Unfolding the high orders of asymptotic expansions with coalescing saddles: singularity theory, crossover and duality

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We study the leading behaviour of the late coefficients (high orders r) of asymptotic expansions in a large parameter k, for contour integrals involving a cluster of coalescing saddles, and thereby establish the form of the divergence of the expansions. The two principal cases are: 'saddle-to-cluster', where the integral is through a simple saddle and its expansion diverges because of a distant cluster; and 'cluster-to-saddle', where the integral is through a cluster and its expansion diverges because of a distant simple saddle. In both, the large-r coefficients are dominated by the 'factorial divided by power' familiar in asymptotics, but this changes its form as the saddles in the cluster are made to coalesce and separate by varying parameters $\boldsymbol{A} = \{A_1, A_2 \ldots\}$ in the integrand. The 'crossover' between different forms is described by a series of canonical integrals, built from the cuspoid catastrophe polynomials of singularity theory that describe the geometry of the coalescence. The arguments of these integrals involve not only the $\boldsymbol{A}$ but also fractional powers of r, which by a curious duality replace the powers of the original large parameter k which occur in uniform approximations involving these integrals. A by-product of the cluster-to-saddle analysis is a new exact formula for the coefficients of uniform asymptotic expansions.

1. Introduction

We shall study integrals of the form

$$I_C(k, \boldsymbol{A}) = \int_C \mathrm{d}z\, g(z) \exp\{-kf(z, \boldsymbol{A})\} \tag{1}$$

for large real k. Here C is an infinite contour connecting valleys of the exponential, and f and g are analytic functions of z near C. In addition, f depends on parameters

$$\boldsymbol{A} = \{A_1, A_2, \ldots\}, \tag{2}$$

whose variation causes some of its saddles $z_n(\boldsymbol{A})$ (defined as zeros of $f' \equiv \mathrm{d}f(z, \boldsymbol{A})/\mathrm{d}z$) to coalesce (that is, to become degenerate), and to unfold into a cluster of simple saddles.

The method of steepest descent (Dingle 1973; Olver 1974; De Bruijn 1958; Wong 1989) is based on deforming C through some of the saddles. It generates factorially divergent asymptotic expansions in descending powers of k (with complicated coefficients whose calculation can now be facilitated by symbolic manipulation

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packages such as *Mathematica*, making the method practical). The divergences are caused by 'distant saddles' through which C does not pass (Berry & Howls 1991, hereinafter called I).

Our purpose here is to study the high orders of such expansions (i.e. the 'asymptotics of the asymptotics'), and thereby establish the nature of the divergence, as $\boldsymbol{A}$ varies and clusters of saddles degenerate and unfold. The divergences for simple and degenerate saddles are different, and we will obtain the crossover between them.

There are two principal cases. In the first (§2), C passes through a simple saddle, and the divergence in the ordinary steepest-descent expansion arises from a distant cluster; we call this 'saddle-to-cluster'. Here the main result is equation (28). In the second (§3), C passes through a cluster, and the divergence of the uniform asymptotic expansion is caused by a distant simple saddle; we call this 'cluster-to-saddle'. Here the main result is equation (66). We also obtain a new exact formula (equation (58)) for the coefficients in the uniform expansion. A possibility is to study the more general cluster-to-cluster case, but we do not pursue this.

In both cases, we shall invoke singularity theory (Arnold 1986; Poston & Stewart 1978) to map functions ϕ of f (for example $\ln f$, or f itself) in the neighbourhood of a cluster of N saddles onto the polynomial normal form of the cuspoid catastrophe with codimension $N-1$, namely

$$\phi\{f(z,\boldsymbol{A})\} = \phi_0 + F_N(u,\boldsymbol{X}), \tag{3}$$

where $\boldsymbol{X} \equiv \{X_m\} \equiv \{X_1 \ldots X_{N-1}\}$ and $F_N(u,\boldsymbol{X}) = \dfrac{u^{N+1}}{N+1} + \sum_{m=1}^{N-1} X_m \dfrac{u^m}{m}$. (4)

At the origin of the parameters $\boldsymbol{X}$, N saddles coalesce; neighbourhoods of the origin contain all possible stable unfoldings into combinations of simple saddles. The mapping

$$z \rightarrow u(z,\boldsymbol{A}) \tag{5}$$

is guaranteed locally one-to-one by identifying the saddles of f and F. This gives N equations determining the N unknowns $\phi_0(\boldsymbol{A}), \boldsymbol{X}(\boldsymbol{A})$.

