Stokes surfaces of diffraction catastrophes with codimension three

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Abstract. The Stokes set, where exponentially small complex (i.e. evanescent) rays appear and disappear, is the locus of wavefield positions where stationary points of a diffraction integral have equal phase. In three dimensions, it is a surface. Stokes surfaces are calculated and displayed for the diffraction patterns decorating the swallowtail, elliptic and hyperbolic umbilic singularities. The surfaces are smooth apart from cusped edges where they meet the cusp lines of the real bifurcation set (caustic), and finite-angled creases at the complex whiskers of the singularity.

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1. Introduction

Geometrical optics is the short-wave limit of wave optics. It can be derived by applying the method of stationary phase to diffraction integrals representing wavefields. Stationary points of the phase of the integrand give contributions corresponding to rays. The large parameter in geometric asymptotics is the wavenumber $k$. Wavefunctions also depend on a set of $N$ real variables $X = (X_1, X_2, \ldots, X_N)$, representing spatial coordinates of the field point, parameters of refracting objects, etc. A complete asymptotics must include not only the real rays for each $X$ but also the exponentially weak ‘complex’, or ‘evanescent’, rays that correspond to complex stationary points through which the integration contour can be deformed. As $X$ varies in its $N$-dimensional space, complex rays appear and disappear across sets of codimension 1. These were called ‘Stokes sets’ by Wright (1980), in recognition of the discovery of the phenomenon by Stokes (1864) and as a generalisation of the ‘Stokes lines’ familiar in the asymptotics of special functions in the complex plane ($N = 2$).

In Stokes’ phenomenon a ray is born where its small exponential is maximally dominated by another (usually real) ray that corresponds to a different stationary point. In a sense this is the gentlest possible birth, and should not be confused with
the violent birth (or death) of rays where two or more real stationary points coincide. These latter events correspond to caustics, where the wavefield is particularly intense, and their local forms in $X$ space are the bifurcation sets classified by catastrophe theory (Poston and Stewart 1978, Berry and Upstill 1980).

Our purpose here is to determine the forms of the Stokes sets for three of the canonical ‘diffraction catastrophes’ that describe structurally stable wavefields in the short-wave limit. This is a continuation of the programme initiated by Wright (1980). He calculated the Stokes set of the cusp singularity, for which $N = 2$. Earlier (Wright 1977), he had calculated the swallowtail diffraction integral using a program which in the dark region numerically located the Stokes sets, in order to perform a numerical approximation by the method of steepest descents. He displayed part of the Stokes set for the swallowtail. We calculate explicitly and completely Stokes sets for all three of the stable singularities with $N = 3$: the swallowtail and the elliptic and hyperbolic umbilics. Because of the dimensionality, the sets are two-dimensional surfaces.

The immediate motivation for this investigation is renewed interest in Stokes’ phenomenon following recent advances in our understanding of it (Berry 1989a,b, Olver 1990, Jones 1990). Several asymptotic techniques, the most general being analytic resummation of the divergent series whose first term is the dominant ray, have revealed that the birth of a complex ray is not discontinuous but occurs smoothly, in a universal manner, over a region whose width is of order $k^{-1/2}$ and which is centred on the Stokes set.

All the diffraction integrals to be studied here will have the form

$$I(k, X) = \int_{C_0} d\xi \exp[ik\phi(\xi; X)]$$

(1)

where $\xi$ is a complex variable and $\phi$ is an optical distance (or, more generally, action) function analytic in $\xi$. In the method of stationary phase (de Bruijn 1958), the first step is to find all the solutions $\xi_j(X)$ of

$$\frac{\partial \phi(\xi; X)}{\partial \xi} = 0.$$  

(2)

These are the saddles of $\phi$ in the $\xi$ plane. Next, the integration contour is deformed into a series of infinite arcs along lines of steepest descent (i.e. contours of $\text{Re} \phi$), each passing through a single saddle and beginning and ending in valleys (i.e. maxima—often infinite—of $\text{Im} \phi$). For each set of variables $X$, this deformation is unique, and results in a contour $C(X)$ through a subset of the $\xi_j$; these are the contributing saddles.

