# QUANTUM ADIABIATIC ANHOLONOMY

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# Contents



(Illustrations can be found at the end of lectures 1 and 4)

# Lecture 1

My aim is to give a self-contained account of the geometric phase, which is helping to sweep clean a corner of quantum mechanics that was for a long time dusty and obscure. The treatment will be thoroughly elementary - there will be no fibre bundles and Chern numbers - but nevertheless in the spirit of Einstein's injunction: physics should be made as simple as possible, but not simpler.

I do not intend to give comprehensive coverage of what has become a large subject, and so will omit entirely several major topics that have been treated elsewhere, such as nonadiabatic corrections (ref [1] ) and the reaction of quantum systems on their environment [2]. Nor will I give a complete list of references, because with the assistance of Richard Lim I am compiling a comprehensive phase bibliography that will be freely available, and because a reprint collection [3] has just been published.

Details of many of the arguments will be left as exercises for the student. In these notes such exercises will be denoted by {E}.

The geometric phase is based on two ideas: *adiabaticity* and *anholonomy*. Adiabaticity here refers to quantum physics on the border between statics and dynamics. Statics is concerned with *things*, that is persisting entities. In quantumspeak these are eigenstates of the Hamiltonian describing the system's environment. Dynamics is concerned with *happenings*, in this case those induced by changes in the environment. On the border are things in environments that change slowly; such changes are the province of adiabatic theory.

Anholonomy is a geometric concept: the failure of some quantities to come back to their original values when others, which drive them, are forced to return. The failure derives from nonintegrability of the driving law. In this lecture I will concentrate on an example of

anholonomy, introducing it as pure geometry in a way that generalizes easily to quantum mechanics.

Let a vector $\boldsymbol{e}$, lying in the surface of a sphere, be transported round a circuit C (fig.1.1). "Transported" means that the unit radius vector $\boldsymbol{r}$, to which $\boldsymbol{e}$ is attached, is forced round a loop ($\boldsymbol{r}(t)$ with $\boldsymbol{r}(T)=\boldsymbol{r}(0)$) and "in the surface" means $\boldsymbol{e}\cdot\boldsymbol{r}=0$. The law of transport is *parallel transport*: $\boldsymbol{e}$ never twists about $\boldsymbol{r}$. After the circuit, it is found that $\boldsymbol{e}$ has rotated, by an angle $\theta(C)$ that we wish to calculate. $\theta(C)$ embodies anholonomy: $\boldsymbol{e}$ has not returned, even though $\boldsymbol{r}$, which drives it, has.

This parallel transport anholonomy is easy to demonstrate. Hold a pointer at arm's length above your head and pointing forwards. Move your arm down till it is horizontal, then rotate it sideways through a right angle, and finally bring it back up again, taking care never to twist the pointer. You will find that the pointer now points sideways, that is, it has rotated, in spite of never having been turned! I have done this little trick many times, and yet it still seems magical.

To calculate $\theta(C)$ we must give mathematical expression to the law of parallel transport. Let the orthonormal frame $\boldsymbol{r},\boldsymbol{e}_1,\boldsymbol{e}_2$ (with e any fixed combination of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$) rotate with angular velocity $\Omega$, i.e.

$$\dot{\boldsymbol{e}} = \Omega \wedge \boldsymbol{e} \tag{1.1}$$

(the overdot denotes differentiation). $\Omega$ has the general form

$$\Omega = a\boldsymbol{r} + b\dot{\boldsymbol{r}} + c\,\boldsymbol{r} \wedge \dot{\boldsymbol{r}} \tag{1.2}$$

Parallel transport means that $\Omega$ has no component along $\boldsymbol{r}$, so $a=0$. To determine $b$ and $c$ we impose the requirement that $\boldsymbol{e}$ remain perpendicular to $\boldsymbol{r}$, i.e. $(\boldsymbol{e}\cdot\boldsymbol{r})^{\cdot}=0$. This gives {E} $b=0$, $c=1$, so

$$\Omega = r \wedge \dot{r} \tag{1.3}$$

The law of parallel transport is therefore

$$\dot{e} = (r \wedge \dot{r}) \wedge e = - e \cdot \dot{r}\, r \tag{1.4}$$

We express the law in a form suitable for later generalization to quantum mechanics. Define a complex unit vector on the sphere by

$$\phi \equiv (e_1 + \mathrm{i}\, e_2)/\sqrt{2} \tag{1.5}$$

Thus $\phi^* \cdot \phi = 1$ . From the fact that $e_1, e_2$ separately transport according to (1.4), it follows easily {E} that

$$\phi^* \cdot \dot{\phi} = 0 \tag{1.6}$$

Now we can calculate the anholonomy $\theta$(C). Chart the passage of $e_1, e_2$ relative to a *local basis* of unit vectors $u(r)$ ,$v(r)$ defined at each point $r$; (fig.1.2), and so singlevalued round C by construction. For example $u$ and $v$ could point along lines of latitude and longitude. Let the corresponding complex unit vector be

$$n(r) \equiv (u + \mathrm{i}\, v)/\sqrt{2} \tag{1.7}$$

The relation between the transported basis $\phi(t)$ and the local basis $n(r(t))$ is {E} that they differ by a phase factor:

$$\phi(t) = \exp\{- \mathrm{i}\, \theta(t)\}\ n(r(t)) \tag{1.8}$$

where $\theta$ is the angle by which $u$, $v$ must be rotated to coincide with $e_1, e_2$.

The desired $\theta(C)$ is the increment of $\theta$ round C, that is the total rotation of $e_1, e_2$ relative to $u, v$.
We find $\theta$ using (1.6):

$$0 = \phi^* \cdot \dot{\phi} = \exp\{i\theta\}(-i\dot{\theta}\, n^* \cdot n + n^* \cdot \dot{n}) \exp\{-i\theta\} \quad (1.9)$$

Now $n^* \cdot n = 1$ and so $n^* \cdot \dot{n}$ is imaginary, so that

$$\dot{\theta} = \mathrm{Im}\; n^* \cdot \dot{n} \quad (1.10)$$

Thus

$$\begin{aligned} \theta(C) &= \mathrm{Im} \int_0^T n^* \cdot \dot{n}\, dt = \mathrm{Im} \oint_C n^* \cdot d\, n \\ &= -\oint_C v \cdot d\, u \end{aligned} \quad (1.11)$$

where the last equality is a simple {E}.

To obtain the explicit form of $\theta(C)$ for any circuit $r(t)$ we use Stokes' theorem:

$$\theta(C) = \iint_{\partial S = C} V \quad (1.12)$$

where V is the *2-form*

$$\begin{aligned} V &= \mathrm{Im}\, d\, n^* \wedge \cdot d\, n = \mathrm{Im}\, \nabla n^* \wedge \cdot \nabla n \cdot d\, S \\ &= -\nabla v \cdot \wedge \nabla u \cdot d\, S \end{aligned} \quad (1.13)$$

Here $dS$ is an element of area on the sphere, and the integration is over any surface whose boundary is C. The scalar products $\cdot$ act between $n^*$ and $n$, and $u$ and $v$, and the vector products $\wedge$ act between the $\nabla$s. Of course the anholonomy $\theta(C)$ is independent of the basis $u(r)$, $v(r)$. A

convenient choice is the unit vectors corresponding to spherical polar angles, that is

$$u \equiv \frac{r \wedge e_z}{|r \wedge e_z|} \quad , \quad v \equiv \frac{r \wedge u}{r} \tag{1.14}$$

where $e_z$ is a fixed vector and we include the length $r$ of $\boldsymbol{r}$ because it is convenient to have formulae valid in the whole three-dimensional $\boldsymbol{r}$ space rather than just on the unit sphere. From (1.12), a straightforward calculation {E} gives

$$V = \boldsymbol{r} \cdot d\boldsymbol{S} / r^3 \tag{1.15}$$

We thus obtain the final result, from (1.12):

$$\begin{aligned} \theta(C) &= \iint d\boldsymbol{S} \cdot \boldsymbol{r}/r^3 \\ &= \text{flux through C of unit monopole at } \boldsymbol{r} = 0 \\ &= \text{solid angle } \Omega(C) \text{ subtended by C at } \boldsymbol{r} = 0 \end{aligned} \tag{1.16}$$

Anholonomy is the fact that $\theta$ does not vanish. Its origin is the nonintegrability of the parallel transport law (1.4).

An amusing observation (by V J Smith) is that (1.16) equates a plane angle, measured in radians, to a solid angle, measured in steradians=(radians)$^2$. This shows how unnatural is the decision of the International Committee of Weights and Measures [4] that radians and steradians shall be supplementary units in the SI system, with different dimensions, rather than being dimensionless as any physicist would think.

With these preparations we can now consider anholonomy in the transport of quantum states. Let $X \equiv \{X_1, X_2 \ldots\}$ be parameters influencing the quantum state $|\phi\rangle$ of a system. For given $X$, $|\phi\rangle$ is determined up to a phase. Now let $X$ be varied round a cycle C:

($X=X(t)$:($X(T)=X(0)$) (fig.1.3). We seek the phase $\gamma(C)$ accumulated when $|\phi\rangle$ is parallel-transported round C. The definition of transport is obvious from the analogy with vectors: $X$ is analogous to position $\boldsymbol{r}$, and $|\phi\rangle$, a complex unit vector in many-dimensional Hilbert space, is analogous to the two-dimensional complex unit vector $\phi$. Parallel transport of $|\phi\rangle$ is now defined by generalizing (1.6) to

$$\langle \phi | \dot{\phi} \rangle = 0 \tag{1.17}$$

To find $\gamma(C)$we develop the analogy further. Let $|n(X)\rangle$ be a local basis state, defined to be single-valued on and inside C, coinciding with $|\phi\rangle$ up to a phase.Thus $|n\rangle$ generalizes $\boldsymbol{n}$, and instead of (1.8) we now have

$$|\phi(t)\rangle = \exp\{i\,\gamma(t)\}|\,n(X(t))\rangle \tag{1.18}$$

(1.17) gives

$$\dot{\gamma} = i\langle n|\dot{n}\rangle = -\,\mathrm{Im}\langle n|\mathrm{d}n\rangle/\mathrm{d}t \tag{1.19}$$

where we have used the fact that $\langle n|\dot{n}\rangle$is imaginary (which follows from $\langle n|n\rangle=1$. Thus

$$\gamma(C) = -\oint \mathrm{Im}\langle n|\mathrm{d}n\rangle \tag{1.20}$$

(cf. 1.11), and application of Stokes' theorem (cf.1.12) now gives

$$\gamma(C) = -\iint_{\partial S=C} V \tag{1.21}$$

where (cf.1.13)

$$V \equiv \mathrm{Im}\langle d\, n|\wedge|d\, n\rangle \tag{1.22}$$

and $\wedge$ now denotes the wedge product.

