

$$\left.\begin{aligned} &y(X;k) = M_+ \exp(k\phi_+) \sum_{r=1}^{\infty} a_r, \\ &(a_0 = 1; \quad a_r \propto k^{-r}). \end{aligned}\right\} \tag{8}$$

This diverges and so is numerically meaningless, but Dingle (I) explains how it can nevertheless be regarded as a coded representation of y, which can be reconstructed exactly (in principle and sometimes in practice) by proper interpretation of the late terms $r \gg 1$. The interpretation reveals how the subdominant exponential (together with S) originates from the divergence of the late terms. My derivation of the leading-order functional form of S across a Stokes line is a simple development within Dingle's interpretative scheme.

2. Derivation of the Stokes multiplier function

The obstruction preventing the series in (8) from converging is the existence of the subdominant exponential. (In the example (3) the series appended to (4) is obtained by an expansion about the integration limit $t = Z$, and divergence originates in the stationary point at $t = 0$.) This subdominant exponential engenders in the late terms a_r a remarkable universality, best expressed in terms of the complex quantity

$$F \equiv k(\phi_+ - \phi_-). \tag{9}$$

Dingle calls this the *singulant*; on a Stokes line, it is real and positive. He shows that

$$a_r \rightarrow \frac{M_-(r-\beta)!}{2\pi M_+ F^{r-\beta+1}} \quad \text{as} \quad r \rightarrow \infty. \tag{10}$$

For example, if y is defined by an integral for which (as in the example (3)) the dominant exponential comes from a limit of integration and the subdominant one from a stationary point, then $\beta = \frac{1}{2}$ (I, pp. 111, 145). If y is defined by an integral for which both exponentials are associated with stationary points, then $\beta = 1$ (I, pp. 135, 145). If y is a solution of a second-order differential equation, say

$$\partial_z^2 y = k^2 Q(Z)\, y \tag{11}$$

with $\operatorname{Re} Q > 0$ in the region of $Z = X_1 + \mathrm{i}X_2$ being studied, for which the exponentials come from the two primitive phase-integral (JWKB) approximations (giving

$$M_\pm = Q^{-\frac{1}{4}} \quad \text{and} \quad \phi_\pm = \pm \int_a^Z Q^{\frac{1}{2}}\, \mathrm{d}z'$$

where a is a simple zero of $Q^2(Z)$), then again $\beta = 1$ (I, p. 299) (for an nth-order zero, (10) is multiplied by $2\cos(\pi/(n+2))$).

To interpret (8) Dingle employs Borel summation, not for the whole series as is customary, but for the nth term and beyond, where n is close to the value $r \sim |F|$ for which a_r is least. Assuming this formal procedure is valid, we obtain, by using (10),

$$y \approx M_+ \exp(k\phi_+) \sum_{r=0}^{n-1} a_r + \mathrm{i} M_- S_n(F) \exp(k\phi_-), \tag{12}$$

where

$$S_n(F) \equiv \frac{-\mathrm{i}}{2\pi} \exp(F) \sum_{r=n}^{\infty} \frac{(r-\beta)!}{F^{r-\beta+1}}. \tag{13}$$

The interpretation is obtained by writing the factorial as an integral and performing the summation

$$\begin{aligned} S_n(F) &= \frac{-\mathrm{i}\exp(F)}{2\pi F^{1-\beta}} \int_0^\infty \mathrm{d}S \exp(-S)\, S^{-\beta} \sum_{r=n}^{\infty} \left(\frac{S}{F}\right)^r \\ &= \frac{-\mathrm{i}}{2\pi} \int_0^\infty \mathrm{d}t \frac{t^{n-\beta} \exp\{F(1-t)\}}{1-t}. \end{aligned} \tag{14}$$

To complete the interpretation it is necessary to specify the t-contour relative to the pole at $t = 1$. This corresponds to specifying the contour of integration (as will be illustrated later), or the desired solution of a differential equation, when defining y. Different choices differ by real constants, corresponding to the value of the quantity S_- in section 1. We specify that the contour passes above $t = 1$, so that

$$S_n(F) = \tfrac{1}{2} - \frac{\mathrm{i}}{2\pi} \int_0^\infty \mathrm{d}t \frac{t^{n-\beta} \exp\{F(1-t)\}}{1-t}, \tag{15}$$

where now the principal value of the integral is taken. This choice corresponds to the situation in example (3), where the Stokes multiplier switches on from zero ($S_- = 0$) as $\operatorname{Im} F$ increases through zero.

Now we identify S_n as the Stokes multiplier and determine its dominant asymptotics when F is large and nearly real. A crucial simplification (corresponding to the evaluative interpretation of asymptotic series adopted by Stokes) occurs if we truncate the r-sum near its least term, that is at

$$n-1 = \operatorname{Int}(|F| + \alpha), \tag{16}$$

where α is of order unity (as is β). With this truncation, the stationary point of (15) almost coincides with its pole $t = 1$, whose neighbourhood therefore dominates the integral. Let

$$F \equiv A + \mathrm{i}B \tag{17}$$

(where A and B are real with $A \gg 1$ and $B \ll A$) and

$$n - \beta \equiv A + \mu, \quad \text{i.e.} \quad \mu = \operatorname{Int}(|F| + \alpha) - \beta - A + 1, \tag{18}$$