Stokes’ phenomenon is a discontinuous change in $C(X)$ (but not of course in $I(k, X)$), resulting in the sudden capture or loss of one of the saddles. It happens at those $X$ values where there is a saddle connection (see figure 1), i.e. where two contributions have equal phases:

$$\text{Re} \phi[\xi_i(X); X] = \text{Re} \phi[\xi_j(X); X] \quad i \neq j.$$  

(3)

The caustic, on the other hand, corresponds to the catastrophe set, where in addition to (2) the second derivative of $\phi$—or, in the more general case where $\xi$ is multidimensional, its Hessian determinant—vanishes.

Catastrophe theory justifies the replacement of $\phi$ in (1) by one of the standard ‘potential functions’ describing the local topology of stationary points. These are
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Figure 1. Topological change, across a Stokes line in x, y, z space, of steepest-descent integration contours in the $\xi$ plane. The sequence corresponds to points a, b, c in figure 2(b) for the swallowtail. Thin curves are contours of constant phase through saddles leading to heights (H) and valleys (V); thick arrowed curves are contours of integration (deformed from the real axis). In all cases there are two real saddles that always contribute, and a complex one that never does. The second complex saddle contributes at (a) but not at (c), with the change occurring on the Stokes line at (b), where the complex saddle connects with a real one.

polynomials in $\xi$ and linear in the N variables X (N is the codimension), with the property that $k$ can be eliminated from (1) by rescaling (Berry 1977, Berry and Upstill 1980). The resulting canonical integrals are the diffraction catastrophes. They are dominated by their bifurcation sets, whose geometry is thoroughly understood. By contrast, the Stokes sets, which we shall study, have been largely neglected, in spite of being an essential element in the architecture of the singularities.

Our principal technique for determining the Stokes surfaces will be the saddle-connection relation (3), implemented numerically along with a graphical study of the topography of $\phi$ to establish that the connecting saddles both contribute. An important guide to the form of the surfaces is provided by the study by Wright (1980) of the Stokes line for the cusp catastrophe (N = 2). The bifurcation set here is a cusped curve consisting of two smooth ‘fold’ lines connecting at a point. Wright showed that the Stokes line is a widened mirror image of the catastrophe set, and therefore also a cusped curve. For the N = 3 singularities that we will study, the bifurcation sets are smooth surfaces connected along cusp lines. Wright’s result leads to the expectation that the Stokes surfaces will also have cusp lines, sprouting from the cusp lines of the catastrophe set. Other important places are the ‘complex whiskers’ (Poston and Stewart 1976) which are caustics of complex rays (i.e. degeneracies of complex stationary points). For N = 3 these are lines, and analysis in section 2 will show that two Stokes surfaces issue from each whisker, separated by a finite angle. Apart from these cusps and creases, we expect the Stokes surfaces to be smooth.
For the space $X$ inhabited by the singularities we employ the notation
\[ X = (x, y, z). \]  
(4)

The swallowtail diffraction catastrophe (section 2) will appear naturally in the form (1). The elliptic umbilic (section 2) and the hyperbolic umbilic (section 3) first appear as double integrals, and will be transformed into the form (1).

2. Swallowtail

This diffraction catastrophe is the integral
\[ S(x, y, z) = \int_{-\infty}^{\infty} d\xi \exp[i(\frac{1}{3}\xi^3 + \frac{1}{2}z\xi^2 + \frac{1}{2}y\xi^2 + x\xi)]. \]  
(5)

Stationary points are the solutions of
\[ \phi'(\xi; x, y, z) = \xi^2 + z\xi + y\xi + x = 0. \]  
(6)

Up to sign, $z$ can be eliminated by scaling, so that the singularity need be studied only in the three $xy$ planes $z = -1$, $z = 0$, $z = 1$. Bifurcation and Stokes sets for $|z| \neq 1$ are obtained from those for $|z| = 1$ by dilating $x$ by $|z|^2$ and $y$ by $|z|^{3/2}$.

In each of these three planes, (6) was solved numerically for each $X$, obtaining four stationary points. The values of $\text{Re} \phi$ were compared for each pair of these points, using the reduced form of $\phi$ valid there:
\[ \phi - \frac{1}{5}\xi \phi' = \frac{1}{13}z\xi^3 + \frac{1}{10}y\xi^2 + \frac{1}{2}x\xi. \]  
(7)

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Figure 2. Stokes sets (thick curves) and bifurcation sets (thin curves) in three sections of the swallowtail singularity. (a) $z < 0$; (b) $z = 0$; (c) $z > 0$. The sets divide the sections into regions with the indicated numbers of contributing real (R) and complex (C) saddles. In (b) the sequence of points labelled a, b, c corresponds to the integration contours in figure 1. For large $x$, $y$ in (c) (not shown), Stokes lines from the complex whisker eventually cross the smooth bifurcation set inertly.
Stokes sets were determined from the condition (3) (using a simple bisection routine on a unit-spaced square grid), noting the connectivity of the stationary points in $\xi$ space. Each $xy$ plane is partitioned by Stokes lines into regions with different numbers of contributing stationary points.