This is the main result. It gives the phase anholonomy $\gamma(C)$ as the flux through C of the *2-form* V. Do not be afraid of 2-forms; if you are mathematically innocent (like me), think of them as objects which deliver a number when integrated over an area. A more explicit representation of the 2-form can be written by defining coordinates $X_a$,$X_b$ on the spanning surface S and expanding $|n>$ in an $X$-dependent superposition of *fixed* orthonormal basis states $|\chi_m>$,i.e.

$$|\, n(X\,) > = \sum_m a_m(X\,)|\, \chi_m > \tag{1.23}$$

(in position representation, for example, $m$ would label position and $a_m(X)$ would be the $X$-dependent (complex) wavefunction). Then

$$\begin{aligned} V &= \mathrm{Im} \sum_m d\, a_m{}^* \wedge d\, a_m \\ &= dX_a dX_b\, \mathrm{Im} \sum_m \left( \frac{\partial a_m{}^*}{\partial X_a} \frac{\partial a_m}{\partial X_b} - \frac{\partial a_m{}^*}{\partial X_b} \frac{\partial a_m}{\partial X_a} \right) \end{aligned} \tag{1.24}$$

So far our considerations have been rather abstract and mathematical. To turn them into physics we have to specify a realistic mechanism for parallel-transporting the state $|\phi>$. This will be done in the next lecture. For now I simply mention a case where the abstractions can be interpreted literally, namely the Aharonov-Bohm effect [5,6]. Here $|\phi>$ is the state of an electron, the parameters $X$ are its position in ordinary 3-space, the 2-form V is proportional to the field of a

single line of ordinary magnetic flux, and the anholonomy $\gamma$ is the phase change of the electron round a circuit C of the flux line.

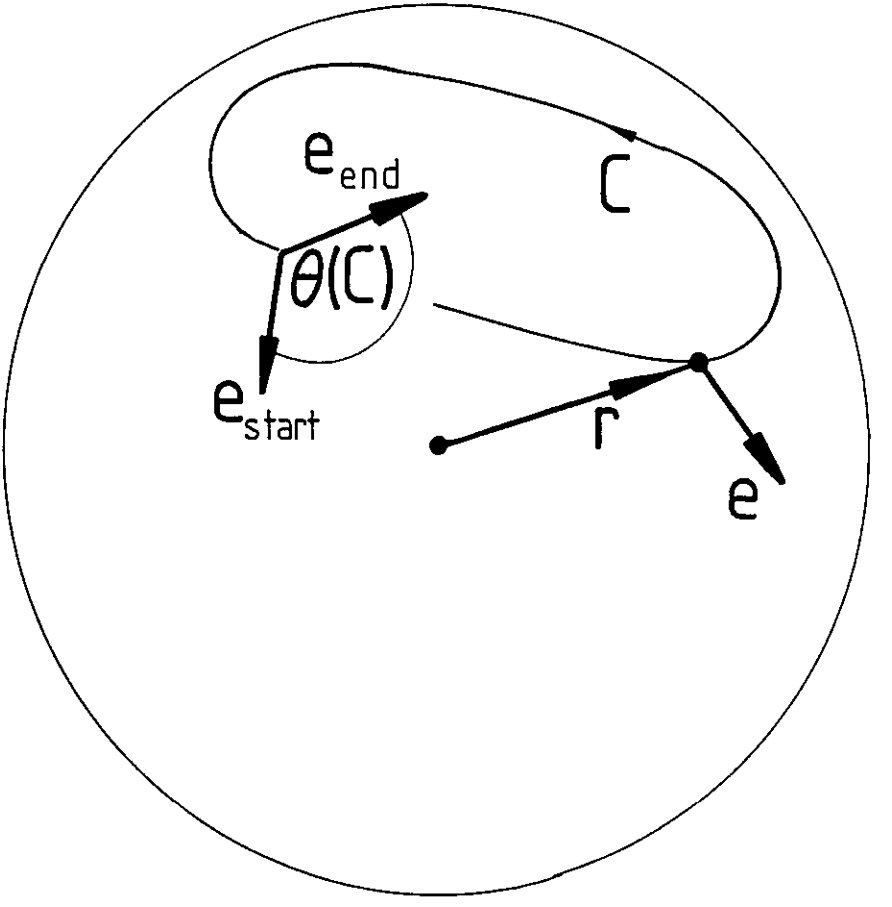


Fig. 1.1

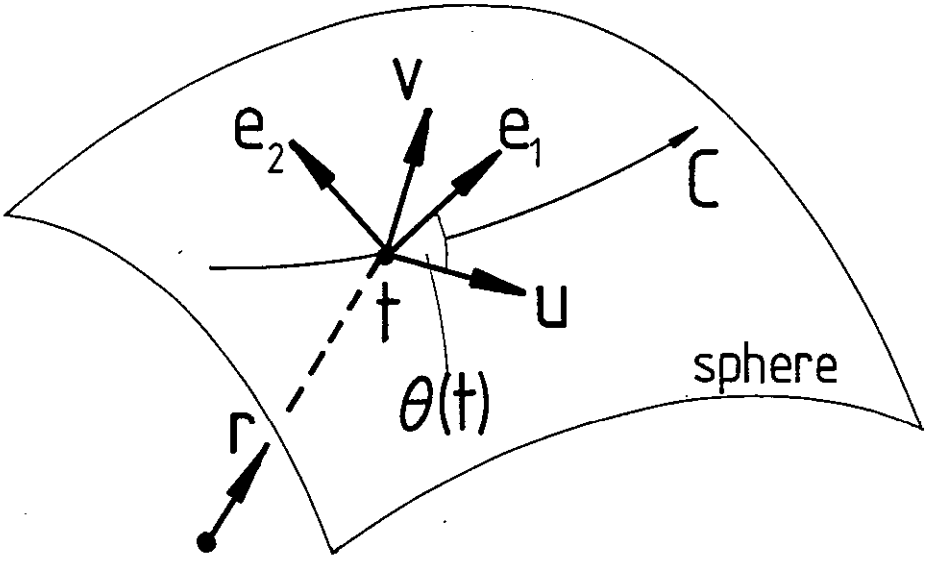


Fig. 1.2

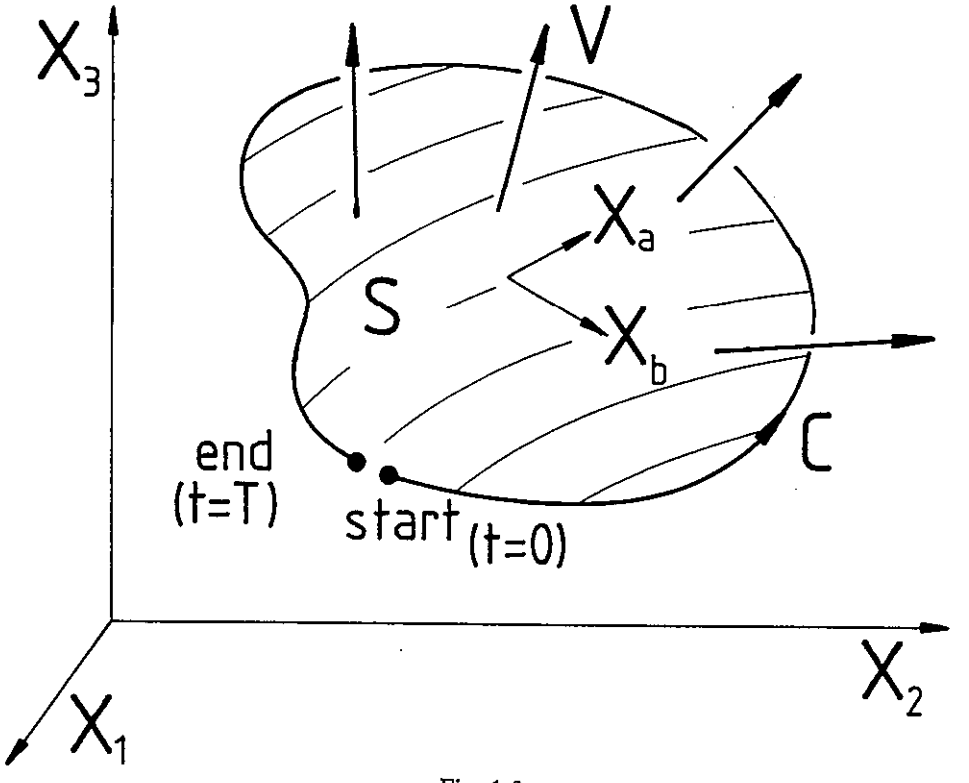


Fig. 1.3

# Lecture 2

In 'reality', quantum states are transported not by any convenient mathematical rule but by a Hamiltonian operator $H$ acting via the time-dependent Schrödinger equation. One way [6] to implement the parallel-transport rule (1.17) is to incorporate the changing parameters $X$ into $H$ and make the changes occur *slowly*. Thus we have the slowly-cycled Hamiltonian

$$H = H(X(t)) \qquad (0 \le t \le T, T \to \infty, \quad X(0) = X(T)) \tag{2.1}$$

and we can invoke the quantum *adiabatic theorem.*

This states that the time-dependent Schrödinger equation is satisfied by eigenstates of the frozen Hamiltonian $H$ at each instant, multiplied by the usual oscillatory time factor. Thus the adiabatic states (labelled $n$) are

$$|\Psi_n(t)> = \exp\{-i\int_0^t dt\, E_n(X(t))/\hbar\}\,|\phi_n(t)> \tag{2.2}$$

where $|\phi_n>$ (assumed nondegenerate) satisfies the eigenequation at $t$:

$$H(X)|\phi_n> = E_n(X)|\phi_n> \tag{2.3}$$

The phase $-\int dt E_n(t)/\hbar$ of the oscillatory factor is the *dynamical phase*; it generalizes the familiar "$-\omega t$" of any oscillatory process, and is present even if $X$ is held fixed. (Of course the simple form (2.2), in which $|\Psi(t)>$ clings to individual eigenstates $|\phi_n>$ ,is an exceptional case, valid only in the adiabatic limit $T\to\infty$. Otherwise the changing $H$ couples different $|\phi_n>$ via transitions, whose study is the usual business of time-dependent quantum mechanics.)

Now comes the central point. Equation (2.2) does not constitute a complete specification of the adiabatic state $|\Psi(t)\rangle$, because the eigenequation (2.3) provides no means to connect its solutions $|\phi\rangle$ at different parameters $X$: we need a transport law. This is provided by the time-dependent Schrödinger equation, projected on the subspace $|\phi_n\rangle$:

$$\begin{aligned} 0 &= \langle \Psi_n(t)|(H - \mathrm{i}\hbar\,\partial_t)|\,\Psi_n(t)\rangle \\ &= \langle \Psi_n(t)|(E_n - \mathrm{i}\hbar\,\partial_t)\exp\{-\mathrm{i}\textstyle\int_0^t \mathrm{d}\,t\,E_n/\hbar\}|\phi_n(t)\rangle \\ &= \langle \phi_n(t)|\,\dot{\phi}_n(t)\rangle \end{aligned} \tag{2.4}$$

This is exactly the parallel-transport law (1.17), which we found to be nonintegrable. At the end of the cycle, $|\Psi\rangle$ as given by (2.2) therefore acquires a phase from the non-return of $|\phi_n(t)\rangle$ as well as the dynamical phase. thus

$$|\,\Psi_n(T)\rangle = \exp\{-\mathrm{i}\int_0^T \mathrm{d}\,t\,E_n(X(t/T))/\hbar\}\exp\{\mathrm{i}\,\gamma_n(\mathrm{C})\}|\,\Psi_n(0)\rangle \tag{2.5}$$

Here $\gamma_n(\mathrm{C})$ is the *geometric phase*, given by (1.21) and (1.22) with $|n(X)\rangle$ now being any solution of (2.2) that is single-valued in $X$ space on C and on the chosen spanning surface S.