Cuspoid catastrophes will govern high-order asymptotic crossover through the canonical integrals

$$W_N(\boldsymbol{X}) \equiv \int_{\infty \exp\{2\mathrm{i}\pi/(N+1)\}}^{\infty} \mathrm{d}u \exp\{-F_N(u,\boldsymbol{X})\}, \tag{6}$$

where the contour joins adjacent valleys of F_N. The two simplest cases are

$$W_1 = \surd(2\pi), \quad W_2(X_1) = 2\pi \exp(-\tfrac{1}{6}\mathrm{i}\pi)\,\mathrm{Ai}\{X_1 \exp(\tfrac{1}{3}\mathrm{i}\pi)\}, \tag{7}$$

where Ai denotes the Airy function (Abramowitz & Stegun 1964).

These canonical integrals are familiar as the principal ingredients in uniform approximations of *low* order for integrals of the type (1) (Chester *et al.* 1957; Bleistein 1967; and §3). In optics, they are visible as diffraction catastrophes (Berry & Upstill 1980). We shall find an interesting duality: W_N reappears in the high-order coefficients, with the order playing the role of k in (1).

We could have chosen other contours in (6), namely

$$W_{N,j}(\boldsymbol{X}) \equiv \int_{\infty \exp\{2\mathrm{i}\pi(j+1)/(N+1)\}}^{\infty \exp\{2\mathrm{i}\pi j/(N+1)\}} \mathrm{d}u \exp\{-F(u,\boldsymbol{X})\}. \tag{8}$$

These integrals can, however, be expressed in terms of $W_N(=W_{N,0})$:

$$W_{N,j}(\boldsymbol{X}) = \exp\{2\pi i j/(N+1)\}\, W_N(\{X_m \exp\{-2\pi i m j/(N+1)\}\}). \tag{9}$$

Moreover

$$\sum_{j=0}^{N} W_{N,j}(\boldsymbol{X}) = 0. \tag{10}$$

To save writing in what follows, we shall frequently omit the $\boldsymbol{A}$ and $\boldsymbol{X}$ dependences of f and F_N. In addition, we shall often make tacit use of the asymptotic relation

$$(r+\lambda)!/r^{\mu} \to (r+\lambda-\mu)! \quad \text{as} \quad r\to\infty, \tag{11}$$

which follows from Stirling's formula.

2. Saddle-to-cluster

(a) *The existence of crossover*

Let C in (1) pass through a simple saddle, at $z = z_0$ (figure 1). Then the usual method of steepest descent (based on the transformation $f(z) = f_0 + \frac{1}{2}u^2$, which is simply the codimension-zero case of (3)) can be applied. The standard result (Copson 1965; Dingle 1973) is the formal expansion

$$I(k,\boldsymbol{A}) = \frac{\exp\{-kf_0(\boldsymbol{A})\}}{k^{\frac{1}{2}}} \sum_{r=0}^{\infty} \frac{T_r(\boldsymbol{A})}{k^r}, \tag{12}$$

where

$$T_r(\boldsymbol{A}) = \frac{(r-\frac{1}{2})!}{2\pi i}\oint_{C_0} dz \frac{g(z)}{[f(z,\boldsymbol{A})-f_0(\boldsymbol{A})]^{r+\frac{1}{2}}}. \tag{13}$$

Here $f_0(\boldsymbol{A}) \equiv f(z_0,\boldsymbol{A})$, and the contour (figure 1) encircles z_0. The integrals can be evaluated as combinations of derivatives of f and g at z_0; these get rapidly more complicated as r increases (Dingle 1973).