In planes with $z < 0$ (see figure 2(a)), the bifurcation set has cusps connecting threefold curves, two of them crossing. As expected, two Stokes lines sprout from each of the cusps; they do not intersect the bifurcation set again.

In the plane $z = 0$ (see figure 2(b)) the cusps coincide at the origin which is the fourfold-degenerate organising centre of the swallowtail. The bifurcation set can be calculated analytically:

$$x = 3(y/4)^{4/3}. \tag{8}$$

The four Stokes lines now sprout from the origin in two symmetrical pairs, and another scaling argument shows that each of these also obeys a $\xi$ power law; the two coefficients (determined numerically) have opposite sign.

In planes with $z > 0$ (see figure 2(c)), the bifurcation set is a smooth curve bending upwards towards positive $x$. The Stokes set separates into two branches. One of these is pinned to the bifurcation set at $x = y = 0$, and curves downwards. The other bends upwards from a vertex at $y = 0$ which recedes up the positive $x$ axis as $z$ increases. Note that our figure 2(c) completes figure 6.7(b) of Wright (1977), who considered only $x > 0$. Local analysis of this upper set shows the vertex to follow the path of the ‘complex whisker’ (Poston and Stewart 1976), which for the swallowtail is

$$x = \frac{1}{4}z^2. \tag{9}$$

Expansion of the integrand of (5) about the whisker shows that $S$ is locally an Airy function (Abramowitz and Stegun 1964)

$$S(x, y, z) \propto \text{Ai} \left( \frac{i\sqrt{2}z^2 - x - iy\sqrt{z/2}}{(2z)^{1/3}} \right). \tag{10}$$

The constant of proportionality conceals a real exponential damping, from which it may be shown that there is no peak in intensity at the complex whisker for any value of $z$. As was discovered by Stokes (1864) himself, the Stokes lines of $\text{Ai}(w)$ are

Figure 3. The Stokes surface (shaded) and the bifurcation set (caustic surface) for the swallowtail diffraction catastrophe, constructed by synthesising the three sections in figure 2.
Arg(w) = ±2π/3, corresponding to the curves

\[ y^2 = \left(\frac{6}{z}\right)(x - \frac{1}{3}z^2)^2 \]  

in the swallowtail, in agreement with Wright (1977). These Stokes lines recede from the whisker and eventually cross the bifurcation set without interacting with it.

From the previously mentioned scaling law, together with continuity, it is now possible to reconstruct the complete Stokes surface in the full three-dimensional space \( x, y, z \) (see figure 3).

3. Elliptic umbilic

This diffraction catastrophe is the integral

\[ E(x, y, z) = \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\zeta \exp\{i[\eta^3 - 3\eta \zeta^2 - z(\eta^2 + \zeta^2) - x\eta - y\zeta]\}. \]  

This function has been extensively studied, theoretically and experimentally, by Berry, Nye and Wright (1979) (hereafter referred to as BNW). It describes, for example, the diffraction from a thin horizontal water-droplet lens whose boundary is an equilateral triangle.

As it stands, (12) is not in the form (1). It can be brought into that form by adapting a technique employed by BNW for numerical purposes. First note that \( \zeta \) occurs quadratically in the phase, and so can be integrated out. Then make the transformation

\[ \xi = (\eta + \frac{1}{3}z)^{1/2}. \]  

This produces

\[ E(x, y, z) = 2\left(\frac{\pi}{3}\right)^{1/2} \exp(i\gamma_E) \int_{C_E} d\xi \exp[i\phi_E(\xi; x, y, z)] \]  

where

\[ \gamma_E = -\frac{1}{3}\pi + \frac{1}{3}xz - \frac{4}{27}z^3 \]

\[ \phi_E = \frac{y^2}{12\xi^2} + \xi^6 - 2z\xi^4 + (z^2 - x)\xi^2 \]  

and the contour \( C_E \) runs from \( -i\infty \) to \( \infty \), passing to the lower right of the essential singularity at \( \xi = 0 \).