The reason for calling $\gamma_n(\mathrm{C})$ geometric is that it depends only on the geometry of C in $X$ space (and on which state $n$ is being transported), and not on the rates with which different parts of C are traversed (assuming of course that the transport is slow). In particular, $\gamma$ is independent of the adiabatic parameter $T$, unlike the dynamical phase which $\{\mathrm{E}\}$ increases linearly with $T$.

It is amusing to see how the geometric phase is contained in the path-integral representation of the evolving state. This was done with J.Hannay and M.Wilkinson in 1983 but never published (but see [45]). Introducing the time-ordered product for the evolution over the cycle, and dividing the time interval into many small steps of duration $\delta \equiv T/N$ where $N \to \infty$, we have

$$\begin{aligned} |\Psi(T)> &= \mathrm{T}\exp\{-\mathrm{i}H(t)/\hbar\}|\Psi(0)> \\ &= \prod_{k=1}^{N} \exp\{-\mathrm{i}\,\delta\, H(t_k)/\hbar\}\,|\Psi(0)> \end{aligned} \tag{2.6}$$

At each $t_k$ we introduce the complete set of instantaneous eigenstates $|m(X_k)>$ of $H(t_k)$ (single-valued solutions of (2.3)). Thus

$$\begin{aligned} |\Psi(T)> = \prod_{k=1}^{N} \sum_{m} \exp\{-\mathrm{i}\,\delta E_m(t_k)/\hbar\}\,|m(X_k)><m(X_k)|\times \\ \times|\Psi(0)> \end{aligned} \tag{2.7}$$

This is exact, but now we invoke the adiabatic approximation to eliminate terms $m \neq n$ in all the intermediate sums, where $|\Psi(0)>=|n(X(0))>$ is the initial state. Then all the factors involving the $E_m$ combine to give the dynamical phase, leaving

$$\begin{aligned} |\Psi(T)> &= |\Psi(0)> \exp\{-\mathrm{i}\int_0^T \mathrm{d}t\, E_n(t)/\hbar\}\times \\ &= \times <n(X_N)|n(X_{N-1})> \ldots <n(X_1)|n(X_0)> \end{aligned} \tag{2.8}$$

Terms in the product have the form

$$\begin{aligned} <n(X_{j+1})|n(X_j)> &= <n(X(t+\delta))|n(X(t))> \\ &\approx (<n| + \delta <\dot{n}|)|n> \\ &= 1 + \delta <\dot{n}|n> \\ &\approx \exp\{\delta <n|\dot{n}>\} = \exp\{-\mathrm{i}\mathrm{Im}<n|\mathrm{d}n>\} \end{aligned} \tag{2.9}$$

By (1.20), the accumulation of these factors gives precisely $\gamma_n(C)$.

By (1.22), the geometric phase for the $n$'th state is the flux through C of a 2-form that we now call $V_n(X)$:

$$V_n(X) = \mathrm{Im} < \mathrm{d}n| \wedge |\mathrm{d}n > \tag{2.10}$$

This is the mathematical object at the heart of the whole subject. It sits in parameter space, waiting to be brought to life as a phase when $H$ is cycled.

Now we describe some interesting properties of $V_n$. The first concerns its singularities. To see where these are (in $X$ space), we introduce the complete set of eigenstates $|m(X)>$ of $H(X)$.

$$V_n = \mathrm{Im} \sum_{m \neq n} < \mathrm{d}n|\, m > \wedge < m|\mathrm{d}n > \tag{2.11}$$

Note the exclusion of the state $m=n$ {E}. From the eigenequation for $H$ it is possible {E} to derive

$$< m|\mathrm{d}n > = - \frac{< m|\mathrm{d}H|\, n >|}{E_n - E_m} \qquad (m \neq n) \tag{2.12}$$

so that

$$V_n = \mathrm{Im} \sum_{m \neq n} \frac{< n|\mathrm{d}H|\, m > \wedge < m|\mathrm{d}H|\, n >}{(E_n - E_m)^2} \tag{2.13}$$

This shows that the singularities of $V_n$ occur where the spectrum of $H(X)$ has *degeneracies* involving the transported state $|n>$. Later we shall determine the precise nature of the singularity.

The other properties of $V_n$ concern *gauge invariance*. In its simplest form, this is the fact that $V_n$ is independent of the choice of single-

valued eigenstates $|n(X)>$. Different choices are related by a single-valued $X$-dependent phase factor, and we have {E}

$$\begin{aligned} &\mathrm{Im} < \mathrm{d}n' \wedge \mathrm{d}n' > = \mathrm{Im} < \mathrm{d}n \wedge \mathrm{d}n > \\ &\quad \text{if } |n'(X)> \equiv \exp\{\mathrm{i}\chi(X)\}|n(X)> \end{aligned} \tag{2.14}$$

By contrast, the 1-form $<n|dn>$ (cf 1.20) does not possess this gauge invariance.

Another quantity is invariant under the gauge transformatiion $|n>\rightarrow|n'>$ To find out what it is, introduce coordinates $X_i$ in parameter space and write

$$V = V_{ij}\ \mathrm{d}X_i \wedge \mathrm{d}X_j \tag{2.15}$$

where $V_{ij}$ is the antisymmetric second-rank tensor {E}

$$V_{ij} = 2\mathrm{Im} < \partial_i n| \partial_j n > \tag{2.16}$$

What about the real (symmetric) part of the tensor? It is easy to show {E} that this is not gauge-invariant, but that the following quantity is:

$$g_{ij} = \mathrm{Re} < \partial_i n\,(1 - |n><n|)\,\partial_j n > \tag{2.17}$$

The interpretation [2] of $g_{ij}$ is as a *metric tensor* in $X$ space, measuring distance $ds$ between states $|n>$ at neighbouring points $X$ and $X+dX$ in the most natural gauge-invariant way:

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}X_i\,\mathrm{d}X_j = 1 - |< n(X)|\, n(X + \mathrm{d}X) >|^2 \tag{2.18}$$

(Both tensors, $V_{ij}$ and $g_{ij}$ also contribute in important ways [2] to the reaction of the quantum system on its environment, in the form of the dy-

namics of the $X_j$ when regarded self-consistently not as parameters but as quantum variables.)

In elementary physics the most familiar gauge invariance is that of *magnetic field*

$$\boldsymbol{B}(\boldsymbol{r}) = \nabla \wedge \boldsymbol{A}(\boldsymbol{r}) \tag{2.19}$$

under the transformation of the vector potential $\boldsymbol{A}$ to $\boldsymbol{A}'$ where

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla_{\boldsymbol{r}} \Lambda(\boldsymbol{r}) \tag{2.20}$$

and $\Lambda$ is a single-valued scalar function of $\boldsymbol{r}$. This type of gauge transformation has to be considered when studying the geometric phase for a charged particle whose slowly-cycled environment (parameterized by $X$) includes a magnetic field.

We expect $\gamma_n(C)$ to depend on $\boldsymbol{A}$ because $\boldsymbol{A}$ occurs in the Hamiltonian. But all physics must be invariant under (2.20), even when this transformation is parameter-dependent, i.e. $\Lambda=\Lambda(r;X)$. The effect of $\Lambda$ is to contribute a phase factor to the wavefunctions $\langle r|n(X)\rangle$ in position representation. This is so similar to the type of transformation in (2.14) - a phase factor multiplying the Hilbert-space vector - that it came as a surprise to find that not only the 2-form $V_n$ but also $\gamma_n(C)$ itself *change* under (2.20), the transformation law for $V_n$ being {E}

$$V'_n(X) = V_n + d \wedge \langle n|d\,\Lambda|n\rangle \tag{2.21}$$

Physics is saved from inconsistency, however, by the fact that a parameter-dependent $\Lambda$ also generates an *electric* field (through the slow change in $X$) unless it is allowed to transform the scalar potential as well. In the resulting complete gauge transformation, the *dynamical*

phase is changed in a way that exactly compensates the effect of (2.21). Details are given in [7].

# Lecture 3

At first the geometric phase appears unobservable because its detection based on (2.5) seems to require superposing the system at $t=T$ on its former self (at $t=0$), which is impossible. There are however at least two ways in which $\gamma_n(C)$ can be - and has been - measured.

The way that was originally suggested [6] was by interferometry. A system in the state $|n\rangle$ (e.g. a coherent beam of particles) is split at $t=0$ into two subsystems. One is slowly cycled and the other not. Both subsystems will acquire dynamical phases, say $\gamma_{1\mathrm{dyn}}$ and $\gamma_{2\mathrm{dyn}}$, but the cycled one will, in addition, gain a geometric phase. If the subsystems are subsequently recombined, the intensity of their superposition is

$$\begin{aligned} I &\propto \left|\exp\{i(\gamma_{1\,\mathrm{dyn}} + \gamma_n(C)\} + \exp\{i\,\gamma_{2\mathrm{dyn}}\}\right|^2 \\ &= 4\cos^2\{\tfrac{1}{2}[\gamma_{1\mathrm{dyn}} - \gamma_{2\,\mathrm{dyn}} + \gamma_n(C)]\} \end{aligned} \tag{3.1}$$

Therefore $\gamma_n(C)$ can be detected as a shift of interference fringes - as in the Aharonov-Bohm experiments [5].

One can say that the interferometric experiments involve the same state and two Hamiltonians (one for each subsystem). The second class of experiments, on the other hand, involves (at least) two states and the same Hamiltonian. Let the initial state be a superposition of two eigenstates $|m\rangle$ and $|n\rangle$ of $H(0)$ :

$$|\Psi(0)\rangle = a_m|m\rangle + a_n|n\rangle \tag{3.2}$$

This is a non-stationary state, which after the cycle has become, in an obvious notation,

$$| \Psi(T) > = a_m | m > \exp\{i[\gamma_{m\mathrm{dyn}} + \gamma_m(C)]\} + \\ + a_n | n > \exp\{i[\gamma_{n\mathrm{dyn}} + \gamma_n(C)]\} \tag{3.3}$$

Now measure the expectation value of some operator $A$ that does not commute with $H$; we find {E}

$$< \Psi(T)|A|\Psi(T) > = |a_n|^2 < n|A|n > + |a_m|^2 < m|A|m > + \\ + 2\mathrm{Re}\, a_n^* a_m < n|A|m > \exp\{i[\gamma_{n\mathrm{dyn}} - \gamma_{m\mathrm{dyn}} + \gamma_n(C)_- \gamma_m(C)]\} \tag{3.4}$$

The interference term reveals the difference of the phase shifts experience by the two constituent states, and of course this includes the difference of their geometric phases.

