Hyperasymptotics

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We develop a technique for systematically reducing the exponentially small ('superasymptotic') remainder of an asymptotic expansion truncated near its least term, for solutions of ordinary differential equations of Schrödinger type where one transition point dominates. This is achieved by repeatedly applying Borel summation to a resurgence formula discovered by Dingle, relating the late to the early terms of the original expansion. The improvements form a nested sequence of asymptotic series truncated at their least terms. Each such 'hyperseries' involves the terms of the original asymptotic series for the particular function being approximated, together with terminating integrals that are universal in form, and is half the length of its predecessor. The hyperasymptotic sequence is therefore finite, and leads to an ultimate approximation whose error is less than the square of the original superasymptotic remainder. The Stokes phenomenon is automatically and exactly incorporated into the scheme. Numerical computations confirm the efficacy of the technique.

1. Introduction

In an asymptotic series, it is common for the terms Y_r to decrease at first but ultimately increase, so that the series diverges. If λ is the (large) asymptotic parameter (i.e. $Y_r \propto \lambda^{-r}$) then the smallest error (relative to the first term $Y_0 \equiv 1$) from the bare series is $\exp(-A\lambda)$, where A is a positive constant, and is obtained by truncation at or near the least term (for example, Olver 1974). The order of this least term is $r \propto \lambda$. Such 'superasymptotics', in which exponential accuracy is achieved by making the number of terms increase with the large parameter (rather than considering a fixed number of terms as in the asymptotics of Poincaré), has a long history. Stokes (1864) used it in the accurate approximation of Bessel functions. Nekhoroshev (1976) made it the basis for estimates of long-time predictability in nonlinear hamiltonian dynamics. And it is an essential component of detailed studies (Berry 1989*a*, *b*, 1990*a*; Olver 1990*a*, *b*; Boyd 1990; Jones 1990) of the Stokes phenomenon, which is the appearance and disappearance of small exponentials as certain variables (not λ) change.

Here we go beyond superasymptotics, and obtain by repeated resummation a sequence of asymptotic series, each approximating the leftover from optimal truncation of its predecessor. We define *hyperasymptotics* as the systematic study of these approximations to the small exponential error left by truncation of the main series. The possibility of such sequential improvement was glimpsed by Stieltjes (1886), explored in particular cases by Airey (1937) and Miller (1952) (reviewed by Olver 1974), and clearly envisaged by Dingle (1973, hereinafter called D), Rakovic & Solov'ev (1989) and Boyd (1990). D developed a systematic method, based on Borel resummation, for the first stage of hyperasymptotics, in which the terms of the first hyperseries for a great variety of functions are expressed as certain standard integrals called 'basic terminants'. Boyd (1990) envisages a hyperasymptotics based on Stieltjes transforms, and studies its first stage for certain Bessel functions.

In exceptional cases, hyperasymptotics is unnecessary because the exponential leftover can be expressed exactly as a single basic terminant (an example is $\mathrm{Erf}(z)$, $|z|$ large), or as a convergent series of basic terminants (an example is $\ln\Gamma(z)$, $|z|$ large). Usually, however, hyperasymptotics is non-trivial in the sense that the resummations persist in generating asymptotic series which themselves require resummation.

Perhaps the simplest case where this happens is in the approximation of solutions $y(z,\lambda)$ of the one-dimensional Helmholtz (or Schrödinger) equation,

$$\mathrm{d}^2y(z,\lambda)/\mathrm{d}z^2 = \lambda^2 Z(z)\,y(z,\lambda), \tag{1}$$

where z is a complex variable and $Z(z)$ is analytic. This is the situation we study here, concentrating on the (generic) case of points z where y is dominated, in a sense to be specified later, by a single transition point (zero of $Z(z)$). We are able to construct the hyperasymptotic series explicitly (§3) and study in detail the asymptotics of the sequence of remainders that it generates (§4). Our hope was that this sequence would converge to the exact solution when each hyperseries is truncated at its least term, but this is not the case. It turns out that the successive hyperseries get smaller, and the process terminates naturally after a number (of order $\ln\lambda$) of stages. The error at this last stage is of order $\exp\{-(1+2\ln 2)A\lambda\}\approx\exp(-2.386A\lambda)$, where A is the positive constant already introduced; this error is smaller than the square of the error at the zeroth (superasymptotic) stage. (In principle, arbitrary accuracy could be obtained by a non-optimal truncation of the hyperseries, but as explained in §3 this procedure would be impractical.)

We achieve this improvement by exploiting a 'resurgence formula' (§2), discovered by D (p. 300) and rediscovered by Rakovic & Solov'ev (1989), for the late terms $Y_r\,(r\to\infty)$ in the asymptotic series for each of the two 'wave' solutions of (1). This is an expansion giving each Y_r for large r as a series over the Y_r for small r, that is the early terms. At first encounter this appears astonishing but in fact it is inevitable, because of the following: (i) each of the two wave series is by itself a complete formal solution of (1); (ii) both series must diverge, to reflect the fact that (unless $Z(z)$ is constant) there are really two waves in the solution of (1); (iii) therefore the divergent tail of each series must encode the early terms of the other series; (iv) the terms in the two series are the same (apart from signs) namely Y_r.

In different regions of the z plane (separated by Stokes lines) the asymptotics of the solutions $y(z,\lambda)$ contain different combinations of the two fundamental wave solutions. This is the Stokes phenomenon. Our hyperasymptotic formalism incorporates these changes of form automatically and exactly (§5).

To illustrate the workings of hyperasymptotics we present (§6) a numerical example. This is the Airy function, in which $Z(z)$ in (1) is linear. The expected sequential improvement over brute asymptotics (stopping at the first term, $Y_0 = 1$, error $\sim\lambda^{-1}$) and superasymptotics (error $\sim\exp(-A\lambda)$), down to the ultimate stage (error $\sim\exp(-2.386A\lambda)$) is achieved in all Stokes sectors.

Such extreme numerical precision is not, however, our main reason for carrying out this study. After all, it is often the case that where a numerical solution of (1) is required, standard differential equation routines (e.g. Runge–Kutta), or convergent series, are more accurate or faster than asymptotics (ordinary, super- and hyper-), even when λ is quite large. Our main motive is rather that the hyperasymptotic solution of (1) provides a testbed for studying in great detail the structure of a physical theory as an important parameter (here λ) takes a limiting value. One such limit, of considerable current interest, is the classical limit of quantum mechanics (Berry 1990*b*).

