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Waves near Stokes lines

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The large-k asymptotics of

$$\mathrm{d}^2u(z)/\mathrm{d}z^2 = k^2R^2(z)\,u(z)$$

are studied near a Stokes line ($w \equiv \int_{z_0}^{z} R\,\mathrm{d}z$ real, where z_0 is a zero of $R^2(z)$, of any order), on which there is greatest disparity between the dominant and subdominant exponential waves in the phase-integral (WKB) approximations. The aim is to establish precisely how the multiplier b_- of the subdominant wave varies across the Stokes line. Although b_- always has a total change proportional to i times the multiplier of the dominant wave (the Stokes phenomenon), the form of the change depends on the convention used to define the two waves. The optimal convention, for which the variation is maximally compact and smooth, is to define them by the phase-integral approximation truncated at its least term, whose order is proportional to k and therefore large ('asymptotics of asymptotics'). Then the variation of b_- is proportional to the error function of the natural Stokes-crossing variable $\mathrm{Im}\,w\sqrt{(k/\mathrm{Re}\,w)}$. This result is obtained without resumming divergent series (thereby avoiding 'asymptotics of asymptotics of asymptotics'). An application is given, to the birth of exponentially weak reflected waves in media with smoothly varying refractive index.

1. INTRODUCTION

Many problems of wave physics require solutions of the one-dimensional Helmholtz equation, which it will be convenient to write in the form:

$$\mathrm{d}^2u(z)/\mathrm{d}z^2 = k^2R^2(z)\,u(z). \tag{1}$$

Here R, and therefore u, are analytic functions of the complex variable z. The asymptotic limit $k \to \infty$ is often important; it corresponds, for example, to the classical limit of quantum mechanics. In this limit, the solutions can be found by the phase-integral, or WKB, method (Heading 1962; Fröman & Fröman 1965 (hereinafter called FF); Berry & Mount 1972 (hereinafter called BM); Dingle 1973 (hereinafter called D); Olver 1974). Emanating from the real or complex zeros z_j (transition points) of $R^2(z)$ are the *Stokes lines*, defined by

$$\mathrm{Im}\,w(z) = 0, \quad \text{where} \quad w(z) \equiv \int_{z_j}^{z} \mathrm{d}z\,R(z). \tag{2}$$

Stokes lines lie at the heart of the asymptotics of (1). They are loci of greatest disparity between the dominant and subdominant fundamental phase-integral approximate solutions attached to z_j:

$$u_{\pm} \approx \exp\{\pm kw(z)\}/R^{\frac{1}{2}}(z). \tag{3}$$

Stokes (1864) discovered that if a solution u of (1) is approximated by a linear combination of u_+ and u_-, the multiplier of the subdominant solution u_- jumps (by i times the multiplier of u_+) across each of these lines. This jump (now called the Stokes phenomenon) is necessary to achieve concordance between different asymptotic representations valid in different regions of the z-plane.

Here I study in detail the behaviour of u across a Stokes line. The precise nature of the jump in the subdominant multiplier depends on the choice of asymptotic representation for u, and so is to some extent a matter of convention. Nevertheless, there is an optimal choice, for which the jump is as smooth and compact as possible, and moreover takes a remarkably simple form given by a universal function of a natural variable describing the crossing of the line.

Elsewhere (Berry 1989*a*, hereinafter called B; Berry 1989*b*) I have obtained this universal smoothing by a formal resummation of the divergent tail of the asymptotic expansion correcting the fundamental exponentials (3). That argument is very general, and not restricted to functions of a complex variable satisfying the differential equation (1). It applies, for example, to the diffraction integrals of catastrophe optics (Berry & Upstill 1980) and to Stokes phenomena in spaces of real variables (Wright 1980; Berry & Howls 1990). Subsequently, Olver (1990) and independently Jones (1990) have provided rigorous derivations of the smoothing formula for certain integrals, without using resummation, and Boyd (1990) has obtained precise error bounds. My reason for rederiving the general formula in the particular context of differential equations is to demonstrate that here, as with integrals, it can be obtained without resummation.

In investigating (1) near a Stokes line, the most efficacious formalism is a synthesis of the techniques of FF and D. From FF we use (§2) a pair of coupled equations, equivalent to (1), for the multipliers of phase-integral approximations of arbitrary order (rather than the lowest-order approximations (3)). From D we use (§3) explicit and universal formulae for high-order phase-integral approximations. Going to high orders enables the variation of the subdominant multiplier to be obtained explicitly. The optimal order (§4) corresponds to the least term in the phase-integral expansion, and leads to the simple universal form for the multiplier. A physical application is the detailed description of the birth of a reflection (§5).

The aim is to understand an individual Stokes line, and so it will not be necessary to consider the complications and subtleties associated with more than one Stokes line and more than one transition point (Heading 1977; Olver 1978). We make use of the variable $w(z)$ defined in (2), taking as origin the transition point z_j giving rise to the Stokes line under study. Without loss of generality we shall henceforth take $z_j = 0$. We assume that this transition point is well separated from any others, and that it is a simple zero of $R^2(z)$. Results for a transition point

of any order can be obtained by an almost identical argument and will be stated in §4 (see also Appendix B).

It will emerge that a more natural variable is the *singulant* (D)

$$F(z) \equiv 2kw(z). \tag{4}$$

This is the difference between the exponents of the two phase-integral approximations (3) attached to z_j, and is positive real on the Stokes line.