Many of the experiments that have been carried out [3] involve the *turning of spinning particles.* I will now work out the underlying theory, which is useful in several other applications as well. Consider a particle with spin $l$ (integer or half-integer), described by the vector of three $(2l+1)\times(2l+1)$dimensional angular-momentum matrices $\sigma$ satisfying the familiar commutation rule

$$\sigma \wedge \sigma = i\sigma \tag{3.5}$$

Let $H$ at each instant be rotationally symmetric about some direction described by a vector in the parameter space $\boldsymbol{R}=(X,Y,Z)$ i.e.

$$H(t) = F\{\boldsymbol{R}(t) \cdot \sigma\} \tag{3.6}$$

An example is a particle with magnetic moment $\mu$ in a magnetic field $\mu\boldsymbol{R}$, for which $F$ is linear and $\boldsymbol{R}=\mu\boldsymbol{B}$.

To find the phase 2-form $V_n(R)$ (here regarded as a vector in $R$ space), we use the sum-over-states (2.13), replacing d by $\nabla$. The states $|n>$ are $(2l+1)$-component spinors labelled by the component of angular momentum along $R$ ($n$ runs from $-l$ to $+l$) and the eigenvalues are $F(nR)$ where $R\equiv|R|$. In (2.13) the dependence on $F$ cancels and we obtain {E}.

$$V_n(R) = \frac{1}{R^2}\mathrm{Im}\sum_{m\neq n=-l}^{l}\frac{<n|\sigma|m>\wedge<m|\sigma|n>}{(n-m)^2} \tag{3.7}$$

Now, the matrix elements are zero unless $m=n$ or $m=n\pm1$ {E} and the contribution $m=n$ is excluded by the $\wedge$. Therefore we can take $(n-m)^2=1$ out of the sum, eliminate the sum-over states and use (3.5) to get

$$V_n(R) = \frac{1}{R^2}\mathrm{Im}<n|\sigma\wedge\sigma|n> = \frac{1}{R^2}<n|\sigma|n>$$
$$= nR/R^3 \tag{3.8}$$

where the last equality follows from the fact that $|n>$ is an eigenstate of the component of $\sigma$ along $R$, in which the perpendicular components have zero expectation value.

The 2-form is therefore the field of a *monopole* of strength $n$, situated at $R=0$. The geometric phase is minus the flux through C of this monopole, that is

$$\gamma_n(C) = -n\,\Omega(C) \tag{3.9}$$

The simplest case is spin 1/2, for which $\sigma$ are the three Pauli matrices. This describes (for example) *neutrons*, whose geometric phase was measured in an ingenious experiment by Bitter and Dubbers [8]. They sent a beam of neutrons along the $x$ direction in a helical magnetic field

$$B = B\{\cos\theta, \sin\theta\cos(2\pi x/L), \sin\theta\sin(2\pi x/L)\} \quad (0 \le x \le L) \tag{3.10}$$

The moving neutrons see a $B$ that changes with time, sweeping out a cone with solid angle {E}

$$\Omega(C) = 2\pi(1 - \cos\theta) \tag{3.11}$$

Their experiment was of the second type described above, with the initial state being polarized along z.. Thus

$$|\Psi(0)> = \begin{pmatrix}1\\0\end{pmatrix} = \cos(\theta/2)|+> + \sin(\theta/2)|->$$ (3.12)

where

$$|+> = \begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)\end{pmatrix} \quad \text{and} \quad |-> = \begin{pmatrix}\sin(\theta/2)\\ -\cos(\theta/2)\end{pmatrix}$$ (3.13)

are the eigenstates of $H = \mu\sigma\cdot B$ along the initial direction of $B$ {E}. At the end of the cycle,

$$|\Psi(T)> = \cos(\theta/2)\exp\{-i\alpha\}|+> + \sin(\theta/2)\exp\{i\alpha\}|-> = \begin{pmatrix}\cos\alpha - i\sin\alpha\cos\theta\\ -i\sin\theta\cos\alpha\end{pmatrix}$$ (3.14)

where

$$\alpha \equiv \mu BT/2\hbar + \Omega/2 \tag{3.15}$$

(the first term is the dynamical phase). Bitter and Dubbers measured $\sigma_z$ by again passing the beam through a polarizer. Its expectation value is {E}

$$< \sigma_z > = < \Psi(T) | \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} | \Psi(T) > = \tfrac{1}{2}(\cos^2\theta + \cos 2\alpha \sin^2\theta) \tag{3.16}$$

They detected $\Omega$ (and verified the formula 3.11) through its effect on $\alpha$ (3.15), by measuring the phase of the oscillations of the $\cos 2\alpha$ term as $B$ was varied.

The spin-1/2 case has the wider importance that it describes any 2-state quantum system. The restriction to rotational symmetry, and the generality embodied in the function $F$ in (3.6), are here unnecessary because the most general 2-state Hamiltonian is (up to a trivial multiple of the identity)

$$H = \sigma \cdot R = \tfrac{1}{2}\begin{pmatrix} Z & X + \mathrm{i}Y \\ X - \mathrm{i}Y & -Z \end{pmatrix} \tag{3.17}$$

An interesting recent application is to the *electron microscopy of crystal dislocations* [9] in the 2-beam approximation: a dislocation causes distortion and disruption of micrograph fringes, which can be interpreted entirely as an effect of the geometric phase.

Another application is to *degeneracies*. From the sum-over-states (2.13) it was clear that degeneracies of the transported state $|n>$ are singularities of $V_n(X)$. We can discover the nature of the singularities by realising that close to the parameters $X^*$ where they occur the dominant contribution $m$ to (2.13) comes from the state degenerating with $n$ (we assume the typical situation where there is only one such state). Therefore we have, locally, a 2-state problem, for which a linear change of parameters brings the non-trivial part of $H$ to the form (3.17). Application of (3.8) for $n$=1/2 shows at once that the singularity of $V_n$ is a monopole with strength $\pm$ 1/2. (The sign depends on whether $|n>$ degenerates with the state above or below, and whether the transforma-

tion to local parameters $\boldsymbol{R}$ is proper or improper. Originally [6] I left out the latter condition; it was stated correctly by Simon [10].) Mondragon and I [7] have explored the details of the monopole singularities in several numerical examples.

An important special case of degeneracy occurs when $H$ is real (for example when the dynamics at each instant has time-reversal symmetry). Then in the local model (3.17), $Y$=0, and circuits C lie in the $XZ$ plane with their spanning surfaces like hemispheres, which have solid angle $\Omega=2\pi$ (if C encloses the degeneracy). The geometric phase $\gamma=n\Omega=\pm(2\pi)/2=\pm\pi$ therefore contributes a sign change (which is of course the only phase change that a real eigenfunction can have). Elsewhere [2] I have described some early history associated with this sign change, in the differential geometry of surfaces and in molecular physics.

Now I want to discuss some experiments involving *photons*, whose interpretation has been controversial. These particles have spin 1, so $\sigma$ are 3x3 matrices; the eigenvalues of $\sigma\cdot\boldsymbol{R}$ are $+R$, 0, $-R$. Photons have no magnetic moment and so cannot be turned with a magnetic field. But they have the property of *helicity*: along their propagation direction $\boldsymbol{e}_k$ they may have states with $\sigma\cdot\boldsymbol{e}_k=\pm1$ but not zero. Therefore the photon spinor can be turned by turning its propagation vector $\boldsymbol{k}$. Chiao and Wu [11] and Tomita and Chiao [12] had the clever idea of achieving this with the light in a coiled optical fibre. $\boldsymbol{k}$ is the forward tangent direction of the fibre, and can be cycled with a coil whose ends are parallel. The geometric phases for the two helicities would then be (cf.3.9)

$$\gamma_{\pm1}(C) = \mp\, \Omega(C) \tag{3.18}$$

where $\Omega$ is the solid angle swept out by $\boldsymbol{e}_k$ on its unit sphere.

Momentarily choosing the $z$ axis along $\boldsymbol{k}$, we can write the helicity states as

$$|+1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.19}$$

Classically these correspond to *beams of circularly polarized light*, whose (generally complex) unit polarization vector $\boldsymbol{e}$, in the electric field

$$\boldsymbol{E} = |\boldsymbol{E}|\,\mathrm{Re}\,\boldsymbol{e}\,\exp\{\mathrm{i}[\boldsymbol{k}z - \omega t]\} \tag{3.20}$$

is related to the spinor $|\Psi\rangle$ by

$$|\Psi\rangle = \begin{pmatrix} e_x - \mathrm{i}\,e_y \\ 0 \\ e_x + \mathrm{i}\,e_y \end{pmatrix} / \sqrt{2} \tag{3.21}$$

The states $|+\rangle$ and $|-\rangle$ correspond to right and left polarization, with $e_y = +\mathrm{i}e_x$ and $-\mathrm{i}e_x$.

Tomita and Chiao fed their fibre with light *linearly* polarized in direction $\alpha$, i.e. $e_x = \cos\alpha$, $e_y = \sin\alpha$. This is a superposition of the two helicity states:

$$|\Psi(0)\rangle = \begin{pmatrix} \exp\{-\mathrm{i}\,\alpha\} \\ 0 \\ \exp\{+\mathrm{i}\,\alpha\} \end{pmatrix} / \sqrt{2} = \exp\{-\mathrm{i}\,\alpha\}|+\rangle + \exp\{\mathrm{i}\,\alpha\}|-\rangle \tag{3.22}$$

They were careful to coil their fibre without twisting it. Because of this, there was no stress-induced circular birefringence and the two helicities propagated at the same speed. In other words, the two components in (3.22) acquire *identical* dynamical phases (equal to $kL$, where $L$ is the length of the fibre). But their geometric phases are equal and opposite, leading to emergent light with state

$$|\Psi(L)> = \exp\{i\,kL\}(\exp\{-i(\alpha + \Omega)\}|+> + \exp\{i(\alpha + \Omega)\}|->) \quad (3.23)$$

This is again linearly polarized, but along $\alpha+\Omega$ rather than $\alpha$.

So the effect of the geometric phase is to rotate the direction of linear polarization by $\Omega$. In other words, the coiling induces 'geometric optical activity' or 'geometric circular birefringence'. Experiment [12] verifies the effect very accurately. Recalling now the geometry of lecture 1, we can rephrase the description in yet another way: quantum anholonomy of photon eigenstates is equivalent to *parallel transport* of linear polarization along the fibre.

This raises the question: is the effect quantum or classical? Several authors have [13,14] argued that it is classical. Chiao and Wu

> "would rather think of these effects as topological features of classical Maxwell theory which originate at the quantum level, but survive the correspondence principle limit ($h\to 0$) into the classical level".

Further questions now arise: *where* in Maxwell's theory is the anholonomy? Why is it so tricky to understand the effect classically, yet so straightforward quantum-mechanically? I have answered the first question [15] by deriving the parallel transport of linear polarization from Maxwell's equations for a fibre (the result cannot be justified by appeal to the known parallel transport along curved *rays* because these

experiments involve monomode fibres, for which geometrical optics is not a valid approximation).