To find the high orders, that is T_r for large r, we first write

$$1/[f(z)-f_0]^{r+\frac{1}{2}} = \exp\{-(r+\tfrac{1}{2})\ln\{f(z)-f_0\}\} \tag{14}$$

and note that for large r this function is concentrated near the saddles of the logarithm, which coincide with those of f. Next we expand C_0 to a series of infinite arcs (figure 1) connecting valleys of some of the distant saddles of f; these are the 'adjacent saddles' discussed in I. Thus T_r is (exactly) a sum over adjacent saddles. (This would not be true if the expanding contour encountered zeros of $f-f_0$, but as shown in I this does not happen.) We are here interested in the leading large-order behaviour, and so consider only the 'nearest' adjacent saddle z_*, namely that with the smallest $|f_*-f_0|$. Thus

$$T_r(\boldsymbol{A}) \approx \frac{(r-\frac{1}{2})!}{2\pi i}\int_{C_*} dz\, g(z) \exp\{-(r+\tfrac{1}{2})\ln\{f(z,\boldsymbol{A})-f_0(\boldsymbol{A})\}\}. \tag{15}$$

Now let $\boldsymbol{A} = 0$ denote particular parameter values for which z_* is an N-fold degenerate saddle, resulting from the coalescence of N simple saddles. Near z_* we have

$$\ln\{f(z,0)-f_0(0)\} = \ln\{f_*-f_0(0)\} + \frac{f_*^{(N+1)}}{(N+1)!\,(f_*-f_0)}(z-z_*)^{N+1}+\dots, \tag{16}$$

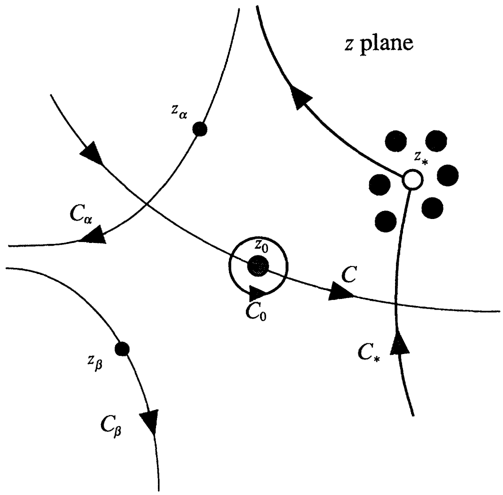

Figure 1. Contours for simple saddle at z_0, dominated by a distant cluster near z_*. The original contour (equation (1)) is C, and the loop defining the coefficients T_r (equation (13)) is C_0. The dominant arc C_* of the expanded loop contour passes through a cluster of distant saddles at z_*. The saddles z_α and z_β are more remote, and their arcs C_α and C_β do not contribute to the late T_r in leading order.

where $(N+1)$ denotes the $(N+1)$st derivative at z_*. Substituting into (15), we obtain the dominant part of T_r as

$$T_r(0) \approx \frac{(r+(N-1)/2(N+1)-1)!}{(f_*-f_0)^{r+(N-1)/2(N+1)}} \frac{g_*}{[f_*^{(N+1)}]^{1/(N+1)}} K_N. \tag{17}$$

Here the number K_N, obtained by evaluating a canonical integral of the form $W_N(0)$ (equation (6)), is

$$K_N = \frac{(-1)^\gamma}{\pi(N+1)}\left(-\frac{N}{N+1}\right)!\,(N+1)!^{1/(N+1)} \sin\left\{\frac{\pi}{N+1}\right\} \exp\left\{\mathrm{i}\frac{\pi}{N+1}\right\}, \tag{18}$$

where $\gamma(=0$ or $1)$ is a contour orientation anomaly (see I). The phase of f_*-f_0 in (17) can be chosen to incorporate the relation between the contour C_* (connecting the pair of adjacent valleys at z_* encountered by the expanding C_0) and the canonical choice (6) (cf. (9)).

From (11) we see that the result (17) implies that the contribution of a late term to the integral (1) scales as

$$k^\alpha \frac{(r+\alpha-1)!}{k^{\frac{1}{2}} F^{r+\alpha}}, \quad \text{where} \quad \alpha = \frac{(N-1)}{2(N+1)} \quad \text{and} \quad F = k(f_*-f_0). \tag{19}$$

This has the 'factorial divided by power' divergence typical of asymptotic series (Dingle 1973). In applications the least term is important. This can be estimated by Stirling's formula as

$$k^{\alpha-\frac{1}{2}} \min\left[\frac{(r+\alpha-1)!}{|F|^{r+\alpha}}\right] \approx k^{\alpha-\frac{1}{2}} \sqrt{\left(\frac{2\pi}{F}\right)} \exp(-F). \tag{20}$$

The importance of saddle degeneration is shown by the prefactor k^α. For example, coalescence of two saddles ($N=2, \alpha=\frac{1}{6}$) increases the least term by a factor $k^{\frac{1}{6}}$ in comparison with a simple saddle ($N=1, \alpha=0$). (The exponents α are the singularity indices of Arnold (1975) and Varchenko (1976); see also Arnold *et al.* (1984).)