2. Coupling

We represent the exact solutions of (1) in terms of multipliers $a_\pm(w)$ describing a superposition of the improved approximate solutions obtained by replacing $R(z)$ in the lowest approximations (3) by an as yet unspecified function $r(w)$:

$$u(z) = \frac{a_+(w)}{r^{\frac{1}{2}}(w)} \exp\left\{k\int_0^z \mathrm{d}z\, r(w)\right\} + \frac{a_-(w)}{r^{\frac{1}{2}}(w)} \exp\left\{-k\int_0^z \mathrm{d}z\, r(w)\right\}. \tag{5}$$

The ambiguity caused by having two unknown functions $a_\pm$, instead of the single u, is resolved by forcing $a_\pm$ to obey the relation that would hold if they were constants:

$$\begin{aligned}\frac{\mathrm{d}u(z)}{\mathrm{d}z} = {} & a_+(w) \exp\left\{k\int_0^z \mathrm{d}z\, r(w)\right\}\left[kr^{\frac{1}{2}}(w) + \frac{\mathrm{d}}{\mathrm{d}z} r^{-\frac{1}{2}}(w)\right] \\ & + a(w) \exp\left\{-k\int_0^z \mathrm{d}z\, r(w)\right\}\left[-kr^{\frac{1}{2}}(w) + \frac{\mathrm{d}}{\mathrm{d}z} r^{-\frac{1}{2}}(w)\right].\end{aligned} \tag{6}$$

A straightforward calculation now shows that $a_\pm$ satisfy the equations (FF)

$$\frac{\mathrm{d}}{\mathrm{d}w}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \mp \frac{kR}{2r}\epsilon(w) \begin{pmatrix} -1 & -\exp\left\{-2k\int_0^z \mathrm{d}z\, r(w)\right\} \\ \exp\left\{+2k\int_0^z \mathrm{d}z\, r(w)\right\} & 1 \end{pmatrix} \begin{pmatrix} a_+ \\ a_- \end{pmatrix}, \tag{7}$$

where

$$\epsilon(w) \equiv \frac{r^2}{R^2} - 1 + \frac{r^{\frac{1}{2}}}{k^2R^2} \frac{\mathrm{d}^2}{\mathrm{d}z^2}(r^{-\frac{1}{2}}). \tag{8}$$

The diagonal coupling can be eliminated by the transformation

$$a_\pm \equiv b_\pm \exp\left\{\mp \frac{k}{2}\int_0^z \mathrm{d}z \frac{R^2}{r}\epsilon\right\} \tag{9}$$

leading to

$$\frac{\mathrm{d}}{\mathrm{d}w} b_\pm(w) = 2kC_\pm(w)\, b_\mp(w) \tag{10}$$

containing the *couplers*

$$C_\pm(w) \equiv \mp \frac{R(w)}{4r(w)}\epsilon(w) \exp\left\{\mp 2k\int_0^w \mathrm{d}w \frac{r}{R}(1 - \tfrac{1}{2}\epsilon)\right\}. \tag{11}$$

We do not need to specify boundary conditions in (10), because the Stokes jumps occur for every solution of (1) when written in the form (5).

The splitting (6) (FF) is a slight generalization of one introduced by Kemble (1935). There are other ways to disambiguate (5) (Bremmer 1951; Van Kampen 1967; BM; Schep 1974), resulting in equations of the same form as (10) but with couplers different from (11). A remarkable fact, which I do not demonstrate here, is that the ultimate formula for the change in the subdominant multiplier b_- is insensitive to the change from (6) to one of these alternatives.

The simplest choice for r is R. Then $a_\pm$ are multipliers of the fundamental phase-integral solutions (3). Although I do not make this choice, there is an important lesson to be learned from it. Near the Stokes line (w positive real), C_+ is exponentially small. This might appear to justify regarding b_+ as constant across the line when solving (10), and hence obtaining the desired change in b_- simply by integration:

$$b_-(w) = b_-(a-\mathrm{i}\infty)+2kb_+\int_{a-\mathrm{i}\infty}^{w} \mathrm{d}w\, C_-(w), \tag{12}$$

where a is positive real. When k is large the integral for the total change in b_- (the 'Stokes constant' times b_+) can be performed analytically by deforming the contour to encircle the transition point. Near this point, C_- can be shown from (11) and (8) (with $r = R$) to have the limiting form

$$C_-(w) \to (5/144k^2w^2)\exp\{2kw\} \tag{13}$$

(the derivation proceeds by approximating $R^2(z)$ by a linear function near the transition point). Thus the Stokes constant would be

$$[b_-(a+\mathrm{i}\infty)-b_-(a-\mathrm{i}\infty)]/b_+ = 5\pi\mathrm{i}/18. \tag{14}$$

This result is wrong; the correct factor should be unity, not $\frac{5}{18}\pi$. Pokrovskii *et al.* (1958), who discovered the error, also understood its source: exponentially small changes in the dominant multiplier b_+ are exponentially amplified by C_- in (13) to produce a finite effect on b_-. Moreover, they appreciated that the correct Stokes constant could be obtained by evaluating the infinite series of multiple integrals generated by solving the pair (10) by iteration. This series is convergent (Atkinson 1960), and consists of terms all of the same order (zero) in k. Choosing a splitting relation different from (6) does not remove the problem, but merely alters the constant. For example, the splitting of Bremmer (1951) gives a '$\frac{1}{3}\pi$ problem' instead of a '$\frac{5}{18}\pi$ problem' (BM). For this case the convergent series was evaluated by Berry (1982); the result is indeed the correct factor of unity.

As a way of calculating the functional form of b_- near the Stokes line, rather than its total change, the series obtained by iteration of (10) is ineffective, even though convergent. Therefore we proceed in a different way. Instead of simply taking $r = R$, we make a sequence of increasingly sophisticated choices, whose effect will be to improve the convergence of the series obtained by iterating (10). Ultimately, even the lowest approximation (12) will generate the correct multiplier, and also the desired accurate approximations to the form of the change in b_-. The first hint that this might happen came in calculations by Lundborg (1979), who showed that Stokes constants much closer to unity than $\frac{5}{18}\pi$ are

obtained from (12) by taking r as R plus the first few corrections generated by the phase-integral method.

3. High orders

Observe that if we could choose the so far unspecified function $r(w)$ so as to make $\epsilon(w)$ vanish in (8), the couplers $C_\pm$ would vanish too (cf. (11)), so that the multipliers b_+ and b_- would be exactly constant. Of course the vanishing of ϵ is just the condition for the exact solutions of (1) to take the phase-integral form (3) with r replacing R. Finding this $r(w)$ would therefore be equivalent to obtaining the exact solution of (1), which we cannot do. We can, however, find a sequence of approximations r_n to the solution of $\epsilon = 0$, accurate to increasingly higher orders in $1/k$. These will generate from (8) a diminishing sequence of functions ϵ_n and thence, by substitution in (11), diminishing couplers $C_\pm$.