The second question is a pseudo-problem, and so in fact is the whole discussion of whether the fibre effect is classical or quantum, because for the optical processes considered here the quantum and classical descriptions are the same! This is the view of Feynman [16]

> "The photon equation is just the same as Maxwell's equations...."

To appreciate the intuition underlying this assertion, consider Maxwell's equations in a fibre:

$$\dot{\boldsymbol{D}} = \nabla \wedge \boldsymbol{H};\ \dot{\boldsymbol{B}} = -\nabla \wedge \boldsymbol{E};$$
$$\boldsymbol{B} = \mu_0 \boldsymbol{H};\ \ \boldsymbol{D} = \varepsilon(r)\boldsymbol{E} \tag{3.24}$$

Here the dielectric permittivity function $\varepsilon(r)$ describes the glass refractive index which confines the light inside the fibre. These equations have the form

$$\partial_t(\text{fields}) = (\text{matrix linear in } \nabla) \times (\text{fields}) \tag{3.25}$$

Multiplying by i$h$ gives

$$\mathrm{i}\hbar\partial_t(\text{fields}) = (\text{matrix linear in } p = -\mathrm{i}\hbar\nabla) \times (\text{fields}) \tag{3.26}$$

and makes Maxwell's classical equations look like Schrödinger's quantum equation.

To make this interpretation legitimate, we have to ensure that the operator on the right of (3.26) is Hermitian. Several authors [17-20] have carried out this programme, but their results are useless here

because they are restricted to propagation in free space, for which the permittivity is constant so there can be no guiding of the light. In discussion with A.Pines I have however found the following exact (and essentially unique) Schrödinger implementation of the fibre equations (3.24) (which also allows the magnetic permeability $\mu_o$ to be reptaced by a function $\mu(r)$.

Define the six-component spinor

$$|\Psi(r,t)> \equiv \begin{pmatrix} M_+ \\ M_- \end{pmatrix} \tag{3.27}$$

where

$$M_\pm = \varepsilon^{1/2} E \pm i \mu^{1/2} H \tag{3.28}$$

In terms of the refractive index

$$n(r) = \left[ \frac{\varepsilon(r)\, \mu(r)}{\varepsilon_0\, \mu_0} \right]^{1/2} \tag{3.29}$$

we define the modified momentum operator

$$\Pi \equiv n^{-1/2}(r)\, p\, n^{-1/2}(r) \tag{3.30}$$

and the 'inhomogeneity vector'

$$\xi \equiv \frac{1}{4n(r)} \nabla \log\left( \frac{\varepsilon(r)}{\mu(r)} \right) \tag{3.31}$$

It can now be shown {E} that $|\Psi>$ obeys a time-dependent Schrödinger equation with Hamiltonian

$$H = c \begin{pmatrix} \Pi \cdot \sigma & i\hbar\xi \cdot \sigma \\ -i\hbar\xi \cdot \sigma & -\Pi \cdot \sigma \end{pmatrix} \tag{3.32}$$

where $\sigma$ is the following vector of spin-1 matrices:

$$\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.33}$$

$H$ is manifestly Hermitian.

To apply this 'Schrödinger lookalike' formalism to the fibre experiments, we replace $|\Psi(r,t)\rangle$ by a wavepacket travelling with the speed of light in the fibre. This allows the replacement

$$c\Pi \rightarrow c\hbar k/n = \hbar\omega e_k(t) \tag{3.34}$$

where $\omega$ is the frequency of the light and $e_k(t)$ the fibre direction at the place reached by the packet at the time $t$. Then we reject the off-diagonal terms in (3.32), on the grounds that the inhomogeneity vector $\xi$ in (3.31) is perpendicular to that helicity component $\sigma$ which in the adiabatic (guiding) approximation is along the fibre. Now $M_+$ and $M_-$ are uncoupled in $|\Psi\rangle$, giving the two separate Schrödinger equations

$$i\hbar\partial_t M_\pm(t) = \pm \hbar\omega\, e_k(t) \cdot \sigma\, M_\pm(t) \tag{3.35}$$

This describes 'photons' (light particles in the sense Newton meant) with energy $E=\hbar\omega$. The eigenstates with $M_\pm$ correspond {E} to light with the two hands of circular polarization. Each of (3.35) also has an eigenstate with $E=-\hbar\omega$. These 'antiphotons' are not new particles but redundant descriptions of the backward-travelling particles with positive energy and $k$ reversed (the redundancy can be traced to the fact that the physical fields are real - cf. 3.20).

With the two equations (3.35) we have distilled from Maxwell's equations a description of light in a fibre as a stream of spin-1 particles driven by a time-dependent Hamiltonian of the form (3.6). The anholonomy (3.18) follows at once, justifying the argument of Chaio and Wu.

Optical anholonomy of a different sort was discovered long ago by Pancharatnam [21] in strikingly original work whose significance went unappreciated until recently [22,23]. Here I have space only for a brief description. Instead of cyclically changing the direction $\boldsymbol{k}$ of light in a fixed state of polarization, Pancharatnam was concerned with cyclic changes in the polarization of light travelling in a fixed direction.

Such light can be represented as a 2-component spinor, whose state of polarization is an eigenstate of (3.17) determined by the direction $\mathbf{R}/R$, which is a point on the Poincaré sphere. A polarization cycle is then a loop C on this sphere, Pancharatnam showed that there is an associated geometric phase $\Omega(C)/2$, where $\Omega$ is the solid angle subtended by C at the centre of the sphere, and thereby anticipated our result (3.9) for $n=1/2$.

I have given elsewhere [23] the 'reconciliation' of the Pancharatnam $\Omega/2$ (on the Poincaré sphere) and the Chiao-Wu-Tomita $\Omega$ (on the sphere of directions $e_k$). It is of course possible to combine polarization and direction cycles; Bhandari [24] describes how to calculate the resulting geometric phases.

# Lecture 4

Suppose the cycled Hamiltonian has a classical limit, corresponding to a system with $N$ freedoms. Then instead of being an operator, $H$ is a function

$$H = H(x; X(t)) \tag{4.1}$$

where

$$x \equiv (q,p) = (q_1 \cdots q_N, p_1 \cdots p_N) \tag{4.2}$$

is position in the $2N$-dimensional phase space. It is natural to expect the quantum phase anholonomy $\gamma_n(C)$, and the underlying 2-form $V_n(X)$, to be mirrored by anholonomy in the classical system. Mystery still shrouds the nature of that anholonomy in the general case, but Hannay [25, see also 26] discovered what it is in the important special case of systems whose motion at each fixed $X$ is *multiply periodic.*

Hannay reasoned as follows. A quantum eigenstate for fixed $X$ is like an oscillator:

$$|\psi> = |n> \exp\{-i\omega t\}, \text{ where } \omega = E_n/\hbar \tag{4.3}$$

We have seen that it exhibits anholonomy when $X$ is cycled. Now there are of course oscillators in classical mechanics too, so we should likewise expect them to possess anholonomy. Instead of $|\Psi>$ we will have the oscillator coordinate, and instead of $|n>$ we will have the oscillator amplitude. The phase will now represent an *angle* $\theta$. This may be literally an angle in space - as with a wheel - or, more commonly, an abstract angle variable [27] chosen to make the motion uniform in phase space - as with a swinging pendulum. In an adiabatically cycled system, the anholonomy should show up as a shift in the total change of $\theta$:

$$\theta(t) - \theta(0) = \int_0^T \mathrm{d}t\, \omega(X(t)) + \Delta\theta(C) \tag{4.4}$$

The first term is the dynamical angle change, which is the obvious generalization of $\omega T$ for changing $\omega$. The second term is the geometric angle shift, now called Hannay's angle. Of course for multiply-periodic motion there is more than one angle, the maximum (fully integrable motion) being $N$; we shall denote the j'th angle by $\Delta\theta_j$.

By geometric arguments, Hannay found a formula [25] for $\Delta\theta_j$ which has spawned a considerable amount of new classical phase-space geometry [28-30]. I reformulated Hannay's derivation, and showed [26] how, in the semiclassical limit, $\Delta\theta_j$ is related to the quantum phase. Here I will not repeat those arguments,but will instead outline a new, general approach to the semiclassical limit of the 2-form, developed last year after conversations with M.Wilkinson. As well as reproducing known results this gives a hint of what might happen in the *nonintegrable* case, where the motion is chaotic rather than periodic, and no angle variables exist.

We start with the following formula (intermediate between 1.21 and 1.22):

$$V_n(X) = \mathrm{Im}\ \mathrm{d} \wedge \langle n|\mathrm{d}n\rangle \tag{4.5}$$

to find $\mathrm{d}|n\rangle$ we differentiate the eigenequation for $H(X)$:

$$(\mathrm{d}E_n - \mathrm{d}H)|n\rangle + (E_n - H)|\mathrm{d}n\rangle = 0 \tag{4.6}$$

where

$$\mathrm{d}H \equiv H(X + \mathrm{d}X) - H(X) \tag{4.7}$$

Thus

$$|\mathrm{d}\, n> = \lim_{\varepsilon \to 0} \frac{1}{H - E_n - \mathrm{i}\,\varepsilon} (\mathrm{d}\, E_n - \mathrm{d}\, H)|\, n> \tag{4.8}$$

where $\varepsilon$ is necessary to provide a temporary resolution of the essential ambiguity in $<n|\mathrm{d}n>$. Substituting into (4.5), and introducing an integral representation for the first operator, we obtain {E}

$$V_n = \lim_{\varepsilon \to 0} \frac{1}{\hbar} \mathrm{Re} \int_0^\infty \mathrm{d}\, t \exp\{ - \varepsilon t/\hbar \}\, \mathrm{d} \wedge < n|[(\mathrm{d}\, H)_t - \mathrm{d}\, E_n]|\, n > \tag{4.9}$$

in which $(\mathrm{d}H)_t$ denotes the Heisenberg-evolved operator, namely

$$(\mathrm{d}\, H)_t \equiv \exp\{\mathrm{i}\, Ht/\hbar\}\, \mathrm{d}\, H \exp\{ - \mathrm{i}\, Ht/\hbar\} \tag{4.10}$$

The purpose of these dubious formal manipulations was to get V in terms of an expectation value. Now we can use the correspondence principle: the classical limit of the expectation $<n|\, A\, |n>$ of any observable $A$ is the average of the corresponding classical phase-space function $A(x)$over the manifold corresponding [31-33] to the state $|n>$. Because $|n>$ is a stationary state, the manifold must be *invariant* under the dynamics. For an integrable system, it is a *phase-space torus* with given quantized values of the actions $I=(I_1,....I_N)$. For an ergodic system, the manifold may be - at least in some averaged sense - the whole *energy surface* with the energy $E_n=H(x)$ of the state. It might also happen that the manifold is a single closed orbit. Henceforth we denote such classical averages, which replace $<n|A|n>$, by

$$< A(X) > = \int \mathrm{d}\, \alpha\, A(x(\alpha)) \tag{4.11}$$

where $\alpha=(\alpha_1,\alpha_2...)$ is a set of coordinates on the manifold with $d\alpha$ an invariant measure; the choice of $\alpha$ will be discussed later.