(so μ is of order unity) and change variables to $x \equiv t-1$. Then expanding the integrand in (15) to third order about $x = 0$ gives

$$\begin{aligned} S_n(F) &= \tfrac{1}{2} + \frac{\mathrm{i}}{2\pi} \int_{-1}^{\infty} \frac{\mathrm{d}x}{x} \exp\{(A+\mu)\ln(1+x) - Ax - \mathrm{i}Bx\} \\ &\approx \tfrac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x (1 + \mu x + \tfrac{1}{3}Ax^3) \exp(-\tfrac{1}{2}Ax^2)(\sin Bx + \mathrm{i}\cos Bx)/x \\ &= \frac{1}{2}\left(1 + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x} \exp(-\tfrac{1}{2}Ax^2) \sin Bx\right) \\ &\quad + \frac{\mathrm{i}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \exp(-\tfrac{1}{2}Ax^2)(\mu + \tfrac{1}{3}Ax^2)\cos Bx \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{B/(2A)^{\frac{1}{2}}} \mathrm{d}t \exp(-t^2) - \mathrm{i}(2\pi A)^{-\frac{1}{2}} (\operatorname{Fract}\{|F| + \alpha\} \\ &\quad + \beta - \alpha - \tfrac{4}{3} - B^2/6A) \times \exp(-B^2/2A). \end{aligned} \tag{19}$$

The real part dominates, and comparison with (1) and (12) yields the change in the Stokes multiplier as

$$S(\sigma) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\sigma} \mathrm{d}t \exp(-t^2) \tag{20}$$

involving the *Stokes variable*

$$\sigma(X;k) = B/(2A)^{\frac{1}{2}} = \mathrm{Im}\,F/(2\,\mathrm{Re}\,F)^{\frac{1}{2}}$$
$$= k^{\frac{1}{2}}\,\mathrm{Im}\,(\phi_+ - \phi_-)/\{2\,\mathrm{Re}\,(\phi_+ - \phi_-)\}^{\frac{1}{2}}. \tag{21}$$

Equations (20) and (21) carry the following implication: under a magnification of order $k^{\frac{1}{2}}$, the multiplier varies smoothly from S_- to S_-+1 across the Stokes line, the functional dependence on the natural variable σ being that of the error function.

According to (19), the imaginary part of $S_n(F)$ is smaller than the subdominant exponential in (12) by a factor $A^{-\frac{1}{2}} \approx |F|^{-\frac{1}{2}} \sim k^{-\frac{1}{2}}$. This gives the assurance that the dominant series has been controlled to better-than-exponential accuracy. Thus to detect the multiplier numerically, as we will do in the next section, it is sufficient to subtract from the exact function y the dominant series taken up to its nearly least term, i.e.

$$\left[y \exp(-k\phi_-) - M_+ \exp(F) \sum_{r=0}^{\mathrm{Int}(|F|+\alpha)} a_r\right] \Big/ M_- \to \mathrm{i}\,(S(\sigma)+S_-) \quad \text{as} \quad |F| \to \infty \tag{22}$$

independently of α provided α is of order unity. This equation embodies our main result.

Any alteration in α changes the number of terms included in the sum (cf. 16), and is compensated by a change in the imaginary part of (19) which is small compared with $S(\sigma)$. Two natural choices for the 'best' α are (i) that which minimizes the imaginary part of (19) on the average, which gives (because the average of Fract$\{x\}$ is $\frac{1}{2}$) $\alpha = \beta - \frac{5}{6}$; (ii) that for which the smallest a_r has $r = n$ or $r = n+1$, which is the case if $\alpha = \beta - \frac{1}{2}$.

The imaginary terms in (19) constitute the lowest-order approximation in the technique of 'terminants' or 'converging factors', which has been employed (see I and Olver (1974)) to correct the dominant series representing y *on* the Stokes line. Here, of course, we are studying the variation of y *across* a Stokes line.

3. Numerical illustrations

According to (22), the Stokes multiplier function, which is of order unity, is the difference of two large quantities (of order $\exp(F)$). To detect S numerically, two conditions must be satisfied: y must be calculable to better-than-exponential precision, and the coefficients a_r in the asymptotic expansion must be known (or calculable) for large r. There follow two examples for which these conditions are met.

The first is *Dawson's integral* (Abramowitz & Stegun 1964)

$$y \equiv \int_0^Z \mathrm{d}t \exp(kt^2) = k^{-\frac{1}{2}} \mathrm{erfi}\,(k^{\frac{1}{2}}Z) \tag{23}$$

which differs by $\mathrm{i}\,(\pi/k)^{\frac{1}{2}}$ from the example (3). We are interested in $Z = X_1 + \mathrm{i}X_2$ near the positive real axis. Obviously y is real on the Stokes line, so that in (7) S must vanish when $X_2 = 0$, implying in turn that in the general expression (14) the principal value must be taken and that $S_- = -\frac{1}{2}$ in (22). Figure 2 shows the