There are two obvious ways to accomplish this. One is to solve $\epsilon = 0$ by iteration, starting from $r_0 = R$ and obtaining higher approximations by

$$r_{n+1}^2 = R^2 - \frac{r_n^{\frac{1}{2}}}{k^2}\frac{\mathrm{d}^2}{\mathrm{d}z^2} r_n^{-\frac{1}{2}}. \tag{15}$$

This has the advantage that ϵ_n is exactly the difference between successive approximants, namely

$$\epsilon_n = -(r_{n+1}^2 - r_n^2)/R^2. \tag{16}$$

Moreover it has an interesting interpretation in terms of repeated renormalization of (1) (Littlewood 1963; Schep 1974) or of the equivalent hamiltonian system (Berry 1989*c*). But it has the disadvantage that the approximants r_n are themselves complicated.

The second way is to take for r_n the partial sums generated by substituting the formal series of the phase-integral method (FF), namely

$$r(w) = R(z) \sum_{n=0}^{\infty} \frac{\phi_{2n}(w)}{k^{2n}} \tag{17}$$

into the equation $\epsilon = 0$. (The lowest-order term is $\phi_0 = 1$.) Now it is the evaluation of ϵ_n that proves tricky. We will require this only to lowest order in $1/k$. If the series (17) were convergent, the following simple argument would be valid.

The equation $\epsilon = 0$ has the form

$$\begin{aligned} 0 = \epsilon &= r^2/R^2 - 1 + A(r)/k^2 \\ &= (r_n + r - r_n)^2/R^2 - 1 + A(r_n + r - r_n)/k^2 \\ &\approx \epsilon_n + 2(r - r_n)\, r_n/R^2 + [A(r_n + r - r_n) - A(r_n)]/k^2, \end{aligned} \tag{18}$$

where A denotes a functional of $r(w)$. Assuming convergence, the last term can be neglected because it is smaller by a factor of order $1/k^2$ than the second term, $r - r_n$ can be approximated by the term of order $n+1$ in the series (17), and r_n itself can be approximated by R. Thus

$$\epsilon_n(w) \approx -2\phi_{2(n+1)}(w)/k^{2(n+1)}. \tag{19}$$

This is the same in lowest order as the result of substituting the partial sums of (17) into the formula (16) from the iteration method.

The argument leading to (19) is deprived of its validity by the fact that the series (17) is not convergent but asymptotic, and we intend to use it to its least term and even beyond. Nevertheless, the result (19) survives, owing to a subtle and surprising combination of circumstances explained in Appendix A.

We wish to evaluate the couplers (11) for high orders $n \gg 1$ of the phase-integral approximation. Here a remarkable simplification enters, discovered by Dingle (1973). In terms of the singulant (4), the late terms ϕ_n take the universal form

$$\phi_{2(n+1)}(w)/k^{2(n+1)} \approx -(2n+1)!/\pi F^{2n+2} \quad (n \gg 1) \tag{20}$$

independent of the details of the underlying function $R(z)$. The derivation of (20), outlined in Appendix B, combines formulae in chapters 13 and 14 of D with an observation of Fröman (1966). In essence, the universality of (20) stems from the inevitable divergence of the series (17) that would, if it converged, describe only the dominant exponential. The series, being a formally exact solution of (1), must also reflect the existence of the subdominant exponential, whose exponent differs from that of the dominant one by F.

It must be emphasized that (20) is not just a transitional approximation, valid only near the transition point where $R^2(z)$ can be approximated by a linear function, but holds uniformly out to distant points z, ceasing to be valid only when dominated by a similar contribution from another transition point that is closer to z (on the singulant scale). A way to see this is to note that the recursion relation generating the ϕ_n produces ever higher singularities at the transition point (cf. (13) which shows the first singularity thus generated, at $n = 1$); thus the range of influence of the transition point spreads outwards as n increases, a fact previously noted in computations (see, for example, Fröman 1970).

In terms of F, the couplers (11) can now be approximated to lowest order in $1/k$; (20) and (19) give

$$C_{\pm}(F) \approx \mp \frac{(2n+1)!}{2\pi F^{2n+2}} \exp(\mp F) \quad (n \gg 1) \tag{21}$$

and generate the changes in the multipliers through the equations

$$\frac{\mathrm{d}}{\mathrm{d}F} b_{\pm} = C_{\pm}(w)\, b_{\mp}. \tag{22}$$

As expected, the couplers between these higher-order phase-integral solutions get smaller at first, and then increase owing to the asymptotic nature of the approximation. We shall see in the next section that near the minimum ($n \approx \frac{1}{2}|F|$), C_- is of order unity, so that exponentially small changes in b_+ are no longer amplified, and moreover the other coupler, C_+, is doubly exponentially small (i.e. of order $\exp(-2F)$). We have therefore circumvented the deficiencies of the simplest approximation (12) when used in conjunction with the simplest coupling (13) (based on $r = R$). It is easy to confirm that the '$\frac{5}{18}\pi$ problem' is thereby

solved: from (12), now employed in conjunction with (21), we obtain for all $n \gg 1$ the correct Stokes constant

$$\frac{[b_-(a+\mathrm{i}\infty)-b_-(a-\mathrm{i}\infty)]}{b_+} = \frac{(2n+1)!}{2\pi}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\frac{\mathrm{d}F}{F^{2n+2}}\exp(F)$$
$$= \mathrm{i} \tag{23}$$

(where $a > 0$). What has been accomplished by employing phase-integral approximations of high order is that b_- is no longer coupled to b_+ but slaved to it. Thus b_+ remains constant across the Stokes line, and drives the variation of b_- as will be explained next.