We also need the classical counterpart of the operator (4.10). this must incorporate the $X$-dependence of the classical manifolds. Let $x(t,\alpha; X)$ (fig.4.1) be the phase point at time $t$ on the orbit which starts at $t=0$ from the point $x$ with coordinates $\alpha$ on the manifold at parameters $X$. Then corresponding to (4.10) we have (cf 4.7)

$$(dH)_t = H(x(t,\alpha;X);X+dX) - H(x(t,\alpha;X);X) \tag{4.12}$$

The quantity $dE_n$ in (4.9) corresponds to the energy difference $dE$ between manifolds at $X$ and $X+dX$. This is the same for any pair of points, one on each manifold, and we choose points with the same $\alpha$ and $t$. Thus

$$dE = H(x(t,\alpha;X+dX);X+dX) - H(x(t,\alpha;X);X) \tag{4.13}$$

Subtracting, we obtain

$$(dH)_t - dE = -\frac{\partial H}{\partial x}dx = \dot{p}_t \cdot dq_t - \dot{q}_t \cdot dp_t \tag{4.14}$$

where

$$dx_t = x(t,\alpha;X+dX) - x(t,\alpha;X) \tag{4.15}$$

Now, the operator d in (4.9) commutes with the average (4.11), so we can allow it to act on (4.14):

$$\begin{aligned} \mathrm{d}\wedge[(\mathrm{d}H)_t - \mathrm{d}E] &= \mathrm{d}\dot{p}_t \wedge\cdot\, \mathrm{d}q_t - \mathrm{d}\dot{q}_t \wedge\cdot\, \mathrm{d}p_t \\ &= \mathrm{d}\dot{p}_t \wedge\cdot\, \mathrm{d}q_t + \mathrm{d}\dot{q}_t \wedge\cdot\, \mathrm{d}p_t \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}p_t \wedge\cdot\, \mathrm{d}q_t \end{aligned} \tag{4.16}$$

The classical limit of the 2-form now becomes

$$\begin{aligned} V_n \underset{h\to 0}{\to} \lim_{\varepsilon\to 0} \frac{1}{h}\mathrm{Re} < \int_0^\infty \mathrm{d}t \exp\{-\varepsilon t/h\} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}p_t \wedge\cdot\, \mathrm{d}q_t > \\ \equiv -\mathrm{W}(X)/h \end{aligned} \tag{4.17}$$

where the *classical 2-form* $\mathrm{W}(X)$ is easily found to be

$$\mathrm{W}(X) = < \mathrm{d}p \wedge\cdot\, \mathrm{d}q > \tag{4.18}$$

The wedge $\wedge$ acts between the d's in $X$ space, and the $\cdot$ acts between the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$. There is no longer any time-dependence: $\mathrm{d}\boldsymbol{p}$ and $\mathrm{d}\boldsymbol{q}$ refer to displacements linking points labelled $\alpha$ at $X$ and $X+\mathrm{d}X$, that is $x(0,\alpha; X+\mathrm{d}X)$ and $x(0,\alpha;X)$ (fig. 4.1).

The result (4.18) has two important invariance properties. First {E}, W is invariant under *canonical transformations* of the phase-space variables $x$, provided the transformation does not involve the parameters $X$. Second {E, not easy} W is invariant under arbitrary $X$-dependent shifts of the manifold coordinates $\alpha$; that is, under the change to coordinates

$$\alpha'_j = \alpha_j + F_j(X) \tag{4.19}$$

This latter invariance is the classical analogue of the quantum gauge invariance (2.14).

Because of the simple appearance of (4.18), and these two invariances, it seems that the result we have found is surely the correct classical limit of the phase 2-form. But appearances can deceive, and in fact (4.18) is a subtle and slippery formula whose meaning is proving hard to extract.

Consider first the manifold corresponding to $|n>$ . We do not know what this is in the general case. The correspondence principle, combined with the quantum adiabatic theorem, strongly suggests that the manifold is labelled by the (quantized) value of some classical adiabatic invariant, which is conserved as $X$ varies slowly. However, no such invariant is known for a general system, whose motion is neither integrable nor ergodic. For ergodic (e.g. completely chaotic) systems there is an adiabatic invariant, namely [34] the phase space volume within the energy surface $H(x)=E$. When quantized, this invariant yields the Weyl rule [31] for the energy levels, which gives quite a good semiclassical description of the average behaviour of the spectrum. But the difficulty in an ergodic system is that there seems no sensible choice of coordinates $\alpha$ on the energy surface; this is necessary in order to be able to associate phase points for different $X$ and so give meaning to $dp$ and $dq$.

No problems arise for integrable systems - the case considered by Hannay, where motion is multiply periodic. Corresponding to $|n>$ is a phase-space torus, labelled by the $N$ action variables $I$, which are not only invariant under the motion for fixed $X$ but adiabatically conserved when $X$ changes slowly. And the natural coordinates $\alpha$, labelling each torus with an invariant measure, are the $N$ angular variables $\theta = (\theta_1 \ldots \theta_N)$ conjugate to $I$. Therefore the classical 2-form $W(X)$ is given unambiguously for this case.

Using semiclassical analysis that I will not repeat here, it is possible to show [26] that the j'th Hannay angle (cf.4.4) corresponding to a circuit C is the following flux through C:

$$\Delta\theta_j(C) = -\frac{\partial}{\partial I_j}\iint W(X) \tag{4.20}$$

We can express this very simply in terms of the quantum phase by noting (4.17) and the fact that for integrable systems states are labelled by $N$ quantum numbers $n=(n_1...n_N)$, one corresponding to each action $I_j$, which is quantized in units of $\hbar$. The result is

$$\Delta\theta_j(C) = -\frac{\partial}{\partial n_j}\gamma_n(C) \tag{4.21}$$

After all this abstraction, it is healthy to do a concrete calculation. I will illustrate the inner workings of (4.18) by evaluating W (and $\Delta\theta$) for a *classical spin*. This application was mentioned briefly in Hannay's paper [25] and worked out in detail later [35]. Consider an angular momentum vector $S$ whose dynamics is determined by an energy function $E(S)$ through the equation of motion

$$\dot{S} = \nabla_S E(S) \wedge S \tag{4.22}$$

This conserves the length $S=|S|$, so $S$ moves on the surface of a sphere.

To make contact with our earlier analysis of quantum spins, we choose $E$ as a function of $S\cdot R$, where $R$ is a given vector whose components $(X,Y,Z)$ are the parameters, soon to be cycled. For fixed $R$ the component of $S$ along $R$ is conserved {E}. Therefore the motion is *precession* about $R$, and $S$ moves uniformly round a circle

(fig.4.2) on its sphere. This motion can be described by the evolution of an angle $\theta$.

Now let $\boldsymbol{R}$ be slowly cycled. At the end of the cycle $C$, $S$ is back on its original circle, at a position shifted by a Hannay angle $\Delta\theta$. These assertions follow from the fact that (4.22) describes a Hamiltonian system with one freedom, whose phase space is the $S$ sphere. To see this, choose a fixed direction $z$ in $S$ space and canonical variables

$$p = S_z; \quad q = \tan^{-1}\{S_y/S_x\} \\ = \text{azimuth angle of polar coordinates with axis } z \qquad (4.23)$$

Thus d$q$ d$p$ is the area element on the $S$ sphere. As Hamiltonian choose the energy in $q,p$ variables, i.e.

$$H(q,p) = E\left(\sqrt{S^2 - p^2}\cos q, \sqrt{S^2 - p^2}\sin q,\ p\right) \qquad (4.24)$$

Then Hamilton's equations reproduce the dynamics (4.22) {E}. Moreover in our case where $E$ is a function of $\boldsymbol{S}\cdot\boldsymbol{R}(t)$, the adiabatically conserved action variable is {E}

$$I = \boldsymbol{S}(t)\cdot\boldsymbol{r}(t) \qquad (4.25)$$

where $\boldsymbol{r}$ is the unit vector $\boldsymbol{R}/R$. $I$ confines $S$ to a 1-torus on its sphere; this is just the circle in fig.4.2, and conjugate to $I$ is the angle $\theta$.

It is worth remarking that the anholonomy $\Delta\theta$ involves a hierarchy of three levels of rotation: the *spin* (axis $S$), the *precession* (of $S$ round the axis $\boldsymbol{R}$), and the *turn* (of $\boldsymbol{R}$ around $C$).

Now we calculate W($\boldsymbol{R}$) from (4.18). We need

$$\begin{aligned} \mathrm{d}p \wedge \mathrm{d}q &= \mathrm{d}S_z \wedge \mathrm{d}\left(\tan^{-1}\{S_y/S_x\}\right) \\ &= \frac{\mathrm{d}S_z \wedge \left(S_x \mathrm{d}S_y - S_y \mathrm{d}S_x\right)}{S_x^{\,2} + S_y^{\,2}} \end{aligned} \tag{4.26}$$

Introduce a unit triad $\boldsymbol{r},\boldsymbol{u},\boldsymbol{v}$ on the $\boldsymbol{R}$ sphere. With these local axes (fig.4.3)

$$\boldsymbol{S} = I\,\boldsymbol{r} + \sqrt{S^2 - I^2}\cos\theta\,\boldsymbol{u} + \sqrt{S^2 - I^2}\sin\theta\,\boldsymbol{v} \tag{4.27}$$

The aim now is to express $\langle \mathrm{d}p \wedge \mathrm{d}q \rangle$ in terms of $\mathrm{d}\boldsymbol{u}$ and $\mathrm{d}\boldsymbol{v}$ (and ultimately in terms of $\boldsymbol{R}$ through $\boldsymbol{r} = \boldsymbol{u} \wedge \boldsymbol{v}$).

It helps to choose instantaneous axes $x,y,z$ along $\boldsymbol{u},\boldsymbol{v},\boldsymbol{r}$, i.e.

$$\boldsymbol{u} = (1,0,0); \quad \boldsymbol{v} = (0,1,0); \quad \boldsymbol{r} = (0,0,1) \tag{4.28}$$

Then

$$\begin{aligned} \mathrm{d}\boldsymbol{u} &= (0, \mathrm{d}u_y, \mathrm{d}u_z) \\ \mathrm{d}\boldsymbol{v} &= (\mathrm{d}v_x, 0, \mathrm{d}v_z) \\ \mathrm{d}\boldsymbol{r} &= (-\mathrm{d}u_z, -\mathrm{d}v_z, 0) \end{aligned} \tag{4.29}$$

From (4.27) with $\theta$ fixed, we find

$$\begin{aligned} \mathrm{d}S_x &= -I\,\mathrm{d}u_z + \sqrt{S^2 - I^2}\sin\theta\,\mathrm{d}v_x \\ \mathrm{d}S_y &= -I\,\mathrm{d}v_z + \sqrt{S^2 - I^2}\cos\theta\,\mathrm{d}u_y \\ \mathrm{d}S_z &= \sqrt{S^2 - I^2}(\cos\theta\,\mathrm{d}u_z + \sin\theta\,\mathrm{d}v_z) \end{aligned} \tag{4.30}$$

Substitution into (4.26) and averaging over $\theta$ leads to {E}

$$< \mathrm{d}p \wedge \mathrm{d}q > = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \, \mathrm{d}p \wedge \mathrm{d}q = -I \, \mathrm{d}u_z \wedge \mathrm{d}v_z \tag{4.31}$$

Reinstating general axes gives the result:

$$\mathrm{W} = < \mathrm{d}p \wedge \mathrm{d}q > = -I \, \mathrm{d}u \wedge \cdot \mathrm{d}v \tag{4.32}$$

Apart from the factor $-I$, this is the same as the 2-form (1.13) that occurred in our earlier study of parallel transport. Therefore we can use (1.15) to give the monopole formula, which (with d replaced by $\nabla_R$) is

$$\mathrm{W} = -I \; R/R^3 \tag{4.33}$$

For quantized actions $I=n\hbar$ this immediately confirms the correctness of the relation (4.17), because it reproduces the quantum spin 2-form (3.8) (here 4.17 is exact, rather than being a semiclassical approximation).