2. Dingle's resurgence formula

The two lowest-order 'wave' solutions of (1) are given by the phase-integral (i.e. WKB) method (Heading 1962; Fröman & Fröman 1965) as

$$y(z) \approx \exp\left[\pm\lambda\int_{z^*}^{z} \mathrm{d}\zeta\, Z^{\frac{1}{2}}(\zeta)\right]\Big/ Z^{\frac{1}{4}}(z) \tag{2}$$

in which z^* is an arbitrary reference point. A natural variable will be the difference between the two exponents, namely

$$F(z) \equiv 2\lambda \int_{z^*}^{z} \mathrm{d}\zeta\, Z^{\frac{1}{2}}(\zeta). \tag{3}$$

It is convenient to perform the analysis for the solution which is subdominant (i.e. exponentially small) when $\mathrm{Re}\, F > 0$, and write this as

$$y(z) = (\exp[-\tfrac{1}{2}F]/Z^{\frac{1}{4}}(z))\, Y(F). \tag{4}$$

A formal asymptotic series for Y, in descending powers of λ, can be found by substitution into (1). We write

$$Y(F) = \sum_{r=0}^{\infty} (-1)^r Y_r(F), \quad \text{where} \quad Y_0 = 1 \quad \text{and} \quad Y_r \propto \lambda^{-r}. \tag{5}$$

Then the Y_r satisfy the recurrence relation (D, p. 296)

$$Y'_{r+1}(F) = -Y''_r(F) + G(F)\, Y_r(F), \tag{6}$$

where primes denote derivatives with respect to F and

$$G(F) \equiv (Z^{\frac{1}{4}})''/Z^{\frac{1}{4}}. \tag{7}$$

Note that the large parameter λ no longer appears explicitly. Its role has been to define the terms Y_r in the series representing Y.

An important role is played by the transition points, that is the zeros z_j of $Z(z)$. At the corresponding points F_j, the function $G(F)$ has double poles, whose strengths depend only on the order of the zero: for an mth order zero, it follows from (3) and (7) that

$$G(F) \rightarrow -m(m+4)/4(m+2)^2\,(F-F_j)^2 \quad \text{as} \quad F \rightarrow F_j. \tag{8}$$

To avoid complicating the analysis, we henceforth consider only the generic case $m = 1$; generalization to arbitrary m is straightforward. When iterating (6) to obtain the Y_r, the derivatives magnify the singularities (8), leading as explained by D (p. 299) to the following simple approximate formula for the late terms:

$$Y_r(F) \rightarrow (r-1)!/2\pi(F-F_0)^r \quad \text{as} \quad r \rightarrow \infty, \tag{9}$$

where F_0 denotes the transition point F_j which is closest to F in the sense of having the smallest value of $|F-F_j|$. For large $|F-F_0|$ (ensured by large λ), the terms decrease until $r \approx |F-F_0|$ and then increase.

The resurgence formula which will be central to all our subsequent analysis is a formally exact representation of Y_r with (9) as its leading term. By direct substitution, it can be confirmed that

$$Y_r(F) = \frac{1}{2\pi}\sum_j \frac{1}{(F-F_j)^r}\sum_{s=0}^{\infty} Y_s(F)[-(F-F_j)]^s\,(r-s-1)! \tag{10}$$

satisfies the recurrence relation (6). The function $G(F)$ enters only through the positions F_j of its poles. It should be remarked that (6) is formally satisfied by much more general relations, in which (i) F_j are any points whatever, (ii) on the right-hand side r is replaced by $r+\alpha$ with α arbitrary, (iii) $1/(2\pi)$ can be any constant, and (iv) the Y_s can be any solutions of (6) (rather than those with the same integration constants as the Y_r on the left-hand side). The particular choice (10) reproduces the limiting form (9), which corresponds to the term $s = 0$, and also satisfies certain analytic requirements, as will be explained in §5. The higher terms $s > 0$ provide a formal asymptotic expansion for the late Y_r in terms of successive early Y_r.

In what follows we will use the simplified form of (10) obtained by neglecting all transition points other than the closest, F_0, which we will henceforth take as the origin $F = 0$ (this is equivalent to choosing $z^* = z_0$ in (3)). This gives

$$Y_r(F) = \frac{1}{2\pi F^r}\sum_{s=0}^{\infty}(r-s-1)!\,(-F)^s Y_s(F). \tag{11}$$

As they stand, (10) and (11) are numerically meaningless because the factorials for $s > r-1$ are infinite. They can, however, be made meaningful by Borel summation, and the hope is that these resummed versions will prove to be exact. We can justify this hope in the case of (11), by showing (Appendix A) that its resummation is equivalent to the Stieltjes transform relation between Bessel functions that forms the basis of the rigorous analysis of Boyd (1990).

Thus (11) will be exact (when suitably interpreted) if there is only one transition point. If more transition points are present, their neglect in (10) amounts to neglect of small exponentials whose order is

$$\exp\{-r\ln|(F-F_1)/F|\}, \tag{12}$$

where F_1 is the distance (3) between the transition points nearest and next nearest to F (by hypothesis, $|F-F_1| > |F|$). We will use (11) only for orders $r \geqslant |F|$, so that this neglected exponential will never be larger than

$$\exp\{-|F|\ln|(F-F_1)/F|\}. \tag{13}$$

Now, the smallest small exponentials in hyperasymptotics are of the order of $\exp\{-(1+2\ln 2)|F|\}$ (§4) so we can expect our analysis to be valid provided (13) is smaller than these, that is if

$$|F-F_1| > 4\mathrm{e}|F| \tag{14}$$

Equation (11) is Dingle's formula (D, p. 300). We will adopt his term 'singulant' to denote F, the distance to the dominant transition point. Rakovic & Solov'ev (1989) rediscovered (11), and presented (10) (also without proof) for the case of two transition points.

3. Hyperasymptotic resummations

We seek accurate approximations to $Y(F)$, as defined by the formal series (5), when the distance $|F|$ to the nearest transition point is large (cf. (3) with λ large and z fixed).

Retain the first N_0-1 terms Y_r as given by the solution of (6), where N_0 is the least term, and replace the remaining terms by (11). Thus (interchanging the r and s labels)

$$Y = S_0 + \sum_{r=0}^{\infty} Y_r(-F)^r \sum_{s=N_0}^{\infty} (-1)^s \frac{(s-r-1)!}{2\pi F^s}, \tag{15}$$

where
$$S_0 \equiv \sum_{r=0}^{N_0-1} (-1)^r Y_r. \tag{16}$$

The truncated sum S_0 is the zeroth level of hyperasymptotics, which we have called superasymptotics.