4. Optimal coupler

Now we study in detail the universal high-order couplers C_- as given by (21). In the complex singulant plane we introduce polar coordinates

$$F = |F|\exp(\mathrm{i}\theta) \tag{24}$$

so that the Stokes line is $\theta = 0$ and large-k asymptotics means large $|F|$ (cf. (4)). Defining $N \equiv 2n+1$ and using Stirling's formula because N is large we obtain, from (21),

$$C_-(F) \approx \frac{(N/|F|)^{N+\frac{1}{2}}\exp(|F|\cos\theta - N)}{\sqrt{(2\pi|F|)}}\exp\{\mathrm{i}[|F|\sin\theta-(N+1)\theta]\}. \tag{25}$$

In general this is a complex function, rising to very large values near the Stokes line, with rapid oscillations that reduce its integral across the line to unity (cf. (23)). Remarkable simplifications occur when N is close to $|F|$, corresponding to the least term in the phase-integral series (17). First we find

$$C_-(F) \approx (2\pi|F|)^{-\frac{1}{2}}\exp\{-[|F|\theta+\mathrm{i}(N-|F|)]^2/2|F|\}, \tag{26}$$

if $|N-|F\,\| \ll |F|^{\frac{2}{3}}$ and $\theta \ll (6/|F|)^{\frac{1}{3}}$. Even simpler is

$$C_-(F) \approx (2\pi|F|)^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}|F|\theta^2\} \quad (\text{if } |N-|F|\,\| \ll |F|^{\frac{1}{2}}). \tag{27}$$

This is the main result: the coupler is simply a real gaussian with integral unity (because near the line $\mathrm{d}F = \mathrm{i}|F|\,\mathrm{d}\theta$), ensuring the correct Stokes constant). It is valid not only at the phase-integral approximation with precisely optimal order (for which N is the closest integer to $|F|$), but for neighbouring orders in a range $|F|^{\frac{1}{2}}$ about $|F|$. The natural Stokes-crossing variable is

$$\sigma \equiv \theta\sqrt{(\tfrac{1}{2}|F|)} = \mathrm{Im}\,F/\sqrt{(2\,\mathrm{Re}\,F)} \tag{28}$$

and leads via (22) to the following formula giving the change in the subdominant multiplier as an error function:

$$b_-(\sigma) = b_-(-\infty) + \mathrm{i}\,\mathrm{Erfc}(-\sigma)\,b_+. \tag{29}$$

The freedom to choose $b_-(-\infty)$ reflects the freedom to choose solutions of (1) satisfying different boundary conditions. Previously the result (29) was derived (B) by resumming the divergent tail of the asymptotic series multiplying the dominant exponential.

The coupler takes its maximum value at $\theta = 0$. For the optimal N, this is $1/\sqrt{(2\pi|F|)}$. Away from optimality, this maximum value increases exponentially fast: from (26), its value, relative to the optimal, is $\exp\{(N-|F|)^2/2|F|\}$. So the optimal coupler is the smallest, and, because the coupler controls the rate of change of the subdominant multiplier (cf. (22)), optimality means the smoothest implementation of the Stokes jump.

It is interesting to calculate the range of values $\Delta\theta$ over which b_- is significantly changing. We call this the *width* of the Stokes line. A natural definition is the θ for which the envelope of $\mathrm{Re}\, C_-$ is greater than $1/e$ of the maximum of the optimal coupler (27). To estimate this it is sufficient to use (26), which gives

$$\Delta\theta = \sqrt{[2/|F| + (N/|F| - 1)^2]}. \tag{30}$$

The optimal coupler has the narrowest Stokes line, so that optimality means the most compact implementation of the Stokes jump, with an angular width of order $1/\sqrt{|F|}$. From (4) this implies a width in the original z plane of order $1/\sqrt{k}$. Away from optimality the width increases, becoming of order $1/k^{\frac{1}{3}}$ at the limit of validity

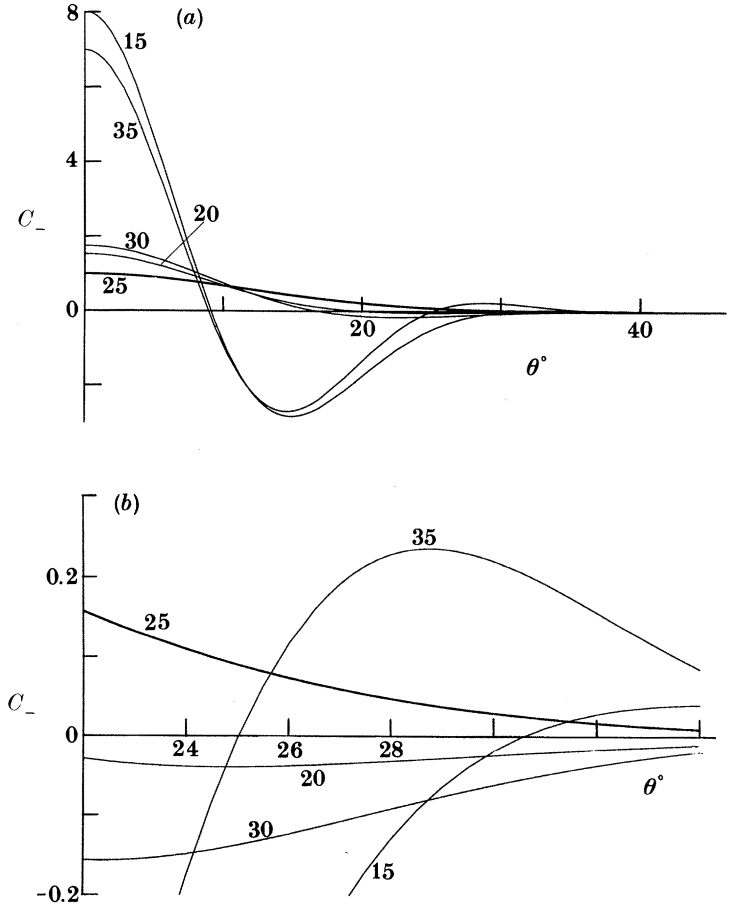

FIGURE 1 *a, b*. For description see opposite.