From (4.20), Hannay's angle is the flux through C of a unit monopole, namely

$$\Delta\theta = \Omega(\mathrm{C}) \tag{4.34}$$

This dynamical angle anholonomy is exactly the same as the geometric anholonomy in the parallel transport of a vector (lecture 1). We encountered a similar identity in the optical fibre experiment (lecture 3) in the duality between photon spin phase anholonomy and the parallel transport of linear polarization. A purely mechanical illustration of the duality is the *Foucault pendulum.*

Imagine first that the pendulum is rotating conically, with angular velocity $\omega$ ('circular polarization'), rather than swinging to and fro ('linear polarization') as it usually does. Our general classical spin analysis can be applied, with the local upward vertical $-g$ playing the role of the parameter $R$. As the earth turns, the vertical turns with it.

After a day ($t=T$) $\boldsymbol{R}$ has cycled, and the angle $\theta$ of the conical pendulum has increased by the solid angle

$$\Omega = 2\pi\,(1 - \sin(\text{latitude})) \tag{4.35}$$

as well as the dynamical $\omega T$. This anholonomy is the same for both senses $\pm\,\omega$ of conical rotation.

Now let the pendulum swing linearly, and regard this as the superposition of two opposite conical rotations. If the bob swings in the $xy$ (horizontal) plane, and is initially $x$ polarized, we have

$$\begin{aligned}
&\text{at the start: } x + \mathrm{i}\,y = \exp(\mathrm{i}\,\omega t) + \exp(-\mathrm{i}\,\omega t),\\
&\qquad\qquad \text{i.e. } (x\,,y\,) = 2\cos\,\omega t\;(1,0)\\
&\text{at the end: } x + \mathrm{i}\,y = \exp(\mathrm{i}\,\Omega)2\cos\,\omega t,\\
&\qquad\qquad \text{i.e. } (x\,,y) = 2\cos\,\omega t\;(\cos\,\Omega,\sin\,\Omega)
\end{aligned} \tag{4.36}$$

The effect of cycling is therefore to rotate the direction of swing by $\Omega$. In other words, the direction is *parallel-transported*. From the rotating earth this appears as a slow rotation, at a rate

$$\omega_{\text{Foucault}} = (2\pi - \Omega)/T = \omega_{\text{earth}}\,\sin(\text{latitude}) \tag{4.37}$$

(in Bristol this is 11.7° an hour). A clever mechanical analogue of the Foucault pendulum, whose anholonomy can be seen without waiting a day, was developed by Kugler and Shtrikman [36].

It seems that we have come full circle in these lectures. We started with parallel transport on a sphere, and now once again we encounter parallel transport on a sphere. But as with our other cycles, the end is subtly different from the beginning. In lecture 1, parallel transport was introduced as a purely mathematical construction. Now we find that what was mathematically natural is enforced physically by the laws of Nature (in this case Newton's).

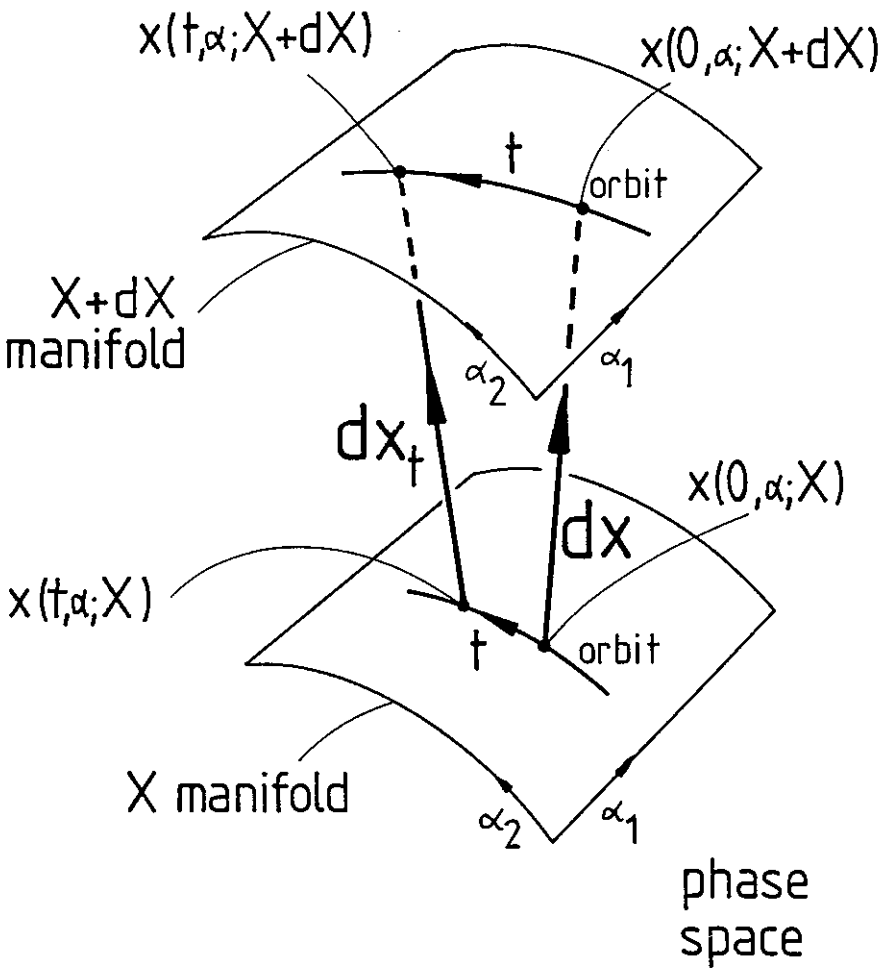
x(t,α;X+dX)
x(0,α;X+dX)
t
orbit
X+dX
manifold
α2
α1
dxt
dx
x(0,α;X)
x(t,α;X)
t
orbit
X manifold
α2
α1
phase
space

Fig. 4.1

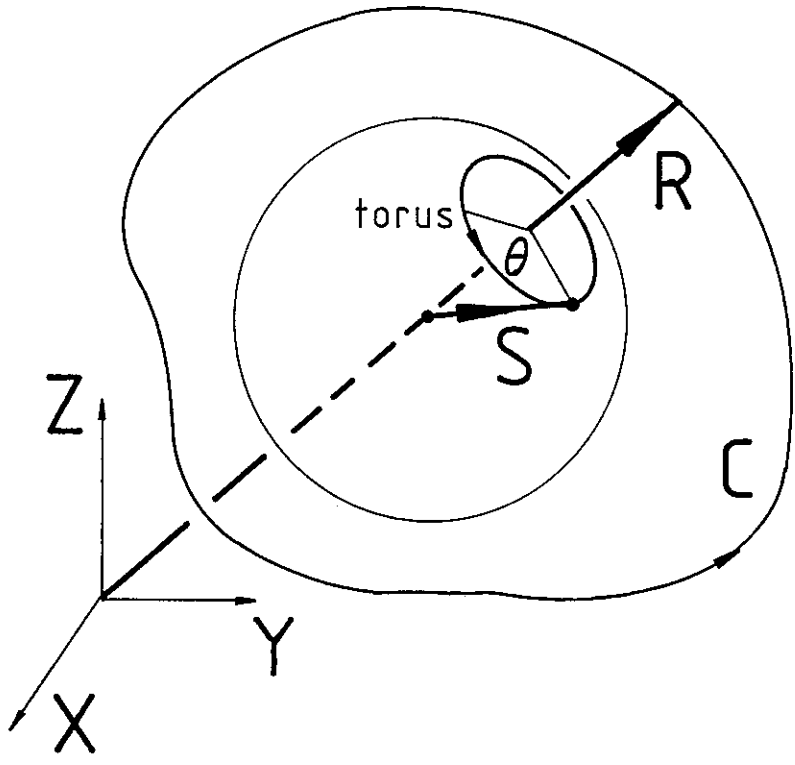


Fig. 4.2

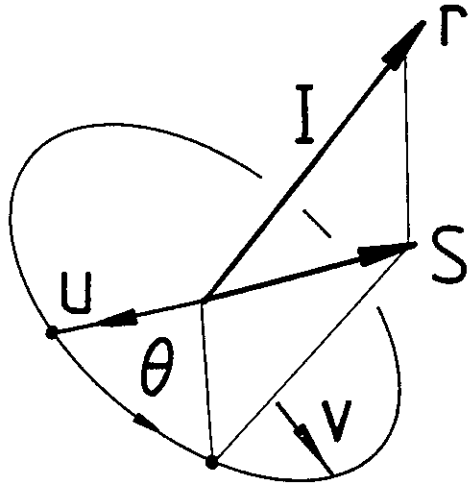


Fig. 4.3

# Lecture 5

Here I will describe three generalizations of quantum adiabatic anholonomy. The first, from Wilczek and Zee [37], allows the transported states to be degenerate. Consider a group of $N$ (orthonormal) states $|1(x)>$, $|2(x)>$ ..... $|N(X)>$ which are degenerate for all $X$ on $C$, with energy $E(X)$. Such a situation usually arises when $H(X)$ has some symmetry. Because of the degeneracy, adiabatically evolving states $|\Psi(t)>$ will not cling to individual members $|n>$ of the group. All that can be said is that the $|\Psi>$ will remain a superposition of members of the group, (i.e. there will be no transitions to states outside the group) so that the adiabatic ansatz generalizing (2.2), is

$$|\Psi(t)> = \exp\left\{-i\int_0^t d\,t\, E(X(t))/\hbar\right\}\sum_1^N a_n(t)|\, n(X(t))> \tag{5.1}$$

To find the evolution of the coefficients $a_n(t)$, it is necessary to use the Schrödinger equation. The result is that the final superposition is the result of a unitary matrix acting on the initial superposition:

$$a_n(T) = \sum_{m=1}^{N} U_{nm}(C)\, a_m(0) \tag{5.2}$$

where {E}

$$U_{nm}(C) = P\exp\{i\oint A_{nm}\} \tag{5.3}$$

in which $P$ denotes path-ordering and $A_{nm}(=A^*_{mn})$ is the Hermitian matrix 1-form

$$A_{nm} = i<n|d\,m> \tag{5.4}$$

This generalises our previous case, where $N=1$ and $U$ is a unit complex number whose phase is $\gamma_n(C)$. Even when $N>1$ it can

happen, exceptionally, that $A_{nm}$ is diagonal; then the constituent states remain uncoupled and so acquire separate geometric phases $\gamma_n(C)$ just as though they were non-degenerate. This occurs in the optical fibre experiment (lecture 3), because the change in one helicity state, induced by change in the fibre direction, has no overlap with the other {E}.