The next stage is Borel summation of the s sum in (15), that is replacement of the factorial by its integral representation, followed by interchange of summation and integration and evaluation of the sum. After an elementary change of variable, we obtain

$$Y = S_0 + \sum_{r=0}^{\infty} (-1)^r Y_r K_{r1}, \tag{17}$$

where
$$K_{r1} = \frac{(-1)^{N_0}}{2\pi F^{N_0-r}} \int_0^{\infty} \mathrm{d}\xi \exp(-\xi) \frac{\xi^{N_0-r-1}}{1+\xi/F}. \tag{18}$$

In §4 we shall see that the second series in (17) also diverges, with a least term at N_1, say. Therefore as before we retain the first N_1-1 terms and replace Y_r in the remaining terms by the resurgence relation (11), to get

$$Y = S_0 + S_1 + \sum_{r=0}^{\infty} Y_r(-F)^r \sum_{s=N_1}^{\infty} (-1)^s \frac{(s-r-1)!}{2\pi F^s} K_{s1}, \tag{19}$$

where
$$S_1 = \sum_{r=0}^{N_1-1} (-1)^r Y_r K_{r1}. \tag{20}$$

The truncated sum S_1 gives the first level of hyperasymptotics. This was studied in great detail by D (chapters XXI and XXIV), who called the integrals K_{r1} 'terminants', by Boyd (1990) (for Bessel functions) and by Olver (1990*b*) (for confluent hypergeometric functions).

Again we can apply Borel summation to the s sum in (19), and obtain

$$Y = S_0 + S_1 + \sum_{r=0}^{\infty} (-1)^r Y_r K_{r2}, \tag{21}$$

where

$$K_{r2} = \frac{(-1)^{N_0+N_1}}{(2\pi)^2 F^{N_0-r}} \int_0^{\infty} \mathrm{d}\xi_1 \exp(-\xi_1) \frac{\xi_1^{N_0-N_1-1}}{(1+\xi_1/F)} \int_0^{\infty} \mathrm{d}\xi_2 \exp(-\xi_2) \frac{\xi_2^{N_1-r-1}}{(1+\xi_2/\xi_1)}. \tag{22}$$

Now the third sum in (21) diverges, with a least term at N_2, say. Therefore, we again retain the first N_2-1 terms and replace the Y_r in the tail by the resurgence relation (11); the truncated sum constitutes the second level of hyperasymptotics.

Now the pattern is clear: truncation, resurgence and Borel summation can be repeated, leading to the hyperasymptotic sequence

$$Y = S_0 + S_1 + S_2 + \ldots, \tag{23}$$

where $$S_n = \sum_{r=0}^{N_n-1} (-1)^r Y_r K_{rn}. \tag{24}$$

The Y_r are the terms in the original asymptotic series. They are specific to the given problem as embodied in the function $Z(z)$ in (1), which appears in the recursion (6) through $G(F)$ as defined in (7). The K_{rn} constitute the hyperasymptotic generalization of Dingle's terminants K_{r1}. They are universal functions – that is, independent of the detailed form of $Z(z)$ – given by the n-fold multiple integrals

$$\left.\begin{aligned} K_{r0} &= 1, \\ K_{rn} &= K_{rn}(F, N_0, \ldots, N_{n-1}) \\ &= \frac{1}{(2\pi)^n F^{N_0-r}} \prod_{i=1}^{n} \int_0^\infty \mathrm{d}\xi_i \exp(-\xi_i) \frac{\xi_i^{N_{i-1}-N_i-1}(-1)^{N_{i-1}}}{(1+\xi_i/\xi_{i-1})} \quad (\xi_0 \equiv F, N_n \equiv r). \end{aligned}\right\} \tag{25}$$

For these integrals to converge, $N_i < N_{i-1}$, so that successive hyperseries contain fewer terms and the process of hyperasymptotics must eventually stop, at the stage where $N_n = 1$.

It is clear from equations (23) and (24) that the series of K_{rn} provide a sort of universal renormalization of the Y_r, enabling the information contained in them to be decoded, via resurgence, to yield much higher precision in the function being approximated. This information is however finite, because, since successive hyperseries get smaller, only the first N_0 terms Y_r will participate in the hyperasymptotics. Therefore we can expect the ultimate error of the scheme, when it stops, to be finite and not zero.

Of course, the actual numerical extraction of the information contained in the first N_0 terms Y_r requires knowledge of the K_{rn} to sufficient accuracy. In a thoroughgoing asymptotics, the multiple integrals (25) would themselves be expanded asymptotically (and where necessary hyperasymptotically) for large F, and the form of the expansions would depend on r and the N_i (cf. the formulae for K_{r1} in D, p. 415ff). We do not carry out this programme, but simply assume that the K_{rn} are known; in our numerical example (§6) we reduce the required K_{rn} to certain special functions which can be computed 'exactly' (see Appendix B).

4. Optimal truncations; estimates of ultimate accuracy

In choosing $N_0, N_1, \ldots$, the guiding principle is that the successive series $S_0, S_1, \ldots$, are truncated near their least terms. To implement this principle for S_0 we use the estimate (9) for Y_r, replacing $(r-1)!$ by Stirling's formula, and thus obtain the well-known result

$$N_0 = \mathrm{Int}\,|F|. \tag{26}$$

For the higher sums we need estimates of the K_{rn}. These we obtain by replacing the ξ_i in the slow varying denominators in (25) by their values at the maxima of the rest of the integrands, namely

$$\xi_i^* = N_{i-1} - N_i - 1 \quad (N_n = r). \tag{27}$$

This uncouples the integrals and gives

$$K_{rn} \approx \frac{1}{(2\pi)^n F^{N_0-r}} \prod_{i=1}^{n} (-1)^{N_{i-1}} \frac{\xi_i^*!}{(1+\xi_i^*/\xi_{i-1}^*)}. \tag{28}$$