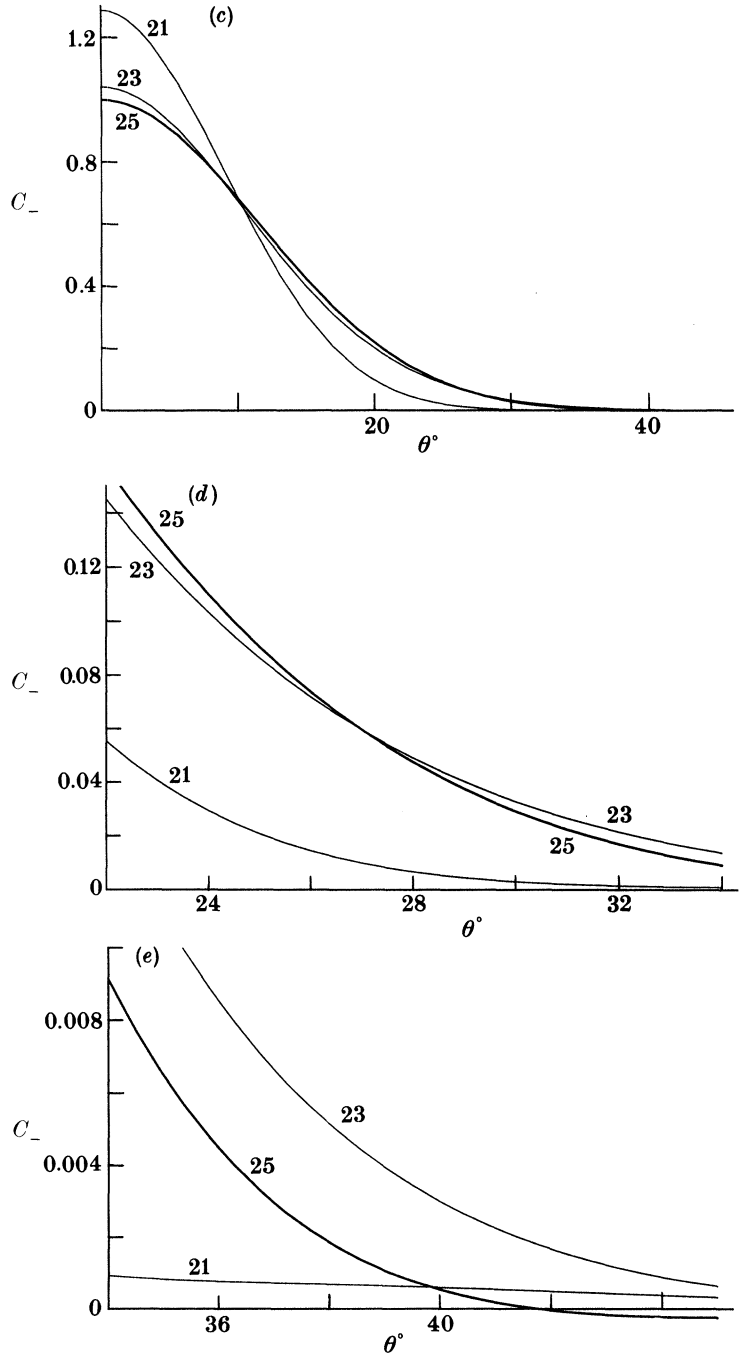

FIGURE 1. Real part of coupler C_- (equation (25)) across a Stokes line, for singulant modulus $|F| = 25$ and the indicated values of phase-integral truncation order N. The optimal coupler (bold curves) has $N = 25$. (*a*) Optimal and far-from optimal N; (*b*) magnification of tail of (*a*); (*c*) optimal and near-optimal N; (*d*) magnification of tail of (*c*); (*e*) continuation of (*d*). The optimal coupler is always the smallest at $\theta = 0$. For intermediate θ, some non-optimal orders are smaller, but the magnifications show that the optimal coupler always decays fastest for large θ. (The formula (25) has been multiplied by $\exp(\mathrm{i}\theta)$ because $\mathrm{d}F = \mathrm{i}\exp(\mathrm{i}\theta)\,\mathrm{d}\theta$.)

of (26), i.e. when N deviates from $|F|$ by $|F|^{\frac{2}{3}}$. Eventually, the width becomes of order unity, indicating that the multiplier is changing even in the neighbourhood of the *anti*-Stokes line $\theta = \frac{1}{2}\pi$, so that the concept of a localized jump ceases to apply.

Figure 1 shows how the coupler (25) oscillates wildly when $N \ll |F|$, hovers about the gaussian (27) when $N \approx |F|$ and dissolves again into large oscillations when $N \gg |F|$. The form of the oscillations is different on the two far sides of optimality. As expected, the increasing strength of the coupler is a more dramatic effect than its increasing width.

Suppose now that the transition point is not a simple zero of $R^2(z)$, as we have assumed heretofore, but is of order M (which may be negative, corresponding to a pole). This causes only a slight alteration in our formulae: Dingle's universal expression (20) for the late terms in the phase-integral expansion is multiplied by

$$G_M = 2\sin\{M\pi/2(M+2)\} \tag{31}$$

(see Appendix B). The same factor multiplies the couplers in (21) and (25)–(27), the Stokes constant in (23), and b_+ in (29). In other words, the width and form of the change across the Stokes line is unaltered, but the strength is different. (Note that for $M = 0$, corresponding to no transition point at all, G_M vanishes, so there is, as expected, no Stokes phenomenon.)

5. Birth of reflections

In (1) let $R(z) = \mathrm{i}\mu(z)$, where $\mu(z)$ is analytic in a strip about the real axis $z = x$, on which it is real and positive, and tends to unity as $x \to \pm\infty$. Then the equation becomes

$$\mathrm{d}^2u(z)/\mathrm{d}z^2 + k^2\mu^2(z)\,u(z) = 0 \tag{32}$$

describing the encounter of waves whose wavenumber at infinity is k with a medium of varying refractive index $\mu(x)$, in which the local wavenumber is $k\mu(x)$. Alternatively, it describes the encounter of quantum particles with an underdense (FF) potential barrier. As is well known, an incident wave will undergo exponentially weak reflection. If the wave comes in along the positive direction from $-\infty$, the boundary condition is that there be no backward-travelling wave at $+\infty$. At $-\infty$ there is a backward-travelling wave, produced by reflection in the medium. So the natural question arises: where and how does the reflected wave arise on the x axis?

We are able to answer this question, because the birth of a reflection is simply the switching-on of a subdominant multiplier. The formalism is the same as in §2, with the wave represented by (5) and (9), and adopting the convention that

$$\int_a^b \mathrm{d}x\, r(x) \quad \text{and} \quad \mathrm{i}\int_a^b \mathrm{d}x\, \mu(x) \tag{33}$$

are both positive imaginary for $b > a$ for real a, b. Thus the incident wave has multiplier b_+, which we define as unity, and we require the coefficient $b_-(x)$ of the reflected wave. There are no transition points $\mu^2 = 0$ on the real axis. Complex transition points occur in conjugate pairs. We consider the one nearest to the real

axis and in the upper half-plane, at z_+, say (the one in the lower half-plane would generate the complex conjugate solution).