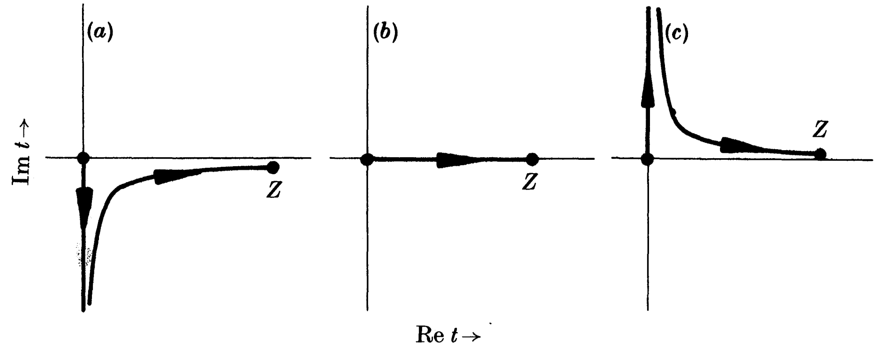

FIGURE 2. Steepest-descent contours of Dawson's integral (23) from $t = 0$ to $t = Z = X_1 + \mathrm{i}X_2$, for three values of Z near the Stokes line $X_2 = 0$.

steepest-descent contours, for comparison with those in figure 1. The singulant is given by (9) and (7) as

$$F = kZ^2 \tag{24}$$

so that the Stokes variable is

$$\sigma = (2k)^{\frac{1}{2}} X_1 X_2/(X_1^2 - X_2^2)^{\frac{1}{2}} \approx (2k)^{\frac{1}{2}} X_2. \tag{25}$$

In the expansion (8) the coefficients a_r can be found by elementary asymptotics, for example (I, p. 5) changing the integration variable in (23) by $kt^2 = kZ^2 + u$ and expanding the jacobian about $u = 0$, with the result

$$a_r = (r-\tfrac{1}{2})!\,F^{-r}/\surd\pi. \tag{26}$$

Thus the limiting form (10) is here exact for all r, and $\beta = \frac{1}{2}$.

Incorporating (24)–(26) into the general result (22) with $S_- = -\frac{1}{2}$ we obtain, as the formula for the asymptotic emergence of the multiplier,

$$\operatorname{Im}\left\{\operatorname{erfi}(F^{\frac{1}{2}})/\surd\pi - (2\pi)^{-1}\exp(F)\sum_{r=0}^{\operatorname{Int}(|F|+\alpha)}(r-\tfrac{1}{2})!/F^{r+\frac{1}{2}}\right\}$$

$$\to \frac{1}{\surd\pi}\int_0^{\sigma} \mathrm{d}t \exp(-t^2) \quad \text{as} \quad |F| \to \infty. \tag{27}$$

We shall here regard σ and $|F|$ as given, and obtain X_1 and X_2 by inversion of (24) and (25):

$$\left.\begin{aligned} k^{\frac{1}{2}}X_1 = |F|^{\frac{1}{2}}\cos\theta; \quad k^{\frac{1}{2}}X_2 &= |F|^{\frac{1}{2}}\sin\theta \\ \text{where} \qquad \theta = \tfrac{1}{2}\arccos\{[1+(\sigma^2/|F|)^2]^{\frac{1}{2}} &- \sigma^2/|F|\}.\end{aligned}\right\} \tag{28}$$

Tables 1–4 show the results of a numerical test of (27) over a range of values of σ, for singulants $|F| = 5$ and $|F| = 25$ and truncation variables $\alpha = -\frac{1}{3}$ and $\alpha = 0$ (these are the two 'best' choices of α described at the end of §2). The integral for erfi was evaluated along a straight-line contour from X_1 to $X_1 + \mathrm{i}X_2$ (the integral from 0 to X_1 being real) by the extended Simpson's rule; sufficient accuracy was achieved with 200 steps for $|F| = 5$ and 2000 steps for $|F| = 25$, and checked by evaluating the convergent series for erfi.

Table 1. Comparison of theory and 'experiment' for the Stokes multiplier function for Dawson's integral (23)

The singulant modulus is $|F| = 5$ and the truncation variable is $\alpha = -\frac{1}{3}$. σ is the Stokes variable (21) and $F = k(X_1 + \mathrm{i}X_2)^2$ with X_1 and X_2 given by (28). (LHS and RHS are left-hand and right-hand sides respectively.)

σ	$2\,\mathrm{Im\,erfi}\,(F^{\frac{1}{2}})/\sqrt{\pi}$	$2 \times$ LHS of (27)	$2 \times$ (LHS − RHS) of (27)
0.2	21.5323356351	0.220432726808	−0.0022770904605
0.4	32.406989074	0.42190599899	−0.0064991774091
0.6	30.3697856073	0.591079840706	−0.012791995848
0.8	21.0291922392	0.722780152631	−0.0193366790011
1.0	11.2241125601	0.818898722383	−0.0238159078063
1.2	4.45519992781	0.88539323385	−0.0249314381838
1.4	0.939754805283	0.929422218948	−0.0228703179355
1.6	−0.43497360601	0.957562519129	−0.018790516202
1.8	−743819764526	0.97506200602	−0.014031146904
2.0	−0.634984356598	0.985737130522	−0.00958651213548
2.2	−0.411962286068	0.992177544661	−0.00596026323016
2.4	−0.190731569089	0.996048680562	−0.00326308993687