In S_1 the terms are

$$|Y_r K_{r1}| \propto \left|\frac{(r-1)!}{F^r}\frac{\xi_1^*!}{F^{N_0-r}}\right| \approx \frac{(r-1)!\,(|F|-r-1)!}{|F|^{|F|}}, \tag{29}$$

whose least term is obvious from symmetry and gives (cf. Rakovic & Solov'ev 1989; Boyd 1990)

$$N_1 = \mathrm{Int}\,(\tfrac{1}{2}|F|). \tag{30}$$

Repetition of this procedure gives

$$N_n = \mathrm{Int}\,(|F|/2^n). \tag{31}$$

Thus each hyperseries S_n is half the length of its predecessor. The natural end of hyperasymptotics comes when the S_n has just one term, which happens after $n_{\max}$ stages, given by $N_{n_{\max}} = 1$ as

$$n_{\max} = \mathrm{Int}\log_2|F|. \tag{32}$$

To study the magnitudes of the terms we use (28) with the factorials replaced by Stirling's approximation and the denominators replaced by $\frac{3}{2}$ (this follows from (27) and (31) for large F). The last term of the nth hyperseries is thus obtained, after some reduction, as

$$(-1)^{N_n} Y_{N_n-1} K_{N_n-1n} \approx \frac{(-1)^{N_0+\dots N_n}\, 2^{n(n+7)/4}}{(2\pi)^{(n+1)/2}\, 3^n |F|} \exp\{-|F|\,[1+2(1-2^{-n})\ln 2]\}. \tag{33}$$

The first term of the $(n+1)$st hyperseries, namely K_{0n+1}, involves exactly the same factors, with the additional denominator

$$1+\frac{\xi_{n+1}^*}{\xi_n^*} = 1+\frac{N_n-1}{N_{n-1}-N_n-1} \rightarrow 2. \tag{34}$$

This shows that not only do the terms decrease within each individual hyperseries but that each new hyperseries begins with a term approximately half the size of the last term in its predecessor. (There is one exception to this rule: when $n = 1$ and F is not real, the factor is not $\frac{1}{2}$ but $1/[1+\exp(-\mathrm{i}\arg F)]$ (cf. Boyd 1990).)

The ultimate accuracy of the hyperasymptotic scheme is expected to be of the same order of magnitude as the single term $K_{0n_{\max}}$ of the last hyperseries. We estimate this by substituting $n_{\max}$ from (32) into (33) and dividing by 2:

$$K_{0n_{\max}} \approx \sqrt{\frac{2}{\pi}}\,|F|^{[\frac{1}{4}\log_2|F|+\frac{3}{4}-\log_2(3\sqrt{2\pi})]} \exp\{-|F|(1+2\ln 2)\}. \tag{35}$$

This confirms that the error is indeed finite, as expected on the basis of the finite number of participating Y_r, and of the order $\exp(-2.386|F|)$ asserted in the Introduction (with $A\lambda$ now identified as $|F|$).

Now we show that it is possible to devise hyperasymptotic schemes of arbitrary accuracy (within the one-transition-point approximation), by abandoning the optimality requirement that the terms must always decrease. For this analysis we need to recast the scheme (23)–(35) so as to yield error bounds. Elementary formal manipulations (cf. Appendix A) give the remainder after n complete resummations, namely

$$R(F; N_0 \dots N_n) \equiv Y - \sum_{m=0}^{n} S_m \tag{36}$$

as

$$R(F;N_0,N_1,\ldots,N_n) = \frac{(-1)^{N_0+N_1+\ldots+N_n}}{(2\pi)^n}\int_0^\infty \frac{\mathrm{d}t_0\, t_0^{N_0}}{t_0(1+t_0)}\int_0^\infty \frac{\mathrm{d}t_1\, t_1^{N_1}}{t_1(1+t_1)}\cdots$$

$$\int_0^\infty \frac{\mathrm{d}t_n\, t_n^{N_n}}{t_n(1+t_n)}\exp\left[-F(t_0+t_0t_1+\ldots+t_0t_1\ldots t_n)\right]Y(Ft_0t_1\ldots t_n). \quad (37)$$

Bounding $|Y(F)|$ by a constant C, we obtain

$$|R(F;N_0,N_1,\ldots N_n)| < C[(N_0-N_1-1)!\,(N_1-N_2-1)!\ldots(N_n-1)!]/(2\pi)^n|F^{N_0}|. \quad (38)$$

For given N_0 the smallest bound on the remainder occurs when all the factorials are unity, and is achieved by making successive truncation limits decrease by 1 (rather than halving as in the optimal scheme). The scheme thus terminates after Int (N_0) stages, leaving a remainder

$$|R(F;N_0,N_0-1,\ldots,1)| < C/|2\pi F|^{N_0}. \quad (39)$$

This can be made arbitrarily small by increasing N_0. None of the truncations are optimal now; the price of arbitrary accuracy is to represent $Y(F)$ by very large cancelling terms, just as in convergent series representations, and all the advantages of asymptotics are lost.

The impracticality of these arbitrarily accurate schemes leads to an increased appreciation of the 'live now, pay later' philosophy underlying the optimal hyperasymptotic (and ordinary asymptotic) schemes, in which ultimate accuracy is sacrificed as a consequence of the requirement that the terms always decrease (within each hyperseries and from one hyperseries to the next).

5. Exact Stokes relations

In the scheme (23)–(25), the singulant F enters through the Y_r and in the denominator of the first integral (over ξ_1) in each K_{rn}. F can take any complex value. We now show that as F makes a circuit around the origin, i.e. as z encircles the transition point, the solution to (1) as given by the hyperasymptotic scheme reproduces the Stokes phenomenon – the birth and death of small exponentials in the presence of large ones – exactly and to all orders.

Consider first F real and positive. Then (4) represents an exponentially decaying (subdominant) solution of (1). Now let $\arg F$ increase to π, so that (4) represents a dominant solution. The replacement $F=-|F|$ in the series S_0 gives late terms $(-1)^rY_r$ which all have the same sign (cf. the limiting form (9)). This is behaviour characteristic of a Stokes line (Stokes 1864; D, pp. 7, 414). Near to the transition point in the original z plane the π rotation of F corresponds to a $\frac{2}{3}\pi$ rotation.

