Quantum Chaology: Our Knowledge and Ignorance

M.V. Berry
H.H. Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TL, UK

Abstract. A brief review is given of what has been achieved in understanding the discrete spectra of classically chaotic bound quantum systems, emphasising the central role of semiclassical techniques and the unsolved problems associated with the clash between the long-time limit and the semiclassical limit.

We are fairly sure that, apart from a curious special class of cases [1,2], there is no chaos in quantum mechanics. It is obvious, however, that in the semiclassical limit $\hbar \to 0$ (e.g. highly excited states) the quantum behaviour must somehow reflect the nature of the classical trajectories, and in particular must exhibit characteristic effects if the orbits are chaotic. For the study of these quantum signatures of classical chaos I proposed the term 'quantum chaology' [3,4]. Here I provide a brief 'readers' guide' to the main advances there have been in the subject, and emphasize the major unsolved problems. I confine myself to the quantum chaology of the discrete spectra (energy levels and wavefunctions) of time-independent bound Hamiltonians with N (>1) freedoms. Therefore I will not discuss the important areas of the time-development of quantum states [5,6] or chaotic scattering [7,8]. It is hardly necessary to say that even for spectra my account will be partial and deliberately selective.

One thing that is very well known about the density $d(E)$ of energy levels is its average value $\langle d(E)\rangle$ (over energy ranges that are classically small but include many quantum levels). This is given by the Weyl rule [9,10]: one quantum state per classical phase space volume $\hbar^N$. Thus d is proportional to $\hbar^{-N}$ and the level spacing proportional to $\hbar^N$. The first few $\hbar$-corrections to $\langle d\rangle$ are known for quantum billiards (vibrating membranes) [10], including the effects of magnetic flux [11] and relativity [12] (Dirac spinorization). These corrections are routinely used in checking computations of quantum billiard eigenvalues, and form the basis of an extremely accurate error-detection scheme [13]. I have not seen explicit general formulae for the $\hbar$-corrections to $\langle d\rangle$ for general smooth potentials, apart for the one-dimensional (WKB) case [14].

Although the first $\hbar$-corrections for the smoothed level density are known for billiards, nobody has studied the late terms of the series. But we know that this must diverge, because the exact $d(E)$ is a series of Dirac δs, which cannot be represented by a power series in $\hbar$. The approach to quantization, starting from $\langle d\rangle$, is given not by $\hbar$-corrections but by a series of contributions, of order $\exp\{iS_n/\hbar\}$ (i.e. nonanalytic as $\hbar \to 0$), from each classical closed orbit, with action S_n. [15,16,17]. Each closed orbit describes oscillatory clustering of the levels, on an energy scale $\hbar/T_n$, where T_n is the period of the orbit. It is important to appreciate that the clustering associated with a given orbit is classically small (of order $\hbar$) but large in comparison with the quantum level spacing (each 'cluster' includes of order $1/\hbar^{N-1}$ levels). Therefore in order to understand spectra on fine scales (that is on the scale of individual levels) within the framework of semiclassical asymptotics, enormously long classical orbits must be involved, with periods of order $1/\hbar^{N-1}$.

There is, however, another framework for studying the fine scales of spectra, namely energy-level statistics [18-22]. Here we abandon the attempt to locate each level precisely, and instead study average properties (fluctuations, correlations, spacing distributions). Now universality emerges: the statistics depend only on the classical chaology, in the following way. For classically integrable systems, the levels are Poisson-distributed [18]; for classically chaotic systems, levels are distributed according to random-matrix ensembles [23] that depend on symmetries - Gaussian unitary (GUE) when there is no symmetry [24-26], Gaussian orthogonal (GOE) when there is time-reversal and the system is Bosonic (e.g. scalar-wave) [21], and Gaussian symplectic when there is time-reversal and the system is Fermionic [27]. Of course there are intermediate cases, where some orbits are regular and some chaotic, and, correspondingly, statistics which interpolate between the four universality classes.

To understand these largely numerical observations about level distributions, the appropriate theoretical framework must be semiclassical, because statistics are not even defined for an individual Hamiltonian unless there are infinitely many levels, and then almost all are highly excited - that is, semiclassical. The semiclassical connection is necessary in order to explain the observed dependence on the dynamics. Without this connection, other theories, such as those based on the 'motion' of levels as parameters vary [28-30], must fail, although they certainly contain interesting mathematics.

The only semiclassical theory available at present is based on the expansion of the spectral fluctuations $d-\langle d\rangle$ as a sum of closed orbits, and we have seen that any explanation of fine-scale spectral structure must involve the very long orbits. At this point we come up against the central theoretical problem in the whole subject [17], namely the fact that our semiclassical formulae break down at long times. This reflects the fundamental quantum-mechanical fact that the semiclassical limit $\hbar\to 0$ and the long-time limit $t\to\infty$ do not commute. We do not know how, or even whether, the closed-orbit sum generates the individual δs in the level density for chaotic systems. This is a serious - perhaps shocking - situation, because it means that we are ignorant of the mechanism of quantization. (Sometimes, singularities in $d(E)$ can be generated by summing selected infinite series of orbits, for example all repetitions of a given one, but these are false singularities [31], not energy levels, even for integrable systems [32].)