Table 2. As table 1 with $\alpha = 0$

σ	$2\,\mathrm{Im\,erfi}\,(F^{\frac{1}{2}})/\sqrt{\pi}$	$2 \times$ LHS of (27)	$2 \times$ (LHS − RHS) of (27)
0.2	21.5323356351	0.242465868574	0.0197560513057
0.4	32.406989074	0.463805332974	0.035400156575
0.6	30.3697856073	0.648329058716	0.0444572221622
0.8	21.0291922392	0.788860592493	0.0467437608605
1.0	11.2241125601	0.886578386719	0.04386375653
1.2	4.45519992781	0.948451996877	0.0381273248433
1.4	0.939754805283	0.983881887394	0.0315893505111
1.6	−0.43497360601	1.00191507031	0.0255620349758
1.8	−0.743819764526	1.00968558697	0.0205924340413
2.0	−0.634984356598	1.01203128278	0.016707640125
2.2	−0.411962286068	1.01184414978	0.0137063418897
2.4	−0.190731569089	1.01067052005	0.0113587495474

Table 3. As table 1 with $|F| = 25$

σ	$2\,\mathrm{Im\,erfi}\,(F^{\frac{1}{2}})/\sqrt{\pi}$	$2 \times$ LHS of (27)	$2 \times$ (LHS − RHS) of (27)
0.2	7833454975.39	0.22225856781	−0.000451249458156
0.4	2631901621.35	0.427119255066	−0.00128592133319
0.6	−4826024257.27	0.600670814514	−0.00320102204009
0.8	−3211113836.27	0.737760543823	−0.00435628780885
1.0	1495138180.67	0.840577602386	−0.00213702780268
1.2	1999214434.28	0.908263206482	−0.0020614655515
1.4	137285256.727	0.946264266968	−0.00602826991531
1.6	−647807326.949	0.970328092575	−0.00602494275588
1.8	−311385089.924	0.987966120243	−0.00112703268075
2.0	42695984.0144	0.99713306129	0.00180941863256
2.2	98642118.0045	0.999141067266	0.00100325937505
2.4	39463275.4458	0.998938336968	−0.000373433530283

TABLE 4. AS TABLE 1 WITH $|F| = 25$ AND $\alpha = 0$

σ	$2\,\mathrm{Im\,erfi}\,(F^{\frac{1}{2}})/\sqrt{\pi}$	$2\times$ LHS of (27)	$2\times$ (LHS $-$ RHS) of (27)
0.2	7833454975.39	0.226700782776	0.00399096550766
0.4	2631901621.35	0.435605049133	0.00719987273419
0.6	−4826024257.27	0.612331390381	0.00845955382661
0.8	−3211113836.27	0.751268863678	0.00915203204589
1.0	1495138180.67	0.854352474213	0.0116378440235
1.2	1999214434.28	0.920836210251	0.0105115382174
1.4	137285256.727	0.956623911858	0.00433137497452
1.6	−647807326.949	0.978073358536	0.00172032320481
1.8	−311385089.924	0.993239462376	0.00414630945182
2.0	42695984.0144	1.00040954351	0.00508590085702
2.2	98642118.0045	1.00099922717	0.00286141927476
2.4	39463275.4458	0.999897457659	0.00058568716054

Successive columns clearly show the decreasing orders of magnitude of the first term on the left-hand side of (27) (i.e. the integral), whose order is $\exp(|F|)/|F|^{\frac{1}{2}}$ (at least for small σ); the whole left-hand side of (27) (i.e. the 'experimental' Stokes multiplier), whose order is unity; and the difference between the two sides of (27), whose order can be shown (by easy extension of the argument leading to (19)) to be $|F|^{-1}$. The two sides of (27) are shown plotted against σ for $|F| = 5$ and 25 in figures $3a, b$, for the two values of α. Evidently the agreement between 'experimental' and 'theoretical' multipliers improves with increasing $|F|$, as it should. The two curves for the 'best' α bracket the theoretical curve.

For the second example we take the *Airy function of the second kind* (Abramowitz & Stegun 1964),

$$y = \int_{\mathrm{C}} \mathrm{d}t \exp\{k(-\tfrac{1}{3}t^3 + tZ)\} = 2\pi k^{-\frac{1}{3}}\,\mathrm{Bi}\,(Zk^{\frac{2}{3}}) \tag{29}$$

where C is the contour shown in figure $4a$. We are interested in $Z = X_1 + \mathrm{i}X_2$ near the positive real axis. The integrand has stationary points at $t_\pm = \pm Z^{\frac{1}{2}}$, so that the dominant and subdominant exponents in (1), and the singulant, are

$$\phi_\pm = \pm\tfrac{2}{3}Z^{\frac{3}{2}}, \quad F = \tfrac{4}{3}kZ^{\frac{3}{2}}, \tag{30}$$

where the roots are positive real when Z is positive real, that is on the Stokes line. Thus the Stokes variable (21) is

$$\sigma = (\tfrac{2}{3}k)^{\frac{1}{2}}\,\mathrm{Im}\,(X_1 + \mathrm{i}X_2)^{\frac{3}{2}}/[\mathrm{Re}\,(X_1 + \mathrm{i}X_2)^{\frac{3}{2}}]^{\frac{1}{2}} \approx (\tfrac{3}{2}k)^{\frac{1}{2}}X_2/X_1^{\frac{1}{4}}. \tag{31}$$

This shows that for fixed k the 'width' of the Stokes zone increases slowly (as $X^{\frac{1}{4}}$) away from the origin.