If now the parameters $\boldsymbol{A}$ are changed so as to completely separate the saddles in the cluster, those which are adjacent to z_0 will contribute individually to T_r, the contributions being of the form (17) with $N=1$. This implies that as $\boldsymbol{A}$ changes there will be a crossover in the form of the T_r, and consequently, a crossover in the size of the contribution to the original integral: from k^{α} to $k^0=1$. We shall now determine the form of this crossover in T_r by obtaining the scaling function of r that uniformly interpolates between the 'coalesced' and 'separated' behaviour.

(*b*) *Uniform approximation for crossover*

When $\boldsymbol{A} \neq 0$, the saddles are completely or partly separated, but can still be considered to constitute a cluster if their separations are much smaller than their distance to other saddles, such as the original z_0. Assuming this, we can evaluate T_r from (15) by mapping the exponent of the integrand onto the cuspoid catastrophe of codimension $N-1$ as in (3)–(5). The only unusual feature in applying the standard uniform approximation procedure (Chester *et al.* 1957; Bleistein 1967) is that here we map the logarithm, rather than the function itself. The mapping is from z to u via

$$\ln\{f(z,\boldsymbol{A})-f_0(\boldsymbol{A}_0)\} = \ln\{f_*(\boldsymbol{A})-f_0(\boldsymbol{A})\}+F_N(u,\boldsymbol{X}). \tag{21}$$

For the mapping to be non-singular near the cluster, the N saddles $z_n(\boldsymbol{A})$ of f and $u_n(\boldsymbol{X})$ of F must correspond. This generates N equations, from which the canonical parameters $\boldsymbol{X}(\boldsymbol{A})$, and also $f_*(\boldsymbol{A})$, can be determined. f_* can be regarded as the average height of the cluster, defined as the value of $f(z)$ at the point z_* corresponding to the centroid $u=0$ of the mapped saddles.

Thus (15) becomes

$$T_r \approx \frac{(-1)^{\gamma}(r-\frac{1}{2})!}{2\pi \mathrm{i}[f_*-f_0]^{r+\frac{1}{2}}}\int \mathrm{d}u\, G(u)\exp\{-(r+\tfrac{1}{2})F_N(u)\}, \tag{22}$$

where $G(u) \equiv g(z(u))\,\mathrm{d}z(u)/\mathrm{d}u$. The contour links neighbouring valleys of the exponential as in (8). To simplify writing we choose the canonical contour as in (6) (this can be justified by careful choices of phase).

To obtain the leading large-r approximation, G is first expanded as follows:

$$G(u) = \sum_{s=0}^{N-1} c_s u^s + F'_N(u)H(u). \tag{23}$$

Since F' vanishes at the saddles, the c_s can be determined by solving

$$G(u_n) = g(z_n)\frac{\mathrm{d}z_n}{\mathrm{d}u} = \sum_{s=0}^{N-1} c_s u_n^s. \tag{24}$$

This involves the derivatives of the mapping at the saddles, which can be found by differentiating (21) twice:

$$\frac{f''_n}{(f_n-f_0)}\left(\frac{\mathrm{d}z_n}{\mathrm{d}u}\right)^2 = Nu_n^{N-1}+\sum_{m=2}^{N-1}(m-1)X_m u_m^{m-2}. \tag{25}$$

Since the mapping is one-to-one, $\mathrm{d}z_n/\mathrm{d}u$ has a finite limit even when $f''_n=0$, that is when some or all of the N saddles are degenerate.