Nevertheless, there has been some progress in incorporating long times into the semiclassical framework. The intensities (squares of amplitudes) of the closed orbits are purely classical quantities which for long times obey a sum rule [33], depending only on the classical chaology and reflecting the uniformity of the distribution of long closed orbits in phase space. The requirement that the long orbits must combine (albeit by some unknown mechanism) to give a sequence of δs with the correct density $\langle d\rangle$ gives another sum rule [34,17], involving the interference between pairs of orbits. With these classical and semiclassical sum rules, it was possible to understand a large class of energy-level statistics [34] in a fundamental way, that is in terms of classical mechanics, by means of semiclassical asymptotics. These are the statistics quadratic in $d(E)$, such as the spectral rigidity, the number variance, the two-level correlation function, and the form factor (all defined in [22]).

From this semiclassical theory of some of the statistics, there emerged as an unexpected bonus the explanation [34] of the breakdown of universality at large energy ranges, first observed in computations [35] on an integrable system. Since the universality is a consequence of classical uniformity, and this only holds for long orbits, it will fail at and beyond energy scales where the spectral fluctuations depend on short orbits. This scale is $\hbar/T_{min}$, where T_{min} is the period of the shortest orbit with energy E.

The difficulties associated with long times disappear if the spectrum is smoothed, for example by giving the energy a sufficiently large imaginary part $i\varepsilon$, because then the sum over closed orbits converges - and indeed provides a useful description of the large-scale spectral fluctuations [41,42]). For chaotic systems, however, ε must exceed a value determined [36] by the entropy of the dynamics. Overcoming this 'entropy barrier' and thereby understanding quantization, is thus a problem of analytic continuation. One possible way to solve this is suggested by the analogy [34, 37,17] between the spectra of classically chaotic systems and the imaginary parts of the zeros of Riemann's zeta function. This analogy has already proved productive in the quantum→Riemann direction, by inspiring an analytical and quantitatively-accurate formula [38] for the number variance of the Riemann zeros. Recently [39] it was employed in the Riemann→quantum direction, to obtain a quantum analogue of the Riemann-Siegel formula [40] which enables efficient computation of the zeros; the idea (already underlying the semiclassical sum rule [34]) is the 'resurgence' [17] of the long orbits to give, by resummation, information related to $<d>$ and the short orbits. This new quantum condition is the vanishing of a sum over a large but finite number of closed orbits. Perhaps it will lead to a semiclassical explanation for the higher-than quadratic level statistics, which the earlier theory [34] did not achieve.

Even if the closed orbits can be resummed to give a quantum condition, either explicitly in the form of the 'Riemann-Siegel look-alike' or implicitly as a compactified determinant [43], there remains the awkward fact that the semiclassical theory on which it is based is only a lowest-order one, in the sense that higher terms in $\hbar$, mutiplying the contributions $\exp\{iS_n/\hbar\}$, are ignored (and unknown). This raises the unexplored question of the asymptotic accuracy and meaning of any 'eigenvalues' generated by these methods: the true levels have spacing $\hbar^N$, but the methods have errors $\hbar^2$. It might be possible to get some insight into this difficult problem by studying systems with special symmetries, such as quantum billiards on tilings of the hyperbolic plane [44,42], or the quantum cat maps [45-7], for which the sum over closed orbits is known to be exact.

For wavefunctions, the mixture of knowledge and ignorance is very similar - although for a long time we did not appreciate how close the analogy is. There is a lowest-order description (analogous to the Weyl rule for the smoothed level density) which states that the probability density for an eigenstate of an ergodic system is the projection of the energy surface from phase space onto coordinate space [48] (for integrable systems, tori must be projected). Then some spectacular computations [49,50] showed wavefunctions looking very different from uniform projections of the energy surface, revealing imprints ('scars') of individual closed orbits. These led to the development of a theory [51] in which, just as for the level density, the lowest-order wavefunctions are corrected by characteristic interference fringes near the closed orbits. The theory can also be formulated in phase space [52,17], and the characteristic fringes there, which take the form of hyperbolae with each orbit's invariant manifolds as asymptotes, could explain certain features of computations [53] of the Husimi (smoothed Wigner) functions for eigenstates. We do not know how to understand the (divergent) superposition of very long and overlapping scar patterns. There is reason to think that their net effect might (after some kind of resummation) be to provide a Gaussian random background decorating the smooth background of the lowest-order description, as conjectured in the old theory [48,9].

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