Figures $4b$–d show the steepest-descent contours for Z nearly positive real. Evidently the contribution of the subdominant exponential reverses across the Stokes line and vanishes on it, as for Dawson's integral, so again $S_- = -\frac{1}{2}$. After taking into account the double contour through the dominant stationary point, the prefactors $M_\pm$ can be obtained from the simplest steepest-descent argument, giving

$$M_+ = 2M_- = 2(\pi^2/k^2Z)^{\frac{1}{4}}. \tag{32}$$

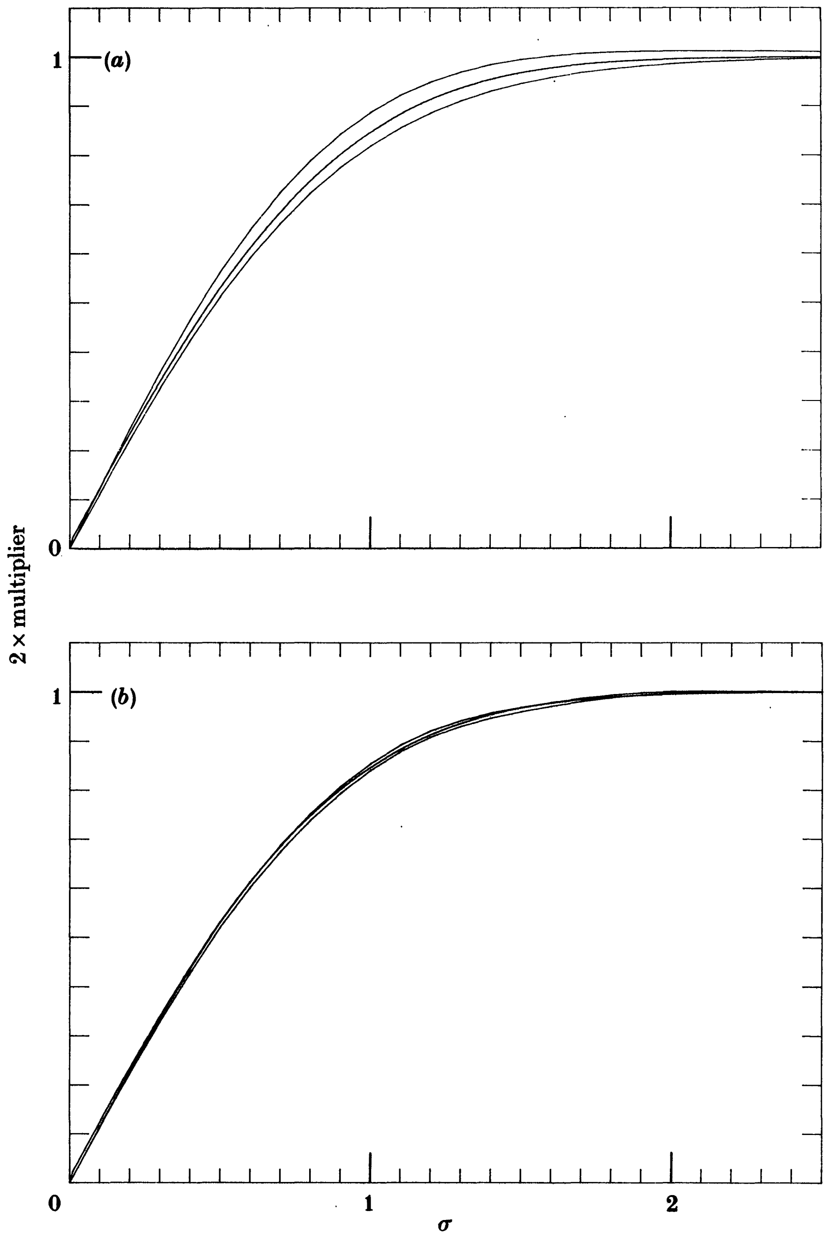

FIGURE 3. Stokes multiplier function for Dawson's integral erfi (equation (23)) for (a) singulant $|F| = 5$; (b) singulant $|F| = 25$. The middle curve is the 'theoretical' multiplier (RHS of (27)) and the upper and lower curves are the 'experimental' multipliers (LHS of (27)) for $\alpha = 0$ and $\alpha = -\frac{1}{2}$ respectively.