Next, the term involving H is neglected because it is of higher order in $1/r$ than the terms involving the c_s (this will become clear in §3, when higher orders in uniform approximations are considered in detail). When substituted into (22), this approximation to G gives the following integrals, which can all be expressed in terms of the canonical W_N defined in (6):

$$\int_{\infty \exp\{2i\pi/(N+1)\}}^{\infty} du\, u^s \exp\{-(r+\tfrac{1}{2})F_N(u,X)\} = \frac{1}{(r+\frac{1}{2})^{(s+1)/(N+1)}}\left(\delta_{s,0} - s\frac{\partial}{\partial\sigma_s}\right) W_N(\boldsymbol{\sigma}), \tag{26}$$

where $\delta_{s,0}$ denotes the Kronecker delta and

$$\boldsymbol{\sigma} \equiv \{\sigma_s\} \equiv \{(r+\tfrac{1}{2})^{(N+1-s)/(N+1)} X_s\}. \tag{27}$$

Thus (22) becomes, to leading order in r,

$$T_r \approx \frac{(-1)^\gamma (r-(N+3)/2(N+1))!}{2\pi i[f_* - f_0]^{r+\frac{1}{2}}} \sum_{m=0}^{N-1} \frac{c_m}{r^{m/(N+1)}}\left(\delta_{m,0} - m\frac{\partial}{\partial\sigma_m}\right) W_N(\boldsymbol{\sigma}), \tag{28}$$

where (11) has been used.

This general formula is the main result of the present section. It gives the coefficients of high orders $r \gg 1$ in the expansion about a simple saddle, in terms of the saddles in a distant cluster which dominates the divergence of the expansion. The description, uniformly valid during the crossover from the coalesced to separated cluster, depends on the canonical integrals (6) and their $N-1$ derivatives with respect to parameters. It might appear from (28) that the derivative terms are of higher order in $1/r$ and so could be omitted, but this is not the case because they are necessary to get the correct separated limit (when the W_N can themselves be represented by their lowest-order saddle-point approximation). The transition occurs over parameter ranges $\Delta \boldsymbol{A}$ which get smaller as r increases, as governed by the scaling (cf. (27))

$$\Delta A_s \propto \Delta X_s \propto r^{-[(N+1-s)/(N+1)]}. \tag{29}$$

(In optics the exponents are the fringe indices governing the scale of detail in diffraction catastrophes (Berry & Upstill 1980).)

(c) *Simple cases*

The simplest case of (28) is $N = 1$. Then the distant 'cluster' in fact consists of an isolated simple saddle, at z_*, say. There are no parameters, so only the term in c_0 occurs. To calculate c_0, (25) and (26) can be used:

$$c_0 = g_*[(f_* - f_0)/f_*'']^{\frac{1}{2}}. \tag{30}$$

Now (7) and (11) give

$$T_r \approx \frac{(-1)^\gamma (r-1)!}{2\pi i[f_* - f_0]^r} g_* \left[\frac{2\pi}{f_*''}\right]^{\frac{1}{2}}. \tag{31}$$

This is the known late-term formula for an isolated saddle (see e.g. I), and coincides with the $N = 1$ case of the coalesced-saddle formulae (17) and (18).

In the next special case N is arbitrary, but the distant 'cluster' is a completely degenerate saddle at z_*. Here it is necessary only to check that the general formula (28) reduces to the coalesced-saddle formulae (17) and (18). The parameters are

$X = 0$, so that the arguments of all the canonical integrals are zero and only the term involving c_0 is important (the derivative terms are of higher order in $1/r$). From (22) and (23), and the $(N+1)$st derivative of (21), we find

$$c_0 = G(0) = g_* \, \mathrm{d}z_*/\mathrm{d}u = g_*(N!(f_* - f_0)/f_*^{(N+1)})^{1/(N+1)}. \tag{32}$$

Now substitution into (28) and evaluation of $W_N(0)$ reproduces (17) and (18) as it must.