Replacement of \( z \) by \( -z \) transforms \( E \) into its complex conjugate, and so turns the bifurcation and Stokes sets into their mirror images. Moreover, scaling shows that the sets for any value of \( z \) can be obtained from those in the plane \( z = 1 \) by dilation of \( x \) and \( y \) by \( z \). Therefore it is necessary to calculate the sets only in the planes \( z = 1 \) and \( z = 0 \). The transformation (13) doubles the number of possible stationary points from the original four, but apart from a curious exception, to be discussed below, none of the extra four can contribute to (14).

Figure 4(a) shows the sets for \( z = \pm 1 \). The bifurcation set is the familiar hypocycloid with three cusps. Two Stokes lines sprout from each cusp and recede to the far field along asymptotic directions making angles \( \pm\pi/3, \pi \) with the \( x \) axis.
From the $z^2$ scaling law it might appear that in the plane $z = 0$ the Stokes set would consist of the three straight semi-infinite lines (figure 4(b)) issuing from the origin in the directions $\pm \pi/3$, $\pi$, each line being the coalescence of two sprouts from different cusps. In fact the two sprouts annihilate when they coalesce, so there are no Stokes lines when $z = 0$: the three semi-infinite lines are merely ghosts of departed Stokes lines. The annihilation can be demonstrated in two differing ways.

First, we can employ the result (BNW)

$$E(x, y, 0) = \frac{2^{5/3}}{3^{2/3}} \pi^2 \text{Re} \left[ \text{Ai} \left( -\frac{x - iy}{12^{1/3}} \right) \text{Bi} \left( -\frac{x + iy}{12^{1/3}} \right) \right].$$

Consider the product

$$F(w) = \text{Re} [\text{Ai}(w) \text{Bi}(w^*)]$$

in the complex $w$ plane. We can study the Stokes set of $F$ in terms of the separate Stokes lines of $\text{Ai}$ and $\text{Bi}$ using the rules of Dingle (1973). $\text{Bi}(w)$ (and also $\text{Bi}(w^*)$) has three Stokes lines, at $\text{Arg}(w) = 0$, $\pm 2\pi/3$, and $\text{Ai}(w)$ has two, at $\text{Arg}(w) = \pm 2\pi/3$. In the product (17) the reality condition ensures cancellation of the Stokes lines along $\pm 2\pi/3$ and the zero value of the Stokes constant of $\text{Bi}$ along $\text{Arg}(w) = 0$ causes the absence of a Stokes phenomenon there.

Second, we can study the stationary points of the integral (14) on the putative Stokes lines. Consider, for example, the line $z = y = 0, x < 0$. There are five stationary points of $\phi_E$ in (15), at

$$\xi = 0 \quad \text{and} \quad \xi = \left( \frac{|x|}{3} \right)^{1/4} \exp[i(2n + 1)\pi/4] \quad (n = 0, 1, 2, 3).$$

The integration contour passes through three of these (figure 5), as would be expected where two Stokes lines coalesce. Note, however, that the exponents of these two points are equal, and that the directions of the contour's entry and exit
from them differ by \(\pi\), so that their contributions (from the semi-infinite contour through each) cancel, as claimed.

By scaling and continuity, it is now possible to build a picture of the complete Stokes surface in the full three-dimensional \(x, y, z\) space (figure 6). There are six smooth branches, intersecting transversely in pairs at the ghost lines in the plane \(z = 0\). The angle between the normals of each surface at an intersection decreases as the origin is approached.

### 4. Hyperbolic umbilic

This diffraction catastrophe is the integral

\[
H(x, y, z) = \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\xi \exp\{i[\eta^3 + 3\eta\xi^2 + z(\eta^2 - \xi^2) - x\eta - y\xi]\}.
\]

(19)

This describes, for example, diffraction by a thin water-droplet lens with a circular boundary, hanging vertically (Berry and Upstill 1980, Nye 1979). It differs from the elliptic umbilic (12) only in several signs, which enables the formal analysis to be carried out in close analogy with the preceding case, although the sign changes drastically alter the bifurcation and Stokes sets. We obtain (cf (14) and (15))

\[
H(x, y, z) = 2\left(\frac{\pi}{3}\right)^{1/2} \exp(i\gamma) \int_{C_H} d\xi \exp[i\phi(\xi; x, y, z)]
\]