In the hyperasymptotic corrections, the rotation produces a pole at $\xi_1=|F|$ in the first integrand of each K_{rn}. As $\arg F$ increases to π, the pole approaches the real ξ_1 axis from below. Thus, by continuity, the integral splits into two contributions: from the principal value and from the negatively traversed infinitesimal semicircle around the pole. Rather than write out these contributions explicitly for each of the stages of hyperasymptotics, we carry out the splitting in one step using the resummed version of the scheme, based on (36) and (37) with $n=0$. This is

$$Y(F) = S_0(F)+\frac{1}{2\pi}\int_0^\infty \mathrm{d}t\left(\frac{-t}{F}\right)^{N_0}\frac{\exp(-t)}{t(1+t/F)}Y(t). \quad (40)$$

The splitting just described can now be easily implemented, and gives

$$Y(|F|\exp(\mathrm{i}\pi)) = S_0(-|F|) + \frac{1}{2\pi} \int_0^\infty \mathrm{d}t \left(\frac{t}{|F|}\right)^{N_0} \frac{\exp(-t)}{t(1-t/|F|)} Y(t) + \tfrac{1}{2}\mathrm{i}\, Y(|F|)\exp(-|F|), \quad (41)$$

that is

$$Y(|F|\exp(\mathrm{i}\pi)) = Y_{\mathrm{p}}(-|F|) + \tfrac{1}{2}\mathrm{i}\, Y(|F|)\exp(-|F|), \quad (42)$$

where the subscript p denotes the hyperasymptotic series for $Y(-|F|)$ with all ξ_1 integrals interpreted as their principal values. Note that (42) is an identity for the function $Y(F)$, in which all reference to the truncation N_0 has disappeared.

Thus on the Stokes line the solution (4) which is subdominant for F positive real is, exactly,

$$\begin{aligned} y(z) &= Z^{-\frac{1}{4}}(z)\exp[+\tfrac{1}{2}|F|]\, Y(|F|\exp(\mathrm{i}\pi)) \\ &= Z^{-\frac{1}{4}}(z)\{Y_{\mathrm{p}}(-|F|)\exp[+\tfrac{1}{2}|F|] + \tfrac{1}{2}\mathrm{i}\, Y(|F|)\exp(-\tfrac{1}{2}|F|)\} \end{aligned} \quad (43)$$

showing that the solution has now acquired a subdominant contribution whose leading term is $\frac{1}{2}\pi$ out of phase and half the size of the dominant exponential, as it must be (D, pp. 8, 414).

The strength of the subdominant contribution grows from 0 to 1 across the Stokes line, being $\frac{1}{2}$ on the line itself, as indicated in (43). Berry (1989*a*, 1990*a*) showed that the strength increases smoothly, according to a uniform approximation involving an error function. In the present hyperasymptotic scheme, this emerges as an approximation to the first term in the first hyperseries, i.e. to the first terminant $K_{01}(F, N_0)$ (equation (18)) for F near the negative real axis. Here – and only here – our previous approximation (28) for the generalized terminants (25) breaks down, because in the ξ_1 integral the pole and the stationary point coincide. For the higher terminants ($r > 0$ or $n > 1$), the pole at $\xi_1 = -|F|$ for F negative real can never coincide with the stationary point at $\xi_1 = \xi_1^*$ (equation (27)) if the N_i are chosen optimally (i.e. as $|F|/2^i$), and so the principal-value pole contribution is exponentially small compared with that from the stationary point. A consequence of this is that the Stokes phenomenon, regarded as the rapid switching-on of the subdominant exponential, is completely described by the error function; further hyperasymptotic corrections vary only slowly across the Stokes line.

Now continue the positive rotation of F until $\arg F = \frac{3}{2}\pi$. We are now on an anti-Stokes line, where both exponentials have equal magnitude and opposite purely imaginary phases. The pole in the ξ_1 integrals has rotated positively up to the imaginary axis, and by continuity has dragged the contour with it. Again these integrals can be split into contributions from the pole and from the original contour among the positive real axis. From (40) we obtain (cf. (41))

$$Y(|F|\exp(\tfrac{3}{2}i\pi)) = S_0(|F|\exp(-\tfrac{1}{2}\mathrm{i}\pi)) + \frac{1}{2\pi} \int_0^\infty \mathrm{d}t \left(\frac{t}{\mathrm{i}|F|}\right)^{N_0} \frac{\exp(-t)}{t(1+\mathrm{i}t/|F|)} Y(t) + \mathrm{i}Y(|F|\exp(\tfrac{1}{2}\mathrm{i}\pi))\exp(-\mathrm{i}|F|), \quad (44)$$

that is (cf. (42))

$$Y(|F|\exp(\tfrac{3}{2}\mathrm{i}\pi)) = Y(-\mathrm{i}|F|) + \mathrm{i}Y(\mathrm{i}|F|)\exp(-\mathrm{i}|F|) \quad (45)$$

in which the functions Y on the right-hand side are evaluated with all K_{rn} contours along the positive real axis.

Thus on the anti-Stokes line the solution (4) is, exactly,

$$y(z) = Z^{-\frac{1}{4}}(z)\exp[+\tfrac{1}{2}\mathrm{i}|F|]\,Y(|F|\exp(\tfrac{3}{2}\mathrm{i}\pi))$$
$$= Z^{\frac{1}{4}}(z)\exp(\tfrac{1}{4}\mathrm{i}\pi)\,2\mathrm{Re}[Y(\mathrm{i}|F|)\exp\{\mathrm{i}(-\tfrac{1}{2}|F|+\tfrac{1}{4}\pi)\}], \tag{46}$$

showing that the subdominant contribution which appeared on the Stokes line is now equal in magnitude to the original exponential, and $y(z)$ is now proportional to a real oscillatory function, as it must be (D, p. 297).

The function $Y(F)$ is not single valued. We can see this by rotating further, to $\arg F = 2\pi$, that is onto the second Stokes line in the z plane, corresponding to a $\frac{4}{3}\pi$ rotation close to the transition point. Similar reasoning to that just used gives (cf. (42))

$$Y(|F|\exp(2\mathrm{i}\pi)) = \tfrac{1}{2}Y(|F|) + \mathrm{i}Y_{\mathrm{p}}(-|F|)\exp(|F|). \tag{47}$$

Finally, we rotate to $\arg F = 3\pi$, corresponding to a complete circuit of the transition point in the z plane. The pole has now dragged the ξ_1 integration contours around a point on the second sheet, above the real axis. There is a contribution from this pole at $\xi_1 = |F|\exp(2\pi\mathrm{i})$, as well as a half contribution from the pole at $\xi_1 = |F|$, leading to

$$Y(|F|\exp(3\mathrm{i}\pi)) = Y_{\mathrm{p}}(-|F|) + \mathrm{i}[Y(|F|\exp(2\mathrm{i}\pi)) + \tfrac{1}{2}Y(|F|)]\exp(-|F|)$$
$$= \mathrm{i}Y(|F|)\exp(-|F|). \tag{48}$$

The solution (4) now becomes

$$y(z) = \frac{\exp[\frac{1}{2}|F|]}{Z^{\frac{1}{4}}(z\to z\exp(2\pi\mathrm{i})}\,Y(|F|\exp(3\mathrm{i}\pi))$$
$$= \frac{\mathrm{i}\exp[-|F|/2]}{Z^{\frac{1}{4}}(z\to z\exp(2\pi\mathrm{i}))}\,Y(|F|). \tag{49}$$

This must be exactly the same as at the start of the circuit, because $y(z)$ is single valued, and indeed it is, because the factor i is cancelled by the change in $Z^{\frac{1}{4}}$ round its zero.