The simplest non-trivial case is $N = 2$. Let the two saddles in the distant cluster have locations $z_\pm$, and heights $f_\pm$. The canonical polynomial (4) (the fold catastrophe) with its single parameter X_1 that henceforth we denote by X, its saddles, and stationary values are

$$F_2(u, X) = \tfrac{1}{3}u^3 + Xu, \quad u_\pm = \pm(-X)^{\frac{1}{2}}, \quad F_\pm = \mp\tfrac{2}{3}(-X)^{\frac{3}{2}}. \tag{33}$$

Equation (21) gives the average cluster height, and the parameter X, as

$$f_* - f_0 = \surd[(f_+ - f_0)(f_- - f_0)], \quad X = \exp\{-\tfrac{1}{3}\mathrm{i}\pi\}\left[\tfrac{3}{4}\ln\left(\frac{f_+ - f_0}{f_- - f_0}\right)\right]^{\frac{2}{3}}. \tag{34}$$

Thus from (7) and (27) the canonical integral is an Airy function with argument

$$\sigma \exp\{\tfrac{1}{3}\mathrm{i}\pi\} = \left[\tfrac{3}{4}(r + \tfrac{1}{2})\ln\left(\frac{f_+ - f_0}{f_- - f_0}\right)\right]^{\frac{2}{3}}. \tag{35}$$

The pair of equations (25) give the coefficients c_0, c_1 as

$$c_0 = \frac{1}{2}\left(g_+\frac{\mathrm{d}z_+}{\mathrm{d}u} + g_-\frac{\mathrm{d}z_-}{\mathrm{d}u}\right), \quad c_1 = \frac{1}{2(-X)^{\frac{1}{2}}}\left(g_+\frac{\mathrm{d}z_+}{\mathrm{d}u} - g_-\frac{\mathrm{d}z_-}{\mathrm{d}u}\right). \tag{36}$$

For the derivatives, (25) gives

$$\mathrm{d}z_\pm/\mathrm{d}u = (-X)^{\frac{1}{4}}(\pm 2(f_\pm - f_0)/f''_\pm)^{\frac{1}{2}}. \tag{37}$$

Substituting these expressions into (28) and using (7), we obtain

$$T_r \approx \frac{(r - \frac{5}{6})!}{\mathrm{i}\surd 2[(f_+ - f_0)(f_- - f_0)]^{r/2 + \frac{1}{4}}} \times \left\{\left[g_+\left(\frac{f_+ - f_0}{f''_+}\right)^{\frac{1}{2}} + \mathrm{i}g_-\left(\frac{f_- - f_0}{f''_-}\right)^{\frac{1}{2}}\right]\Delta^{\frac{1}{6}}\,\mathrm{Ai}\{[(r + \tfrac{1}{2})\Delta]^{\frac{2}{3}}\}\right.$$
$$\left. - \left[g_+\left(\frac{f_+ - f_0}{f''_+}\right)^{\frac{1}{2}} - \mathrm{i}g_-\left(\frac{f_- - f_0}{f''_-}\right)^{\frac{1}{2}}\right] r^{-\frac{1}{3}}\Delta^{-\frac{1}{6}}\,\mathrm{Ai}'\{[(r + \tfrac{1}{2})\Delta]^{\frac{2}{3}}\}\right\}, \tag{38}$$

where $\Delta \equiv \frac{3}{4}\ln[(f_+ - f_0)/(f_- - f_0)]$. Thus the late terms are expressed explicitly in terms of the original saddle z_0 and the two saddles $z_\pm$ in the distant cluster. We expect this formula to hold when $|f_+ - f_-| \ll |f_* - f_0|$, i.e. when the cluster is well defined.

(*d*) *Numerical example*

In this case the distant cluster has two saddles. Let

$$g(z) = 1, \quad f(z, A) = \tfrac{1}{4}z^4 - \tfrac{2}{3}z^3 + \tfrac{1}{2}(1 - A)z^2, \tag{39}$$

where the parameter A is real. Since all derivatives of f at the saddle $z = 0$ are real, all terms in the expansion (12) are real, so that the T_r as given by (38) must be real too. The saddles and heights are

$$z_0 = 0, \quad z_\pm = 1 \mp A^{\frac{1}{2}}; \quad f_0 = 0, \quad f_\pm = \tfrac{1}{12} - \tfrac{1}{2}A - \tfrac{1}{4}A^2 \pm \tfrac{2}{3}A^{\frac{3}{2}} \tag{40}$$

so that the cluster $+,-$ coalesces when $A=0$. The $+$ saddle has been chosen as that which has larger f when $A>0$. The quantities appearing in (38) are

$$\ln\left(\frac{f_+-f_0}{f_--f_0}\right)=\ln\left(\frac{1+\mu}{1-\mu}\right)=\tfrac{4}{3}A, \tag{41}$$

where $\mu\equiv 8A^{\frac{3}{2}}/(1-6A-3A^2)$, and

$$f''_{\pm}=\mp 2A^{\frac{1}{2}}+2A=\mp 2A^{\frac{1}{2}}(1\mp A^{\frac{1}{2}}). \tag{42}$$

The cluster is well defined (as stipulated after equation (38)) if $-0.33<A<0.09$.