(20)

where

\[
\gamma = \frac{1}{4}x - \frac{3}{4}xz + \frac{1}{2}z^3
\]

\[
\phi = -\frac{y^2}{12\xi^2} + \frac{5}{2}\xi^2 + 2z^4 + (z^2 - x)\xi^2
\]

(21)
Figure 7. As for figure 2, but for the hyperbolic umbilic singularity. (a) $|z| 
eq 0$; (b) $z = 0$; (c) $x < 0$. In (a) the Stokes lines have inflections whilst crossing the bifurcation set inertly.
and the contour $C_H$ runs from $i\infty$ to 0 and 0 to $\infty$, passing to the upper right of the essential singularity at $z = 0$. Again we have scaling and mirror symmetry of the sets in $z$, so that it suffices to study the planes $z = 1$ and $z = 0$.

Figure 7(a) shows the sets for $z = \pm 1$. The bifurcation set consists of the familiar smooth outer curve and cusped inner curve. From the cusp there sprout two Stokes lines which cross the outer branch of the bifurcation set and recede to infinity in asymptotic directions $\pm 3\pi/4$ with the positive $x$ axis.

When $z = 0$ (figure 7(b)), the bifurcation set degenerates into the corner $y = \pm x$, $x > 0$, and the Stokes set is its mirror image $y = \pm x$, $x < 0$. Unlike its elliptic counterpart, this Stokes set is not a ghost, but coincides with the complex whisker of the singularity, the coincident roots being at

$$\xi = \left(\frac{1}{6} |x|\right)^{1/4} \exp\left[i(2n + 1)\pi/4\right] \quad n = 0, 1, 2, 3.$$

This can be deduced directly from (20), or from the relation \((BNW)\)

$$H(x, y, 0) = \frac{2^{5/3} \pi^2}{3^{2/3}} \text{Ai}\left(\frac{-x - y}{12^{1/3}}\right) \text{Ai}\left(\frac{-x + y}{12^{1/3}}\right).$$

On each of the four half-lines $y^2 = x^2$, one of the two Airy functions has a zero argument, corresponding to coincident stationary points; for $x > 0$ the other Airy function has negative argument, corresponding to real stationary points and hence to the bifurcation set, and for $x < 0$ the other Airy function has positive argument, corresponding to complex stationary points and hence to the whisker and the Stokes set.

A section through the plane $x = -1$ (figure 7(c)) provides a view of two Stokes lines issuing from the complex whisker at a finite angle (which depends on $x$), in analogy with the swallowtail in planes $z > 0$ (figure 2(c)).

Once again scaling and continuity make it possible to build the complete Stokes surface in the full $x, y, z$ space (figure 8). This has two branches, smooth apart from finite-angled creases on the complex whiskers and a cusped edge at the cusp line of the bifurcation set.

5. Discussion

Stokes surfaces that we have calculated, and the Stokes line for the cusp calculated by Wright (1980), differ in an important respect from more conventional manifestations of Stokes' phenomenon. Our variables $X$ are real and hence accessible to
experiment. Previous mathematical studies have concentrated on Stokes lines in the plane of a complex variable. It is therefore natural to ask whether the birth of an evanescent wave across a real Stokes surface could be detected experimentally. This would be a faint phenomenon, requiring exponentially sensitive measurements, in contrast to the catastrophic birth at a caustic (bifurcation set) where the wavefield is intense.

This is a tricky question, because the separation into exponentially large and small waves near a Stokes set is to some extent a matter of choice. The convention introduced by Stokes (1864), for which Berry (1989a,b) showed the birth to be as unobtrusive and rapid as possible, is to define the dominant wave as the large exponential multiplied by its asymptotic expansion terminated at its smallest term, and the evanescent wave as the remaining contribution to the wavefunction. The question now becomes: are the jumps in this remainder reflected in the exact wavefunction, or are they cancelled by similar jumps in the value of the terminated dominant asymptotic expansion? Preliminary analysis suggests the latter alternative, and hence that the Stokes jumps would not be directly observable (although it is of course essential to incorporate them into any precise evaluation of the wavefunction).

A possible way of detecting the Stokes jumps indirectly would be to physically implement the convention—so natural mathematically—of terminating the asymptotic expansion at its least term. This realisation of the dominant wave could then be subtracted interferometrically from the full wavefunction, leaving the evanescent wave (together with its multiplier switching on across the Stokes surface). We have not found a way to do this.

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