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MULTIPHASE AVERAGING FOR CLASSICAL SYSTEMS
With Applications to Adiabatic Theorems
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By P. LOCHAK and C. MEUNIER: pp. 360. DM.78.–. (Springer-Verlag, 1988)

Most dynamical equations cannot be solved exactly. Therefore approximations are necessary. One of the most important techniques is averaging. This applies when the system is a perturbation of one whose solution is known and for which some ('fast') variables evolve (quasi-) periodically and others are conserved. Under perturbation the formerly conserved quantities change, in ways that can be estimated by averaging over the fast variables.

Many deep and beautiful results have been obtained in this way, mainly by Soviet mathematicians. This literature is scattered and many of the original papers are difficult because their style is terse. In this book the purpose is to provide an organized presentation of these results, with proofs. This undertaking is worthwhile not only for the mathematics but also because the results have important applications (for example, in the magnetic confinement of plasmas in fusion reactors).

The authors begin with some general averaging theorems. The results depend crucially on whether the number n of fast variables (often called angles or—as in the title of this book—phases) is 1, 2 or more. For n phases the unperturbed motion lies on an n-torus. If $n > 1$ there is the possibility that the frequencies might be commensurate, that is, resonant. Resonances are an obstruction to averaging because this works best if the orbit explores all n dimensions of the torus, whereas a resonant orbit is periodic and hence one-dimensional. The difficulties are compounded if $n > 2$ because then the resonances form a connected web. For this reason the results are sharp estimates for $n = 1$ and weak bounds for $n > 2$.

The main applications given here are to Hamiltonian systems. One chapter is devoted to the autonomous case (where the dynamical equations are time-independent). A careful distinction is made between the Kolmogorov–Arnold–Moser theorem (KAM) and Nekhoroshev's theorem (N). KAM asserts that in the perturbed system most orbits continue to lie on tori, with invariants ('actions') close to the unperturbed ones; the excluded orbits are those near resonances. N asserts that when $n > 2$ the actions of any orbit—even a resonant one—remain close to their initial values for a time that is exponentially long (in terms of the perturbation).

Three chapters concern adiabatic theorems, where the perturbation is a slow change in the Hamiltonian. It is remarkable that even though the Hamiltonian may change a lot over infinite time, the actions of the unperturbed (that is, frozen) system are conserved in the limit of infinitely slow change. For finite speed of change the actions do change, by amounts which for $n = 1$ are exponentially small in the slowness and for $n > 1$ are like its square root (the degradation being caused by repeated passage through resonance). A proof is given of the multidimensional adiabatic theorem for chaotic systems, where the frozen motion is ergodic not on n-tori but on the whole energy surface. I was disappointed not to find an estimate for the drift in the single conserved quantity; perhaps none is known. There is a discussion of recent results on adiabatic evolution of the phases and of the analogous adiabatic theory in quantum mechanics.

For physicists the proofs will be pretty hard to follow, but the book is a unique and useful systematic compilation of results and a valuable collection of references. Mathematicians will be interested not only in the proofs but in the clear explanation of the scientific reasons for the importance of these theorems.

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