Usually, though, $A_{nm}(X)$ is not diagonal, and the $A_{mn}$ at different $X$ do not commute, a 'nonAbelian' property that makes it impossible to write $U$ as the flux of anything simple. But it causes the final coefficients $a_n$ to have different amplitudes as well as phases. In other words, the *populations* of degenerate levels can change, without any transitions involving the absorption or emission of energy. Segert [38] has proposed an interesting spectroscopic experiment, involving cycling the direction of parallel electric and magnetic fields whose magnitudes are tuned so as to make an atomic level degenerate (by cancellation of Stark and Zeeman shifts).

The second generalization, from Aharonov and Anandan [39], provides a setting in which the geometric phase can appear in evolutions that are not adiabatic. Let $H(T)$ be chosen to make the state $|\psi(t)>$ return *exactly*, apart from a phase, i.e.

$$|<\psi(0)|\psi(T)>| = 1 \tag{5.5}$$

Such 'cyclic evolution' can be made to occur in many ways [39], even with an $H$ that does not change at all.

It will usually be the case that $|\psi(t)>$ is not an eigenstate of $H(t)$. Nevertheless, it is possible to define the dynamical phase as the integral of the instantaneous expectation value of $H$ . This can be factored out by defining (cf 2.2)

$$|\psi(t)> \equiv \exp\left\{-\mathrm{i}\int_0^t \mathrm{d}t' <\psi(t')|H(t')|\psi(t')>/\hbar\right\}|\phi(t)> \tag{5.6}$$

As in our earlier examples, the anholonomy of $|\phi>$ is determined by the Schrödinger equation, which again gives exactly {E} the parallel-transport law (cf 2.4)

$$< \phi | \dot{\phi} > = 0 \tag{5.7}$$

The anholonomy - that is, the geometric phase - is conveniently calculated in terms of any base state $|\Psi(t)>$ that coincides with $|\psi(t)>$ up to phase but is singlevalued round C. Thus $|\Psi>$ plays the same role as the singlevalued eigenstate $|n>$ in the adiabatic theory, and leads to the same result as before (cf. 1.20) namely

$$| \psi(T) > \equiv \exp\left\{ - i \int_0^T d t' < \psi(t') | H(t') | \psi(t') > / \hbar + i \gamma(C) \right\} | \psi(0) > \tag{5.8}$$

where

$$\gamma(C) = - \mathrm{Im} \oint < \Psi | d \Psi > \tag{5.9}$$

Note that in this formulation there is no parameter space. The circuit C is in the *Hilbert space* of states without phase. Sometimes this is called ray space, or density-matrix space, or projective Hilbert space. It differs from the full Hilbert space of all states $|\psi>$ by regarding as identical any states differing by complex scalar multipliers.

Aharonov and Anandan's theory is both richer and poorer than the adiabatic theory. It is richer in the sense that the adiabatic theory is a special case, where the base states $|\Psi>$ are the $n$'th eigenstates of a family of Hamiltonians labelled by parameters $X$. Thus parameter space is a submanifold of projective Hilbert space. However, in applications (e.g. Born-Oppenheimer theory) parameters occur naturally, and can have a richer geometry (cf. the 2-form and its singularities at

degeneracies, and the geodesics generated by the metric 2.17) than the big Hilbert space - just as geometry on a curved surface can be richer than the geometry of the Euclidian 3-space in which it lives. Moreover, in the adiabatic framework, where the *Hamiltonian* is cycled exactly and states follow as best they can, there is a hierarchy of *corrections* to the geometric phase, of higher order in the adiabaticity parameter, which reveal [1] additional rich anholonomy (of Hamiltonians obtained by successive transformations to moving frames).

The third generalization, from Garrison and Wright [40], removes the restriction to unitary evolution. Consider a vector $|\Psi(t)>$ driven by a first order differential equation with a general time-dependent operator. For convenience we can still write this in 'Schrödinger' form, but $H(X)$ is now not a Hamiltonian but an arbitrary and usually non-Hermitian operator, with eigenvalues $E_n(X)$, possibly complex, corresponding to which are left eigenvectors $<\tilde{n}(X)|$ as well as right eigenvectors $|n(X)>$ , chosen singlevalued on and within the circuit C in $X$ space.

The calculation of adiabatic anholonomy is almost the same as before. We make the adiabatic ansatz (cf 2.2 and 1.18).

$$|\Psi(t)> = \exp\left\{-\mathrm{i}\int_0^t \mathrm{d}t\, E_n(X(t))/\hbar + \gamma(t)\right\}|n(X(t))> \qquad (5.10)$$

with the expectation that now $\gamma(t)$ , and its value at $t=T$ which is the anholonomy, will be complex. A simple argument {E} gives

$$\gamma_n(C) = \mathrm{i}\oint_C \frac{<\tilde{n}|\mathrm{d}\,n>}{<\tilde{n}|n>} \qquad (5.11)$$

To conclude, here is an interesting application of the nonHermitian theory. In the semiclassical asymptotics of the time-*independent* Schrödinger equation, there occur phase shifts of the

WKB-Maslov type [31], appearing as multiples of $\pi/2$ in quantization conditions and reflection amplitudes. People have often wondered whether these phases can be interpreted as anholonomy. Long ago, Voros gave one such interpretation in his thesis [41], and Littlejohn [42] has recently published a similar argument. Here I give a different interpretation, achieved after a conversation with A.Shapere.

A quantum particle with mass $m$ and energy $W$, moving along a line in a potential $V(x)$, satisfies

$$d_z^2 u(z) + \frac{P^2(z)}{\hbar^2} u(z) = 0 \tag{5.12}$$

involving the classical momentum

$$P^2(z) = 2m(W - V(z)) \tag{5.13}$$

We write $z$ rather than $x$ because we want to continue the wavefunction $u$ into the complex plane.

Define the two-component 'spinor' state vector

$$|\Psi(z)\rangle \equiv \begin{pmatrix} u(z) \\ \hbar u'(z) \end{pmatrix} \tag{5.14}$$

Then (5.12) is equivalent to the 'Schrödinger' evolution

$$i\hbar |\Psi'(z)\rangle = \begin{pmatrix} 0 & i \\ -iP^2(z) & 0 \end{pmatrix} |\Psi(z)\rangle \tag{5.15}$$

The 'Hamiltonian' in this equation is not Hermitian, even on the real axis $z=x$, although its eigenvalues

$$E_{\pm}(z) = \pm P(z) \tag{5.16}$$

are real in classically allowed regions of the real axis. The eigenstates $|\pm(z)>$, and their duals $<\tilde{\pm}(z)|$ (corresponding to forward (-) and backward (+) travelling WKB waves in allowed regions), are {E}

$$|\pm(z)> = \begin{pmatrix} 1 \\ \mp i P(z) \end{pmatrix}; \qquad <\tilde{\pm}(z)| = (\mp i P(z), 1) \tag{5.17}$$

Degeneracies ($E_+=E_-$) correspond to *classical turning points*, which are real or complex zeros of $W-V(z)$ . In the usual case of simple zeros, these are *branch points* of $P(z)$ and hence of the spectrum $E_\pm(z)$ and the eigenvectors.

It is easy {E} to show that

$$i\frac{<\tilde{\pm}|\dot{\pm}>}{<\tilde{\pm}|\pm>} = i\frac{P'}{2P} = \frac{i}{4}(\log P(z))' \tag{5.18}$$

Thus the anholonomy (5.11) associated with a complex C can be written as the contour integral

$$\gamma(C) = \frac{i}{4}\oint_C dz\,(\log P)'$$
$$= -\frac{\pi}{2} \times (\text{number of zeros of } P^2(z) \text{ inside C}) \tag{5.19}$$

(If the turning points are not simple, they must be counted with their multiplicity.) Note that $\gamma$ is real, that is the anholonomy takes the form of a phase shift, in spite of $H$ being nonHermitian.

The two-state formalism based on (5.15) bears a superficial resemblance to the spin-1/2 problem considered in Lecture 3, but its nonunitarity is responsible for two important differences. First, the geometric phase is the same for the two states |+> and |-> , in contrast to spin where the $n=\pm 1/2$ have opposite phases (cf 3.9). Second, the phase associated with a planar circuit of a degeneracy is $\pi/2$ rather than $\pi$ . This is because degeneracies are branch points for nonHermitian

operators, and diabolical points ([43], especially the final remarks) for Hermitian ones.

An immediate application is to *oscillators* (e.g. harmonic, which have two real turning points bounding a classically allowed region. For a circuit of this region enclosing both branch points, the states return, with phase shifts (dynamical plus geometric)

$$\mp \frac{1}{\hbar}\oint P \, \mathrm{d}z - \pi \tag{5.20}$$

Single-valuedness of $u$ (or $|\Psi\rangle$ requires that this phase be $2n\pi$. Thus we reproduce the well-known quantization condition

$$\oint P \, \mathrm{d}z = (n + 1/2)\, h \tag{5.21}$$

with the half-integer appearing as a consequence of the nonHermitian geometric phase. In this formulation the '1/2' is the combined effect of the branch points - as though the oscillator were a composite of two particles with spin 1/4, i.e. half-Fermions [46].

Another application is to the amplitude for *reflection above a barrier*. As is well known [44], this process is classically forbidden, and the reflection diminishes exponentially as $h \to 0$. This is because there are no real turning points and we have to take C around the nearest complex one, at $z^*$, say. After the circuit of this branch point, $P$ has changed sign (cf 5.16) and 5.17) and so $|+\rangle$ changes into $|-\rangle$ (and vice versa); physically this means the transformation of an incoming wave into a reflected wave. We immediately identify the reflection amplitude

$$r = -\,\mathrm{i}\exp\left\{2\mathrm{i}\int_{z_0}^{z^*} P \, \mathrm{d}z/\hbar\right\} \tag{5.22}$$

where $z_o$ is the point on the real axis from which the phase of the incident and reflected waves is reckoned. This is exponentially small because the integral in the exponential - the dynamical 'phase' - is in fact

not a phase because it has a positive imaginary part. The phase factor -i is a consequence of nonHermitian anholonomy.

Of course these one-dimensional semiclassical turning-point problems have been solved long ago by other means [44]. Our main result (5.19) is just a fancy way of dealing with the multivaluedness of the amplitude factor $P^{-1/2}$ in the WKB solutions of (5.12). Nevertheless, it is pleasant to discover how easily and naturally these phases appear, without any sign ambiguities, when interpreted as anholonomy.

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