6. Numerical illustration: the Airy function

Here we choose $Z(z) = z$, which is the paradigm for the study of equation (1) near a single transition point. Then without loss of generality we can set $\lambda = 1$, because asymptotics for large λ is equivalent to that for large $|z|$. From (3), the singulant is

$$F = \tfrac{4}{3}z^{\frac{3}{2}}. \tag{50}$$

The solution of (1) which has the form (4), that is decaying as $\mathrm{Re}\,z$ increases, is the Airy function $\mathrm{Ai}(z)$ (Abramowitz & Stegun 1964). Insert the correct constant, we have

$$\mathrm{Ai}(z) = \frac{\exp(-\frac{1}{2}F)}{2\pi^{\frac{1}{2}}z^{\frac{1}{4}}}\,Y(F) \tag{51}$$

so that

$$Y(F) = 2\sqrt{\pi}(\tfrac{3}{4}F)^{\frac{1}{6}}\exp(\tfrac{1}{2}F)\,\mathrm{Ai}\{(\tfrac{3}{4}F)^{\frac{2}{3}}\}. \tag{52}$$

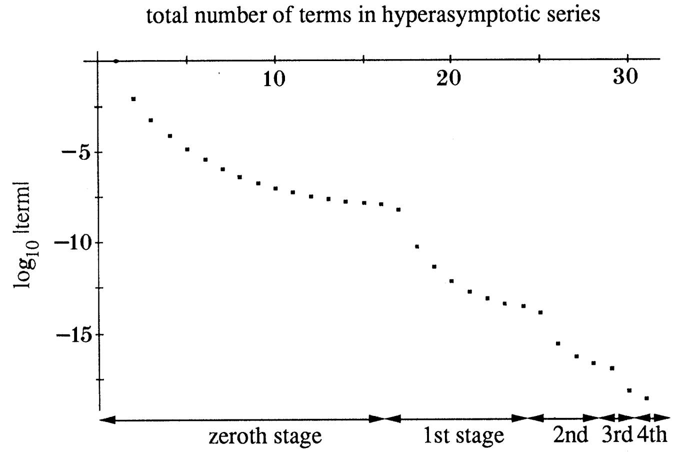

Figure 1. Decrease of the terms in the first five hyperseries of $Y(16)$, for the Airy function Ai.

Table 1. *Hyperasymptotic approximations to the Airy function* Ai (*equation* (51)) *for* $F = 16$

level	approximation to $Y(F)$	approx. − exact
lowest	1	8.163×10^{-3}
S_0	0.991836793511323459 1100	-5.677×10^{-9}
S_0+S_1	0.991836799188251255 0983	-1.134×10^{-14}
$S_0+S_1+S_2$	0.991836799188262590 7500	-8.160×10^{-18}
$S_0+\ldots+S_3$	0.991836799188262599 8682	9.584×10^{-19}
$S_0+\ldots+S_4$	0.991836799188262600 6031	1.151×10^{-18}
exact	0.991836799188262598 9098	0

To compute $Y(F)$ hyperasymptotically we require the terms in its formal asymptotic expansion. These are

$$Y_r(F)=\frac{1}{(27F)^r}\frac{\Gamma(3r+\frac{1}{2})}{\Gamma(r+1)\,\Gamma(r+\frac{1}{2})}, \tag{53}$$

which satisfy $Y_0 = 1$ and the limiting form (9). We also require the generalized terminant integrals (25). K_{r1} and K_{r2} were computed from formulae in Appendix B; for higher hyperseries we used the approximations (28).

All numerical work was performed on an Apple Macintosh II computer using the program Mathematica (Wolfram 1988; Maeder 1990), which has the advantages that it can be configured to work to any specified accuracy and evaluate all its special functions for complex arguments. Mathematica has internal routines for Ai (z), which we checked against the representations in terms of Bessel functions of order $\pm\frac{1}{3}$ and as a convergent series.

First we study $Y(F)$ on the positive real axis, that is positive real z, and choose $F = 16$, that is $z = 5.241482788417793241 3\ldots$. From (35), we can hope to be able to compute $Y(16)$ with an error 8.4×10^{-19}.

Figure 1 shows the decrease of the terms in the first five stages of hyperasymptotics, that is the terms in S_0, S_1, S_2, S_3 and S_4 (the last series having only one term). The optimality of the hyperseries is obvious. Table 1 shows the numerical values of the successive approximants to $Y(F)$, together with their errors. The improvement with hyperasymptotics is dramatic. Even the first level reduces the error of super-

Table 2. *Hyperasymptotic approximations to the Airy function* Bi (*equation* (55)) *for* $|F| = 16$

stage	approximation to $Y_p(-|F|)$	approx. − exact
lowest	1	-9.355×10^{-3}
S_0	1.009354544224441282011 12	-7.389×10^{-9}
S_0+S_1	1.009354551613418769424 46	-4.296×10^{-14}
$S_0+S_1+S_2$	1.009354551613461695449	-3.012×10^{-17}
$S_0+\ldots+S_3$	1.009354551613461721493	-4.078×10^{-18}
$S_0+\ldots+S_4$	1.009354551613461721984	-3.587×10^{-18}
exact	1.009354551613461725570 54	0

asymptotics (S_0) by five orders of magnitude, and at the last stage (here S_4) the reduction is nearly ten orders of magnitude, and we are close to the ultimate error of the method.