From this transition point emanates a Stokes line that crosses the real axis at x_0. (A useful example to bear in mind is $\mu^2 = 1-A/(1+x^2)$, where A is real and less than unity; this has its relevant transition point at $z_+ = \mathrm{i}\sqrt{(1-A)}$, and $x_0 = 0$.) To apply the results of §4 we need the singulant F. At x_0 this is the positive real quantity

$$F_0 = 2\mathrm{i}k\int_{z_+}^{x_0} \mathrm{d}z\,\mu(z) \tag{34}$$

and for x near x_0 we have

$$F = F_0 + 2\mathrm{i}k_0(x-x_0), \tag{35}$$

where k_0 is the wave number $k\mu(x_0)$ at x_0.

Now recall that the integrals in (5) begin at the transition point. By making use of (33) and (34), we can write the wave defined by (5) and (9) to lowest order in $1/k$, after dividing through by $\exp\frac{1}{2}F_0$, as

$$u(x) \approx \frac{1}{|r^{\frac{1}{2}}(w(x))|}\exp\left\{\mathrm{i}k\int_{x_0}^{x}\mathrm{d}z\,[-\mathrm{i}r(w)]\right\} + \frac{b_-(x)}{|r^{\frac{1}{2}}(w(x))|}\exp\{-F_0\}\exp\left\{-\mathrm{i}k\int_{x_0}^{x}\mathrm{d}z\,[-\mathrm{i}r(w)]\right\}, \tag{36}$$

where the integrals are real and $|r^{\frac{1}{2}}| \to 1$ as $|x| \to \infty$.

In (36), the wave u is separated into two terms, representing incident and reflected waves. As is well known, there is no unique way of making this separation in a varying medium. This arbitrariness is manifested in the freedom to choose the function r. We have chosen r by truncating the phase-integral series (17). For any order of truncation, the reflected wave has the same, exponentially small, amplitude as $x \to -\infty$. But the manner in which the reflected wave switches on across the Stokes line $x = x_0$ depends on the order of truncation. Here we make the same, natural, choice as in §4, namely the series given by the phase-integral approximation, truncated near its least term. This gives the most compact switching, and from (28), (29) and (35), together with the boundary condition $b_-(+\infty) = 0$, we obtain

$$b_-(x) = -\mathrm{i}\,\mathrm{Erfc}\{2k_0(x-x_0)/\sqrt{(2F_0)}\}. \tag{37}$$

On a 'macroscopic' scale, the reflection is born suddenly; as we have seen, the width of the Stokes line is of order $1/\sqrt{k}$ (cf. the discussion following (30)). The birth is, however, slow on the 'microscopic' scale defined by the local wavelength $2\pi/k_0$. From (37) we find that it occurs over a distance $\sqrt{F_0}$ local wavelengths. In quantum mechanics these scales can be measured in terms of Planck's constant h; the de Broglie wavelength is of order h, and the width of the Stokes line is of order $\sqrt{h}$. Both are semiclassically small, but on the de Broglie scale even the optimal birth is slow.

6. Concluding remarks

The technique for obtaining the optimal variation of the subdominant multiplier involves truncating the phase-integral approximation at the order n corresponding to its least term. This value of n is proportional to k and therefore large too. Only with such 'asymptotics of asymptotics' does the coupler take the simple universal form (27). Optimal truncation is an idea already present in the original paper of Stokes (1864) on the approximation of Bessel functions; it has also been used by Nekhoroshev (1977); see also Lochak & Meunier 1988) to estimate the accuracy of conservation of adiabatic invariants over long times in nonlinear hamiltonian equations. The novelty of the present application is that it provides a precise formula rather than estimates.

It is, however, only a beginning. The work calls for extension in at least three directions. First, the formal procedures that I have used should be made rigorous, to complete the results by providing error bounds. That would be analogous to what Boyd (1990) has begun to do for integrals.

Secondly, there ought to be a way of streamlining the derivation. The argument in Appendix A for the legitimacy of the high-order approximation (19) to $\epsilon_n(w)$ might be simplified by careful analysis of the recursion (15), and this might at the same time provide a more direct route than Appendix B to Dingle's important formula (20) for the late terms $\phi_n(w)$.

Thirdly, and most important, the effects of more than one transition point should be investigated. One such effect is that Stokes lines from different transition points can approach each other. This happens, for example in the harmonic oscillator ($R^2(z) = z^2 - 1$), where Stokes lines from $z = \pm 1$ have common asymptotes at $z = \pm \mathrm{i}\infty$. Then even the optimal couplers of the Stokes lines can overlap, leading to variations of the subdominant multiplier which are no longer given by the simple error function (29), but lie in a different universality class. This is unexplored territory. We are ignorant even of its codimension, that is the number of parameters in $R^2(z)$ which must be varied to escape the domain of validity of the results reported here.

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Appendix A

This is the demonstration that the function $\epsilon(w)$, defined by (8), can be approximated by (19) when $r(w)$ is approximated by the nth partial sum of the phase-integral series (17). The nonlinear r-dependence of the last term in (8) complicates the analysis, so the first step is to remove it. This is accomplished by choosing a new variable r_1, related to r by

$$r_1 \equiv r - \frac{1}{2kr}\frac{\mathrm{d}r}{\mathrm{d}z}. \tag{A 1}$$

Under this transformation, ϵ becomes

$$\epsilon = \epsilon_1 = \frac{r_1^2}{R^2} - 1 + \frac{1}{kR^2}\frac{\mathrm{d}r_1}{\mathrm{d}z} \tag{A 2}$$

in whose last term r_1 appears linearly. Because ϵ_1 equals ϵ, we can use ϵ_1 instead of ϵ to get approximations to the couplers, by substituting successive approximations to the r_1 that would make ϵ_1 vanish.