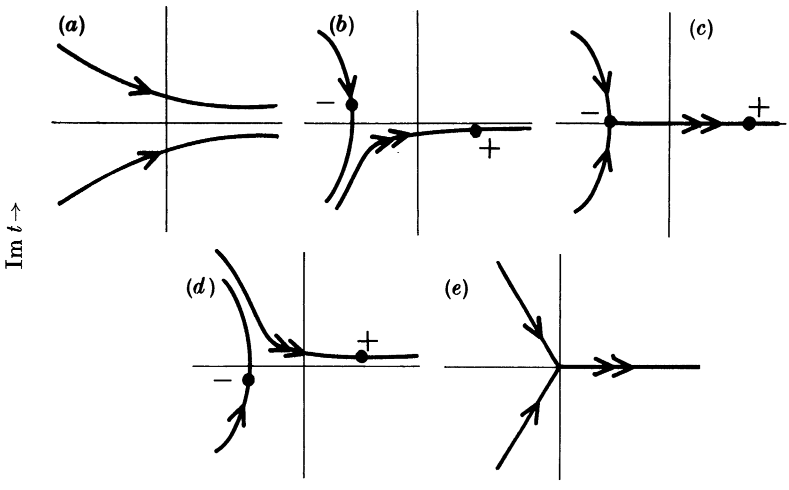

FIGURE 4. Integration contours for the Airy function Bi (equation (29)). (*a*) generic contour; (*b*) steepest-descent contour for $\operatorname{Im} Z < 0$; (*c*) steepest-descent contour for $\operatorname{Im} Z = 0$; (*d*) steepest-descent contour for $\operatorname{Im} Z > 0$; (*e*) contour for numerical integration of Bi. In *b–d* the subdominant and dominant stationary points are labelled + and −. Double arrows mean contours counted twice.

Further steepest-descent analysis (or phase-integral solution of Airy's equation $\partial_Z^2 y = k^2 Z y$) gives the expansion coefficients as

$$a_r = (r-\tfrac{1}{6})!\,(r-\tfrac{5}{6})!/(2\pi r!\,F^r). \tag{33}$$

This satisfies the initial condition $a_0 = 1$ (because $(-\tfrac{1}{6})!\,(-\tfrac{5}{6})! = 2\pi$), and conforms to the limit (10) with $\beta = 1$ (because $(r-\mu)!\,(r-\nu)!/r! \rightarrow (r-\mu-\nu)!$ as $r \rightarrow \infty$).

Incorporating (29)–(33) into the general result (22) with $S_- = -\tfrac{1}{2}$, including the contour doubling, we obtain, as the formula for the asymptotic emergence of the multiplier,

$$2\operatorname{Im}\left\{\pi^{\frac{1}{2}}(\tfrac{3}{4}F)^{\frac{1}{6}}\exp(\tfrac{1}{2}F)\operatorname{Bi}[(\tfrac{3}{4}F)^{\frac{2}{3}}] - \exp(F)\sum_{r=0}^{\operatorname{Int}[|F|+\alpha]}(r-\tfrac{1}{6})!\,(r-\tfrac{5}{6})!/(2\pi r!\,F^r)\right\}$$

$$\rightarrow \frac{2}{\sqrt{\pi}}\int_0^\sigma \mathrm{d}t\exp(-t^2) \quad \text{as} \quad |F| \rightarrow \infty. \tag{34}$$

Again we shall regard σ and $|F|$ as given, obtaining F by solving (30) and (31) for X_1 and X_2;

$$\left.\begin{aligned} k^{\frac{2}{3}}X_1 &= (\tfrac{3}{4}|F|)^{\frac{2}{3}}\cos\phi; \quad k^{\frac{2}{3}}X_2 = (\tfrac{3}{4}|F|)^{\frac{2}{3}}\sin\phi, \\ \text{where} \quad \phi &= \tfrac{2}{3}\arccos\{[1+(\sigma^2/|F|)^2]^{\frac{1}{2}} - \sigma^2/|F|\}. \end{aligned}\right\} \tag{35}$$

Tables 5 and 6 show the results of a numerical test of (34) over a range of values of σ, for singulants $|F| = 5$ and $|F| = 25$. For these integer $|F|$ the two 'best' truncation variables, $\alpha = \tfrac{1}{6}$, and $\alpha = \tfrac{1}{2}$, give identical sums in (34). The function Bi was computed from the extended Simpson's rule, after deforming the contour C

TABLE 5. COMPARISON OF THEORY AND 'EXPERIMENT' FOR THE STOKES MULTIPLIER FUNCTION FOR THE MODIFIED AIRY FUNCTION (29)

The singulant modulus is $|F| = 5$ and the truncation variable is $\alpha = \frac{1}{6}$. σ is the Stokes variable (21) and $F = \frac{4}{3}k(X_1+\mathrm{i}X_2)^{\frac{3}{2}}$ with X_1 and X_2 given by (35).

σ	$\mathrm{Im}\,\{\pi^{\frac{1}{2}}(\frac{3}{4}F)^{\frac{1}{6}}\mathrm{e}^{\frac{1}{2}F}\,\mathrm{Bi}\,((\frac{3}{4}F)^{\frac{2}{3}})\}$	LHS of (34)	(LHS−RHS) of (34)
0.2	86.4604996724	0.221345661477	−0.0121781510106
0.4	123.97607385	0.424404297191	−0.013568051778
0.6	105.332800481	0.595782204695	−0.0159029679234
0.8	59.8291753225	0.729510930471	−0.0184963021774
1.0	18.7836794412	0.826574068164	−0.0202398523799
1.2	−5.02003487888	0.892523281855	−0.0204348576503
1.4	−13.9548273995	0.934750647189	−0.0191036079774
1.6	−14.786787214	0.960427535479	−0.0167804344415
1.8	−12.6179983574	0.975396073945	−0.0141291135887
2.0	−9.96309293078	0.98386023376	−0.0116649481508
2.2	−7.69826258958	0.98856662358	−0.00965797161679
2.4	−5.97784412023	0.991179082126	−0.00816718750941

