GENERALIZED RAINBOWS IN WAVE PHYSICS: HOW CATASTROPHE THEORY HAS HELPED

M. V. Berry

H. H. Wills Physics Laboratory, University of Bristol, Bristol, U. K.

Catastrophe theory has been immensely useful in understanding the connection between waves and rays. Two important examples are the passage from physical optics to geometrical optics, and from quantum to classical mechanics. These are *short-wave limits*, involving the asymptotics of vanishing wavelength λ.

The limit is nontrivial because $\lambda = 0$ is a highly nonanalytic singular point in wave physics. Such nonanalyticities are the rule rather than the exception when one theory is reduced to another by making a parameter vanish, and they always point to new physics in the borderland between the theories. For example, the reduction of viscous flow (Navier-Stokes equation) to inviscid flow (Euler equation) is obstructed by the singularity at zero viscosity, and the borderland physics is turbulence. And critical phenomena are the borderland physics in the reduction of statistical mechanics to thermodynamics as the number of particles diverges, because of the nonanalyticity at the critical point.

In waves, nonanalyticities abound. The interference of two waves produces

$$\Psi(x;\lambda) = \exp\left[\frac{2\pi i x}{\lambda}\right] + \exp\left[-\frac{2\pi i x}{\lambda}\right] = 2\cos\left[\frac{2\pi x}{\lambda}\right], \qquad (1)$$

whose observable intensity

$$I(x;\lambda) = |\Psi(x;\lambda)|^2 = 4\cos^2\left(\frac{2\pi x}{\lambda}\right) \quad (2)$$

has an essential singularity of oscillatory exponential type at $\lambda = 0$, where $I(x)$ is pathological and takes all values between 0 and 4. Only after infinitesimal smoothing does one obtain the "classical" intensity relation 1 + 1 = 2. Another nonanalyticity happens when instead of two rays there are no rays; for example, the reflection coefficient above a quantum barrier scales as

$$r = \exp\left(-\frac{\text{constant}}{\lambda}\right), \quad (3)$$

which is an essential singularity of real exponential type.

The two previous nonanalyticities occur together at caustics, which are the envelopes of families of rays and in the simplest case separate a two-wave region from a no-wave region. Caustics are singularities of geometrical optics and hence singularities of the short-wave singularity. The correct synthesis of (1) and (3) was accomplished by Airy (1838); the wave is

$$\Psi(x;\lambda) = \frac{C_1}{\lambda^{1/6}} \mathrm{Ai}\left(-\frac{C_2 x}{\lambda^{2/3}}\right), \quad (4)$$

where x is a coordinate increasing into the two-wave region, C_1 and C_2 are constants and Ai is the Airy function (Abramowitz and Stegun, 1964). Asymptotics of Ai away from the caustic regenerates (1) and (3). Close to the caustic, we get extra information: the prefactor shows that the intensity on the caustic grows as $\lambda^{-1/3}$, and the argument gives the size of the interference fringes as $\lambda^{2/3}$ (i.e., larger than the λ far away).

At this point catastrophe theory enters. It has contributed in two ways. First, to teach us to seek singularities that are *structurally stable*, that is, unaffected in their essential geometry by well-defined deformations (here diffeomorphisms). Thus we seek caustics in the absence of symmetry, describing focusing as it can occur in nature.

Second, it provides a list of structurally stable caustics (Poston and Stewart, 1978). These are organized by codimension K, that is, the number of parameters C_1, C_2, ..., C_K that must be explored in order to find them. These caustics are the catastrophes: fold, cusp, swallowtail, umbilics, etc. (Thom, 1975) [(4) corresponds to the fold, with K = 1].

Generalizing (4) are a series of diffraction catastrophes (Berry and Upstill, 1980, Berry, 1981) $\Psi(C_j;\lambda)$. These obey scaling laws generalizing the λ-dependences in (4):

$$\Psi(C_j;\lambda) = \frac{1}{\lambda^{\beta}}\,\Psi\left[\frac{C_j}{\lambda^{\sigma_j}};1\right]. \tag{5}$$

The β are the Arnold indices (Varchenko, 1976), describing the wave intensification at caustics: the intensity scales as $\lambda^{-2\beta}$. The σ_j are the fringe indices (Berry, 1977, 1986), describing the shrinking of diffraction features in C space near the caustic: these scale as λ^{σ_j}.

These ideas have been instrumental in explaining a great variety of natural phenomena involving caustics (with and without diffraction), as described (partly) elsewhere in this book. Here is a partial list. For photons: rainbows, twinkling starlight, bright shadows of floating objects, disrupted reflections of things, gravitational lensing, mirages, focusing by irregular water droplets, swimming-pool caustics, radio-wave ducting, ... For atoms and molecules: rainbow scattering, surface scattering. For nuclei: rainbow scattering. For protons, positrons, electrons: channeling in crystals. For phonons: whispering gallery modes, oceanic ducting (whale talk), focusing by crystal anisotropy.

Caustics are the violent births of rays in pairs. But these births are really transformations from exponentially small rays into real ones. It is possible to follow the complex rays and determine how they are born. These births are individual and occur gently at the moment of extreme evanescence, when the complex ray is maximally dominated by another ray. This birth phenomenon was discovered by Stokes (1864) and long regarded as a discontinuity, albeit exponentially small. In reality, it happens smoothly, in a way described recently by Berry (1989a, b) in a formula which is the analogue for the Stokes jump of Airy's smoothing (4) for a caustic. The jumps occur on surfaces that connect with the caustics in interesting ways (Berry and Howls, 1990).

Acknowledgements

I am most grateful to Professor N. Nešković and his colleagues for conceiving and organizing the meeting at which this talk was given, and for the warmth and abundance of their hospitality. No military agency supported the work reported here.

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