Now we move onto the Stokes line, where $\arg z = \frac{2}{3}\pi$ and $\arg F = \pi$. Because of the identity (42), it is necessary only to study $Y_p(-|F|)$, which is the real function obtained when all ξ_1 integrals in the generalized terminants (25) are interpreted as principal values. This is equivalent to studying the exponentially increasing Airy function Bi(z) (Abramowitz & Stegun 1964) for z positive real, because (cf. (51) and (52))

$$\mathrm{Bi}\,(z) = \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \exp\left(\tfrac{1}{2}|F|\right) Y_p(-|F|), \tag{54}$$

i.e.

$$Y_p(-|F|) = \sqrt{\pi}\left(\tfrac{3}{4}|F|\right)^{\frac{1}{6}} \exp\left(-\tfrac{1}{2}|F|\right) \mathrm{Bi}\,\{(\tfrac{3}{4}|F|)^{\frac{2}{3}}\}. \tag{55}$$

The approximations to $Y_p(-|F|)$ are shown in table 2, again for $|F| = 16$. Here the improvement over superasymptotics (S_0) with the first three stages of hyperasymptotics is nine orders of magnitude.

Gailitis & Silverstone (1988) have studied the asymptotics of Bi (2.5), whose exact value is 6.4816607.... With Padé summation of the first 40 Y_r, they obtain for this real quantity the complex value 6.48166 − 0.00001 i. With three stages of hyperasymptotics – nine terms in all – based on only the first five Y_r, we find 6.4816598, thereby achieving the ultimate error of the method, which from (35) is 0.000001.

Finally, we examine the anti-Stokes line, where $\arg z = \pi$ and the Airy function is oscillatory. From equation (46),

$$\mathrm{Ai}\,\{-(\tfrac{3}{4}|F|)^{\frac{2}{3}}\} = \mathrm{Re}\,[Y(\mathrm{i}|F|)\exp\{\mathrm{i}(-\tfrac{1}{2}|F| + \tfrac{1}{4}\pi)\}]/\sqrt{\pi}(\tfrac{3}{4}|F|)^{\frac{1}{6}}. \tag{56}$$

Here we choose to study the zeros, rather than the values, of Ai.

The lowest few stages of hyperasymptotic approximation for the first three zeros are shown in table 3, with errors displayed as fractions of the asymptotic mean zero spacing 2π. For the crudest approximation ($Y \approx 1$), the errors are of order $1/|F|$ and are roughly the same for the three zeros. Superasymptotics ($Y \approx S_0$) gives exponentially small errors which therefore diminish considerably from the first to the third zero. Hyperasymptotics gives further, enormous, improvements: even for the first zero, where $|F| \approx 4.8$ is hardly large, the error diminishes from 10^{-4} (S_0) to 10^{-7} ($S_0+S_1+S_2$); for the third zero ($|F| \approx 17.3$) the additional improvement is nearly eight orders of magnitude.

7. Discussion

We have given an explicit description of hyperasymptotic corrections, that is corrections beyond the accuracy $\exp(-|F|)$ obtained by the usual (superasymptotic) procedure of summing the primary asymptotic series down to its least term. The

Table 3. *Hyperasymptotic approximations to the values of* $|F| = (3|z|/4)^{\frac{2}{3}}$ *for the first three zeros of the Airy function* Ai$(-|z|)$ (*equation* (56))

Table 3*a*

stage	$\lvert F\rvert$ (1st zero)	(approx.–exact)/2π
$Y \approx 1$	4.712388980384689857694	-8.67×10^{-3}
S_0	4.765396652270803735893	-2.38×10^{-4}
S_0+S_1	4.766878592385215365778	-2.33×10^{-6}
$S_0+S_1+S_2$	4.76689444457039445853	1.94×10^{-7}
exact	4.766893225061654917199	0

Table 3*b*

stage	$\lvert F\rvert$ (2nd zero)	(approx.–exact)/2π
$Y \approx 1$	10.99557428756427633462	-3.95×10^{-3}
S_0	11.02039325027731244171	3.28×10^{-7}
S_0+S_1	11.02039119052794602966	-3.17×10^{-11}
$S_0+S_1+S_2$	11.02039119072562870	-2.84×10^{-13}
exact	11.02039119072741674417	0

Table 3*c*

stage	$\lvert F\rvert$ (3rd zero)	(approx.–exact)/2π
$Y \approx 1$	17.27875959474386281154	-2.54×10^{-3}
S_0	17.29471532320673859433	-4.98×10^{-10}
S_0+S_1	17.29471532633719810422	-5.94×10^{-16}
$S_0+S_1+S_2$	17.29471532633720182004	-2.20×10^{-18}
exact	17.29471532633720183386	0

structure of the scheme is particularly interesting. Each correction term is the product of a universal factor, namely a generalized terminant integral K_{rn} (equation (25)), and a non-universal factor, namely a term Y_r of the original asymptotic series for the particular function being approximated. In addition, we have investigated the asymptotics of the hyperasymptotic sequence itself, and shown that the method terminates naturally, leaving an ultimate error of order $\exp(-2.386|F|)$.

Much remains to be done. Our arguments have been based on the resurgence formula (11), which applies when the point z in the solution of (1) is dominated by a single transition point. While this is the generic case in the limit $\lambda \rightarrow \infty$, it is easy to envisage other situations, in which additional transition points are near enough to make appreciable contributions. Presumably hyperasymptotics should then be based on the more general resurgence formula (10) (cf. Rakovic & Solov'ev 1989), but we emphasize that a proper mathematical foundation for this formula is lacking.

Any such generalization of hyperasymptotics, applicable to clusters of transition points, would have to be compatible with the corresponding generalizations of the Stokes relations of §5 (cf. Heading 1977; Olver 1978; Voros 1983; Sibuya 1977).

We thank Dr W. G. C. Boyd for helpful discussions. C.J.H. acknowledges a Research Studentship from the U.K. Science and Engineering Research Council.