When substituted back into the dominant exponential in the representation (5) for $u(z)$, the effect of r_1 is to remove the prefactor $r^{-\frac{1}{2}}$. This led Fröman (1966) to the useful observation that in the phase-integral series for r_1, which contains both odd and even powers of $1/k$, the even terms simply reproduce those of the series (17) for r, while the sum of the odd terms, by (A 1), depends only on the even ones. Therefore we write

$$r_1(w) = R(z) \sum_{n=0}^{\infty} \frac{\phi_n(w)}{k^n} \tag{A 3}$$

and define the nth approximant ϵ_{1n} as the result of substituting the nth partial sum r_{1n} of this series into (A 2).

From $\epsilon_1 = 0$ we have, from (A 2), without approximation,

$$\epsilon_{1n} = -\frac{2r_{1n}(r_1 - r_{1n})}{R^2} - \frac{1}{kR^2}\frac{\mathrm{d}}{\mathrm{d}z}(r_1 - r_{1n}) - \frac{(r_1 - r_{1n})^2}{R^2}. \tag{A 4}$$

For $r_1 - r_{1n}$ we take the tail of the formal series (A 3), namely

$$r - r_{1n} = R \sum_{n+1}^{\infty} \frac{\phi_j}{k^j}. \tag{A 5}$$

We can eliminate the second term in (A 4) with the aid of the recurrence relation imposed on the ϕ_n by the defining equation $\epsilon_1 = 0$; this is easily found to be

$$\sum_{m=0}^{n} \phi_m \phi_{n-m} + \frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}z}(R\phi_{n-1}) = 0. \tag{A 6}$$

After some algebra, (A 4) becomes

$$\epsilon_{1n} = -2 \sum_{j=0}^{n} \sum_{m=n+1}^{\infty} \frac{\phi_j \phi_m}{k^{j+m}} + \sum_{j=n+2}^{\infty} \sum_{m=0}^{j} \frac{\phi_m \phi_{j-m}}{k^j} - \sum_{j=n+1}^{\infty} \sum_{m=n+1}^{\infty} \frac{\phi_j \phi_m}{k^{j+m}}. \tag{A 7}$$

Now we extract from this formally exact result the leading terms in $1/k$. Denote the three double sums in (A 7) by S_1, S_2 and S_3. Only S_1 contains the lowest-order term, proportional to $1/k^{n+1}$. In the next order, $1/k^{n+2}$, there are two terms in S_1 and $\mathrm{Int}\,(n/2)+1$ terms in S_2; the two terms in S_1 are cancelled by the first two of those in S_2, leaving

$$\epsilon_{1n} = -\frac{2\phi_{n+1}}{k^{n+1}} + \frac{2\phi_2 \phi_n}{k^{n+2}} + \frac{2\phi_3 \phi_{n-1}}{k^{n+2}} + \ldots. \tag{A 8}$$

There are no contributions from S_3. The correction terms in (A 8), of order $1/k^{n+2}$, are small even for large n when the asymptotic nature of the series (A 5) becomes apparent. This is clear from the limiting form (20) (with n replacing $2n+2$): the

second and third terms in (A 8) are smaller than the first by factors of order $1/(nF)$ and $1/(n^2F)$ respectively, which are both negligible. By contrast, the first pair of cancelling terms from S_1 and S_2, although also formally of order $1/k^{n+2}$, are not individually negligible, because the result of division by the first term in (A 8) is a factor of order n/F, which is large beyond the least term. Because of the cancellation, then, we have

$$\epsilon_{1n} \approx -2\phi_{n+1}/k^{n+1}. \tag{A 9}$$

Apart from the replacement of $2n+1$ by n, which is a trivial consequence of the replacement (A 1) of r by r_1, this is the same as (19), which was obtained by pretending that the series (17) was convergent. This result would not have been obtained without the cancellation just explained.

Appendix B

This is an outline of the derivation of the universal formula (20) for the late terms of the phase-integral series, together with the modifying factor (31) that is necessary if the transition point is a zero of Mth rather than first order. We begin by making use of the result of Fröman (1966) to replace the series (20) for the function r, which involves even-order terms $\phi_{2(n+1)}$, by the series (A 3) for the function r_1 defined in (A 1), which involves both even and odd terms ϕ_n. This is the formal series obtained by representing $u(z)$ as

$$u(z) = \exp\left\{k\int_0^z \mathrm{d}z\, r_1(w)\right\}. \tag{B 1}$$

The first two terms of (A 3), in k^0 and k^{-1}, reproduce the elementary phase-integral approximation (3) for the dominant solution, leaving

$$\begin{aligned} u(z) &= \frac{\exp\{kw(z)\}}{R^{\frac{1}{2}}(z)} \exp\left\{\sum_{n=2}^{\infty} \frac{1}{k^{n-1}} \int_0^z \mathrm{d}z R(z)\, \phi_n(w)\right\} \\ &\equiv \frac{\exp\{kw(z)\}}{R^{\frac{1}{2}}(z)} \exp\left\{\sum_{n=1}^{\infty} \frac{B_n(w)}{k^n}\right\}. \end{aligned} \tag{B 2}$$

Thus finding the late terms ϕ_n is equivalent to finding the late terms B_n. Dingle (ch. 14 of D) accomplishes this by relating the B_n of the nonlinear representation (B 2) with the coefficients Y_n of the linear representation, defined by

$$u(z) = \frac{\exp\{kw(z)\}}{R^{\frac{1}{2}}(z)} \sum_{n=0}^{\infty} \frac{1}{k^n} Y_n(w) \tag{B 3}$$

with $Y_0 = 1$. Thus we have to solve

$$\sum_1^{\infty} \frac{B_n}{k^n} = \ln \sum_0^{\infty} \frac{Y_n}{k^n}. \tag{B 4}$$

Differentiating with respect to k and defining the shift operator E by

$$\mathrm{E} f_n \equiv f_{n+1} \tag{B 5}$$

acting to the right, we obtain the exact formal solution

$$B_n = \frac{1}{n}\left(1+\sum_1^{n-1} Y_m \,\mathrm{E}^{-m}\right)^{-1} nY_n. \tag{B 6}$$