TABLE 6. AS TABLE 5 WITH $|F| = 25$

σ	$\mathrm{Im}\,\{\pi^{\frac{1}{2}}(\frac{3}{4}F)^{\frac{1}{6}}\mathrm{e}^{\frac{1}{2}F}\,\mathrm{Bi}\,((\frac{3}{4}F)^{\frac{2}{3}})\}$	LHS of (34)	(LHS−RHS) of (34)
0.2	68717981224.8	0.218231201172	−0.0152926113155
0.4	19591748977.8	0.422904968262	−0.0150673807073
0.6	−44424281459.3	0.599472045898	−0.01221312672
0.8	−24788677708.8	0.739608764648	−0.00839846799997
1.0	16363698359.3	0.840934753418	−0.00587916712617
1.2	16545841811.1	0.908313751221	−0.00464438828493
1.4	−1014224788.75	0.949258327484	−0.00459592768219
1.6	−6181168808.05	0.972358703613	−0.00484926630728
1.8	−2073873258.79	0.984602928162	−0.00492225937189
2.0	856749591.829	0.990872144699	−0.00465303721144
2.2	909857385.36	0.993932247162	−0.00429234803532
2.4	237699764.863	0.995340585709	−0.00400568392631

(figure 4*a*) of the integral (29) to that in figure 4*e*, and checked by evaluating the convergent series for Bi.

As for Dawson's integral, successive columns clearly show the expected decreasing orders of magnitude. In the second column, containing the values of the first term on the left-hand side of (34), the order is now $\exp(|F|)$, which is larger by $|F|^{\frac{1}{2}}$ than the corresponding order for Dawson's integral. The left and right sides of (34) are plotted against σ for $|F| = 5$ and $|F| = 25$ in figures 5*a*, *b*.

It is again evident that the agreement between 'experimental' and 'theoretical' multipliers is excellent and improves with increasing $|F|$. A feature of this example, not present for Dawson's integral, is that it illustrates the general case in which the limit (10), forming the basis of our theory of the multiplier, is an approximation valid for the late terms a_r, rather than being exact for all r.

Stokes (1964) himself illustrated the change in the multiplier, by computing an Airy function at two complex arguments with phases $\pm 30°$ from a Stokes line, and a common modulus for which $|F| = \surd 128$. Then (30), (21) and (20) give $\sigma = -2$ and $S(\sigma) = 0.005$, for which the change has barely begun, and $\sigma = +2$ and $S(\sigma) = 0.995$, for which it is virtually complete.