Appendix A. Resumming resurgence

Summing both sides of (11), we obtain $Y(F)$ from (5) (after an interchange of summation labels) as

$$Y(F) = \frac{1}{2\pi}\sum_{r=0}^{\infty} Y_r(F)\,(-F)^r \sum_{s=0}^{\infty} \frac{(s-r-1)!}{(-F)^s}. \tag{A 1}$$

Next, the sum is interpreted by the Borel procedure, which gives

$$Y(F) = \frac{1}{2\pi}\int_0^{\infty} \frac{\exp(-\xi)}{\xi(1+\xi/F)} \sum_{r=0}^{\infty} Y_r(F)\left(-\frac{F}{\xi}\right)^r$$
$$= \frac{1}{2\pi}\int_0^{\infty} \mathrm{d}x \frac{\exp(-Fx)}{x(1+x)} \sum_{r=0}^{\infty} Y_r(F)\left(-\frac{1}{x}\right)^r. \tag{A 2}$$

Because our treatment is restricted to a single first-order transition point, well separated (in F) from any others, it is consistent to regard $Z(z)$ as linear (we exclude such cases as $Z(z) = z\exp(z)$, where the transition point at infinity has finite F). Then $Y_r(F)$ is proportional to F^{-r} (cf. (53)), so that

$$Y_r(F)\,(-1/x)^r = (-1)^r\, Y_r(Fx). \tag{A 3}$$

Again using (5) we obtain the resummed resurgence formula

$$Y(F) = \frac{1}{2\pi}\int_0^{\infty} \mathrm{d}x \frac{\exp(-Fx)}{x(1+x)}\, Y(Fx) \tag{A 4}$$

corresponding precisely to the exact Stieltjes transform for the Bessel function of order $\frac{1}{3}$, as used by Boyd (1990) in his study of the Stokes phenomenon.

In the more general situation, where there is more than one transition point, Y_r will not be precisely proportional to F^{-r}, and the scaling (A 3), necessary to obtain (A 4), cannot be used. But by construction Y_r is precisely proportional to the original large parameter λ. We can exploit this by redefining the singulant as (3) without the factor λ, that is as

$$\mathscr{F}(z) \equiv 2\int_{z^*}^{z} \mathrm{d}\xi\, Z^{\frac{1}{2}}(\zeta). \tag{A 5}$$

Then we can resum the conjectured more general relation (10), involving all the transition points F_j, without assuming (A 3). Thus we obtain a putative resurgence formula for the multiplier $Y(\mathscr{F}, \lambda)$ in the solution (4):

$$Y(\mathscr{F}, \lambda) = \frac{1}{2\pi}\sum_j \int_0^{\infty} \mathrm{d}x \frac{\exp\{-\lambda(\mathscr{F}-\mathscr{F}_j)\,x\}}{x(1+x)}\, Y(\mathscr{F}, \lambda\, x) \tag{A 6}$$

relating the solutions of (1) for different values of λ. We can, however, neither prove nor exploit this intriguing relation.

Appendix B. Calculation of generalized terminants

Our aim here is to express the terminants for the first two stages of hyperasymptotics, that is the integrals (18) and (22), in terms of functions that can be evaluated by Mathematica with sufficient precision (up to 24 digits for the calculations presented in §6).

The first-stage terminant integrals K_{r1} can be expressed in terms of incomplete gamma functions (cf. D, p. 415, Abramowitz & Stegun 1964):

$$K_{r1}(F,N_0) = ((-1)^{N_0}/2\pi)\exp(F)\,\Gamma(N_0-r)\,\Gamma(r-N_0+1,F). \qquad \text{(B 1)}$$

Now we use

$$\Gamma(-N,F) = \frac{(-1)^N}{N!}\left[\Gamma(0,F)-\exp(-F)\sum_{m=0}^{N-1}\frac{(-1)^m m!}{F^{m+1}}\right] \qquad \text{(B 2)}$$

and obtain

$$K_{r1}(F,N_0) = \frac{(-1)^{r+1}}{2\pi}\left[\exp(F)\,E_1(F)-\sum_{m=0}^{N_0-r-2}\frac{(-1)^m m!}{F^{m+1}}\right], \qquad \text{(B 3)}$$

where E_1 denotes the exponential integral function.

To simplify the second-stage terminant integrals K_{r2}, the first step is to evaluate the ξ_2 integral in (22):

$$K_{r2}(F,N_0,N_1) = \frac{(-1)^{N_0+N_1}\Gamma(N_1-r)}{(2\pi)^2}\int_0^\infty \mathrm{d}t\,\frac{\Gamma(r-N_1+1,Ft)\,t^{-r+N_0-1}}{(1+t)}. \qquad \text{(B 4)}$$

Next we use (B 2), which gives

$$K_{r2} = (-1)^{N_0-r-1}/(2\pi)^2\,(I-J),$$

where

$$\left.\begin{aligned} I &\equiv \int_0^\infty \mathrm{d}p\,\frac{\exp(-p)}{p}\int_0^{p/F}\mathrm{d}t\,\frac{t^{-r+N_0-1}}{(1+t)},\\ J &\equiv \sum_{m=0}^{N_1-r-2}\frac{(-1)^m m!}{F^{m+1}}\int_0^\infty \mathrm{d}t\,\frac{\exp(-Ft)\,t^{-r+N_0-m-2}}{(1+t)}. \end{aligned}\right\} \qquad \text{(B 5)}$$

For I we substitute the expansion

$$\frac{t^N}{(1+t)} = \frac{(-1)^N}{(1+t)}-(-1)^N\sum_{r=0}^{N-1}(-t)^r \qquad \text{(B 6)}$$

and thereby obtain

$$I = \int_0^\infty \mathrm{d}p\,\frac{\exp(-p)}{p}\ln\left(1+\frac{p}{F}\right)-\sum_0^{N_0-r-2}\frac{(-1)^q q!}{(q+1)F^{q+1}}. \qquad \text{(B 7)}$$

The integral can be expressed in terms of Euler's constant γ and the digamma function ψ (Prudnikov *et al.* 1986, vol. 1, p. 530), leading to

$$I = \tfrac{1}{2}[(\ln F+\gamma)^2+\tfrac{1}{2}\pi^2]-\sum_{k=1}^{\infty}\left[\frac{2/k+\psi(k)-\ln F}{k!\,k}\right]F^k$$
$$-\sum_0^{N_0-r-2}\frac{(-1)^q q!}{(q+1)\,F^{q+1}} \quad (\operatorname{Re} F \geqslant 0). \qquad \text{(B 8)}$$

The sum over k converges adequately fast for our purposes.

To simplify J we notice that the integral has the same form as K_{r1} and then substitute (B 1) and (B 2)

$$J = (-1)^{N_0-r-2}\left[\exp(F)\,E_1(F)\sum_{m=0}^{N_1-r-2}\frac{m!}{F^{m+1}}+\sum_{m=0}^{N_1-r-2}\sum_{s=0}^{N_0-r-m-3}\frac{(-1)^s m!\,s!}{F^{m+s+2}}\right). \qquad \text{(B 9)}$$

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Received 1 May 1990; accepted 6 June 1990