This generalizes the approximate equation (23a) of chapter 14 of D. Expanding the denominator, we find

$$B_n = Y_n - Y_1 Y_{n-1} + (Y_1^2 - Y_2) Y_{n-2} + \ldots \tag{B 7}$$

giving the late Bs in terms of the late and early Ys. Now, as we shall soon see, late Y_ns behave like $(n-1)!$, so the first term dominates. Thus we find, from (B 2),

$$\phi_n(w) \approx \mathrm{d}Y_{n-1}(w)/\mathrm{d}w. \tag{B 8}$$

To find the late Ys we follow chapter 13.7 of D, generalizing slightly to allow for the transition point being of arbitrary order. Substitution of (B 3) into (1) gives the recurrence relation

$$\frac{\mathrm{d}}{\mathrm{d}w} Y_{n+1} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}w^2} Y_n + \frac{R^{-\frac{1}{2}}}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}w^2} R^{\frac{1}{2}}\right) Y_n. \tag{B 9}$$

Anticipating an inverse power form when n is large, we see that the last term is then negligible, yielding the limiting solution

$$Y_n \approx \alpha(n+\beta)!/(2w)^{n+\beta+1} \quad (n \gg 1). \tag{B 10}$$

To identify α and β we have to connect to $Y_0 = 1$, and here the third term of (B 9) plays a crucial role. Because of the increasing zone of influence of the transition point as n increases, we can replace the third term by its form close to the transition point. This is obtained from

$$R^2 \propto z^M, \quad \text{i.e.} \quad w(z) \propto z^{1+M/2} \tag{B 11}$$

so that

$$\frac{R^{-\frac{1}{2}}}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}w^2} R^{\frac{1}{2}}\right) \to -\frac{M(M+4)}{8w^2(M+2)^2} \quad \text{as} \quad z \to 0. \tag{B 12}$$

Substituting into (B 9) we can then solve exactly, and thereby obtain the following formula for $Y_n(w)$, which is valid for all n (w small) and large n (all w):

$$Y_n(w) \approx \frac{(n-\lambda_M)!\,(n-1+\lambda_M)!}{(2w)^n\, n!\,(-\lambda_M)!\,(-1+\lambda_M)!}, \tag{B 13}$$

where

$$\lambda_M \equiv M/2(M+2). \tag{B 14}$$

For large n, Stirling's formula and the reflection formula for factorials give

$$Y_n \approx 2\sin\left\{\frac{M\pi}{2(M+2)}\right\}\frac{(n-1)!}{2\pi(2w)^n} \quad (n \gg 1). \tag{B 15}$$

This has the conjectured form (B 10) with α and β now identified.

Now (B 8) gives the final result

$$\phi_n(w) \approx -2\sin\left\{\frac{M\pi}{2(M+2)}\right\}\frac{(n-1)!}{\pi(2w)^n} \quad (n \gg 1). \tag{B 16}$$

This is the desired result (20) (cf. (4)), with n replacing $2n+2$ and including the factor G_M of (31). There ought to be a simpler derivation of this formula (complete with prefactor) directly from the nonlinear recurrence relation (A 6), but I have not been able to find it.

References

Atkinson, F. V. 1960 *J. Math. Anal. Applic.* **1**, 255–276.

Berry, M. V. 1982 *J. Phys.* A **15**, 2735–2749.

Berry, M. V. 1989*a* *Proc. R. Soc. Lond.* A **422**, 7–21. (Ref. B of the text.)

Berry, M. V. 1989*b* *Publ. Mathématiques IHES, Paris* **68**, 211–221.

Berry, M. V. 1989*c* In *Geometric phases in physics* (ed. A. Shapere & F. Wilczek), pp. 7–28. Singapore: World Scientific.

Berry, M. V. & Howls, C. J. 1990 *Nonlinearity.* (In the press.)

Berry, M. V. & Mount, K. E. 1972 *Rep. Prog. Phys.* **35**, 315–397. (Ref. BM of the text.)

Berry, M. V. & Upstill, C. 1980 *Prog. Optics* **18**, 257–346.

Boyd, W. G. C. 1990 *Proc. R. Soc. Lond.* A. (In the press.)

Bremmer, H. 1951 *Communs pure. appl. Math.* **4**, 105–115.

Dingle, R. B. 1973 *Asymptotic expansions: their derivation and interpretation.* New York and London: Academic Press. (Ref. D of the text.)

Fröman, N. 1966 *Ark. Fys.* **32**, 541–548.

Fröman, N. 1970 *Ann. Phys.* **61**, 451–464.

Fröman, N. & Fröman, P. O. 1965 *JWKB approximation; contributions to the theory.* Amsterdam: North-Holland. (Ref. FF of the text.)

Heading, J. 1962 *An introduction to phase-integral methods.* London: Methuen.

Heading, J. 1977 *Q. Jl Math.* **30**, 281–302.

Jones, D. S. 1990 *Proc. Int. Symp. Asymptotic and Computational Analysis (Manitoba, Winnepeg 1989).* New York: Marcel Dekker.

Kemble, E. C. 1935 *Phys. Rev.* **48**, 549–561.

Littlewood, J. E. 1963 *Ann. Phys.* **21**, 233–242.

Lochak, P. & Meunier, C. 1988 *Multiphase averaging for classical systems.* New York: Springer.

Lundborg, B. 1979 *Math. Proc. Camb. phil. Soc.* **85**, 493–522.

Nekhoroshev, N. N. 1977 *Russ. math. Surv.* **32**, 1–65.

Olver, F. W. J. 1974 *Asymptotics and special functions.* London: Academic Press.

Olver, F. W. J. 1978 *Phil. Trans. R. Soc. Lond.* A **289**, 501–548.

Olver, F. W. J. 1990 *Proc. Int. Symp. Asymptotic and Computational Analysis (Manitoba. Winnipeg 1989).* New York: Marcel Dekker.

Pokrovskii, V. L., Savvynikh, S. K. & Ulinich, F. R. 1958 *Soviet Phys. JETP* **34**. 879–882.

Schep, T. J. 1974 *Physica* **74**, 397–415.

Stokes, G. G. 1864 *Trans. Camb. phil. Soc.* **10**, 106–128. (Reprinted in *Mathematical and physical papers by the late Sir George Gabriel Stokes*, vol. IV, pp. 77–109. Cambridge University Press, 1904.)

Van Kampen, N. G. 1967 *Physica* **35**, 70–79.

Wright, F. J. 1980 *J. Phys.* A **13**, 2913–2928.