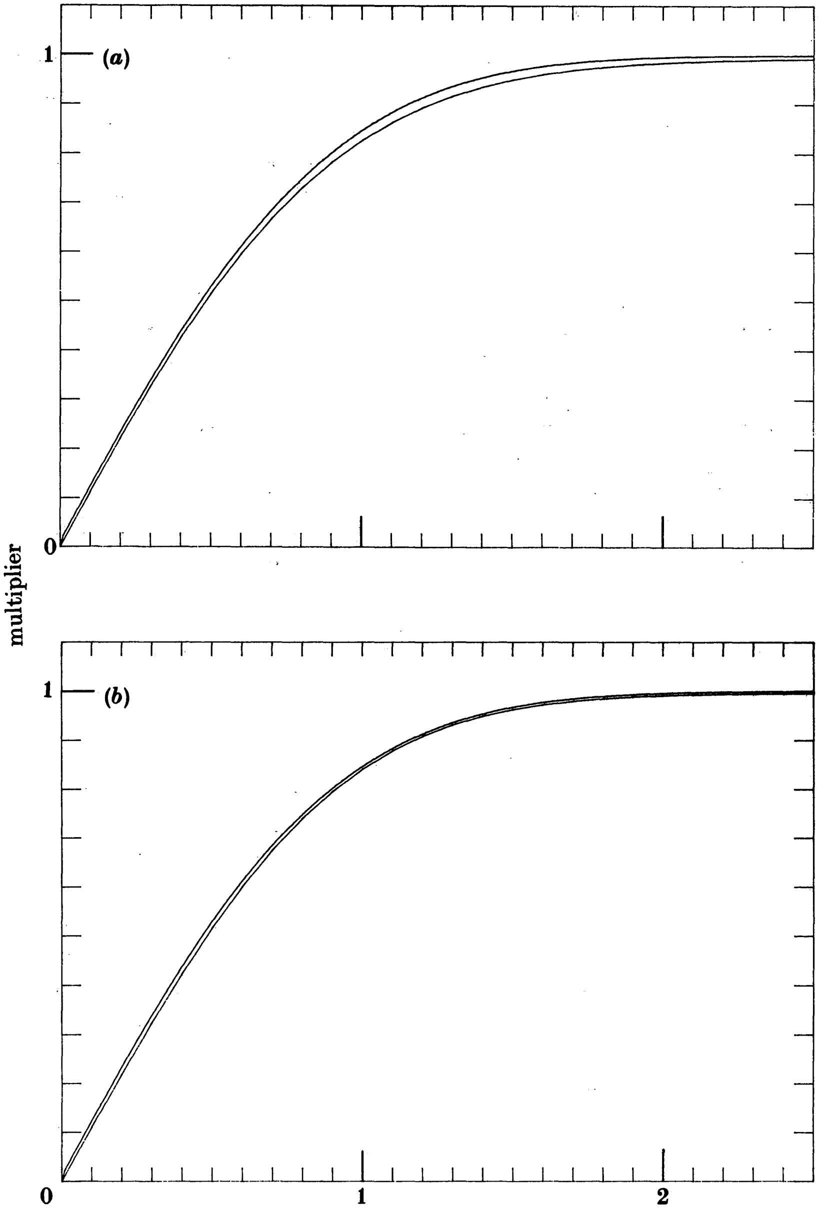

FIGURE 5. Stokes multiplier function for Airy function of the second kind (Bi) (equation (29)) for (*a*) singulant $|F| = 5$; (*b*) singulant $|F| = 25$. The upper curves are the 'theoretical' multipliers (RHS of (34)) and the lower curves are the 'experimental' multipliers (LHS of (34)) for $\alpha = \frac{1}{6}$.

4. Concluding remarks

We have found that across the Stokes line, on which the large exponential is maximally dominant, the multiplier of the small exponential varies rapidly but smoothly in a universal manner that is almost the simplest imaginable; the error function of a natural variable (equation (21)) depending on the singulant (difference between exponents). In numerical computations the Stokes multiplier (of order unity) emerged stably, and in agreement with theory, as the difference between two large quantities (of order up to 10^{10}, cf. table 6).

The universality class appears to be large. Our derivation depended on Dingle's (I) factorial formula (10) for the late terms a_r of the dominant asymptotic series. This type of asymptotics applies (at least) to integrals of functions involving $\exp(k\phi)$ as $k \to \infty$, whether dominated by an end-point or a stationary point; and to the solutions of ordinary differential equations with first-order turning points (for a turning point of order n the multiplier (20) is simply magnified by $2\cos(\pi/(n+2))$). But the results remain valid for some kinds of superfactorial divergence, for example $a_r \to (r!)^m/k^r$ as $r \to \infty$ (as can be shown by multiple Borel summation or, more easily, with multiplication formulae for factorials). It would be interesting to establish the limits of universality.

A trivial way to lose both the universality and simplicity of the main result is truncate the dominant series far from the least term, so that in (16) $|\alpha| \gg 1$. In these circumstances a general theory is still possible provided the limit (10) remains valid for all the Borel-summed terms, that is for all $r > n$, but is unsatisfactory in two respects. First, the multiplier S acquires an imaginary part comparable to its real part. Second, the multiplier now depends on $D \equiv (|F|-n)/(2\,\mathrm{Re}\,F)^{\frac{1}{2}}$ as well as σ, and when $|D|$ is large S rises to values of order $\exp(D^2)$ and oscillates with period of order D^{-1} over a total σ-range of order $|D|$ (rather than unity).

Our treatment is limited by being restricted to the situation where there is just one dominant and one subdominant exponential (equation (1)). What happens when there are more? The following conjecture is natural. Let the exponents be $k\phi_1, k\phi_2, \ldots$, ordered so that $\mathrm{Re}\,\phi_i > \mathrm{Re}\,\phi_j$ if $i < j$, and define for each pair the singulant $F_{ij} \equiv k(\phi_i - \phi_j)$ with $i < j$. Then across the line $\mathrm{Im}\,F_{ij} = 0$ the jth exponential is maximally dominated by the ith and its switching-on is described by the universal Stokes multiplier function that we have obtained, the appropriate singulant being $F = F_{ij}$. A proof (or disproof) is desirable.

It is worth pointing out that although in our examples (erfi and Bi) the parameters X were components of a complex variable $X_1 + \mathrm{i}X_2$, the parameter space need not possess a complex structure. To illustrate this, consider the oscillatory integral describing the cusp diffraction catastrophe (Pearcey's integral)

$$y(X_1, X_2; k) = \int_{-\infty}^{\infty} \mathrm{d}t \exp\{\mathrm{i}k(\tfrac{1}{4}t^4 + \tfrac{1}{2}X_1 t^2 + X_2 t)\} \tag{36}$$

as studied by Wright (1980). The asymptotics are dominated by real stationary points giving oscillatory contributions (waves). In the (real) parameter space X_1, X_2 there are three real stationary points inside the cusp $27X_2^2 + 4X_1^3 = 0$, and one real and two complex stationary points outside. In this outside region the real stationary point always contributes to (36), and one of the complex ones never does. Wright discovered that the (exponentially small) contribution of the second complex stationary point switches on across the *Stokes set*, which is the different cusp $27X_2^2-(5+3\sqrt{3})X_1^3=0$.

An immediate application of the main result is to the *birth of reflected waves* as described by $y''(x)+k^2n^2(x)\,y(x)=0$, where $n(x)$ is a real non-zero refractive index profile. The reflected wave is exponentially small and (if the incident wave is defined by the dominant W.K.B. expansion truncated at its least term) switches on where the Stokes line from the nearest complex turning point crosses the x-axis. The switch occurs over $|F|^{\frac{1}{2}}$ wavelengths, where the singulant $|F|$ is the exponent in the reflection amplitude. (This interpretation originated in conversation with Professor R. G. Littlejohn.)

Finally, there ought to be connections between this work (and, more generally, Dingle's interpretative theory of asymptotic series (I)) and Écalle's recent doctrine of *résurgence* (Écalle 1981, 1984; see also Voros 1983; Pham *et al.* 1989).

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