Substitution into (38) gives the late terms as

$$T_r(A)\approx\frac{(r-\frac{5}{6})!}{2(\frac{1}{12}-\frac{1}{2}A-\frac{1}{4}A^2)^r(1-\mu^2)^{\frac{1}{2}r+\frac{1}{4}}A^{\frac{1}{4}}}\left\{\left[\left(\frac{1+\mu}{1-A^{\frac{1}{2}}}\right)^{\frac{1}{2}}+\left(\frac{1-\mu}{1+A^{\frac{1}{2}}}\right)^{\frac{1}{2}}\right]A^{\frac{1}{6}}\,\mathrm{Ai}\{[(r+\tfrac{1}{2})A]^{\frac{2}{3}}\}\right.$$
$$\left.-\left[\left(\frac{1+\mu}{1-A^{\frac{1}{2}}}\right)^{\frac{1}{2}}-\left(\frac{1-\mu}{1+A^{\frac{1}{2}}}\right)^{\frac{1}{2}}\right]r^{-\frac{1}{3}}A^{-\frac{1}{6}}\,\mathrm{Ai}'\{[(r+\tfrac{1}{2})A]^{\frac{2}{3}}\}\right\}. \tag{43}$$

Equation (41) shows that $\mu>0$ when $0<A<2/\sqrt{3}-1\approx 0.155$, so that the T_r are obviously real as they must be. The Airy argument is positive, so that Ai is asymptotically a pure exponential; this corresponds to the late terms being dominated by the saddle z_+, nearest to the saddle $z_0=0$. When $0>A>-1-2/\sqrt{3}\approx -2.15$, the phase of μ is $\frac{3}{2}\pi$; this is also the phase of the logarithm in Ai, so the argument of Ai is negative and Ai is oscillatory. This corresponds to the late terms being dominated by both saddles in the cluster, because they are equidistant from that at $z=0$ (both in z and in absolute height). The quantities $1\pm\mu$ are complex conjugates, as are $1\pm A^{\frac{1}{2}}$; it follows that the T_r are real in this case too, as they must be.

When $|A|\ll 1$, the uniform approximation (43) can be replaced by the transitional approximation

$$T_r\approx 12^{r+\frac{1}{6}}\exp\{6rA\}(r-\tfrac{5}{6})!\,\mathrm{Ai}\{(12r)^{\frac{2}{3}}A\}. \tag{44}$$

This simple expression captures the behaviour of the coalesced cluster $A=0$ (it agrees exactly with (17) when phases are correctly chosen) and the behaviour of the separated cluster qualitatively. It fails when $\ln f$ can no longer be approximated by a cubic function of z near $z_{\pm}$; then the full uniform approximation must be used, in which $\ln f$ is *mapped* onto a cubic (equation (33)).

To test the approximations, it is necessary to calculate the exact coefficients T_r, defined by (13) and (39). In this case the T_r can be expressed exactly in terms of Gegenbauer polynomials (§22.9 of Abramowitz & Stegun 1964):

$$T_r(A)=\frac{\sqrt{2}(r-\frac{1}{2})!}{(1-A)^{2r+\frac{1}{2}}}C_{2r}^{(r+\frac{1}{2})}\left\{\frac{2}{3}\Big/\sqrt{\frac{2}{1-A}}\right\}. \tag{45}$$

It is convenient to divide by the dominant factors in T_r, and display the scaled coefficients defined by

$$T_r^{\mathrm{sc}}(A)\equiv T_r(A)[(f_+-f_0)(f_--f_0)]^{r/2+\frac{1}{4}}/(r-\tfrac{5}{6})!. \tag{46}$$

Figure 2 shows the comparison of several different approximations for $T_r^{\mathrm{sc}}(A)$ with the coefficients calculated exactly from (45). Clearly the uniform approximation (43) works very well even outside the range $-0.33<A<0.09$ where the cluster is well

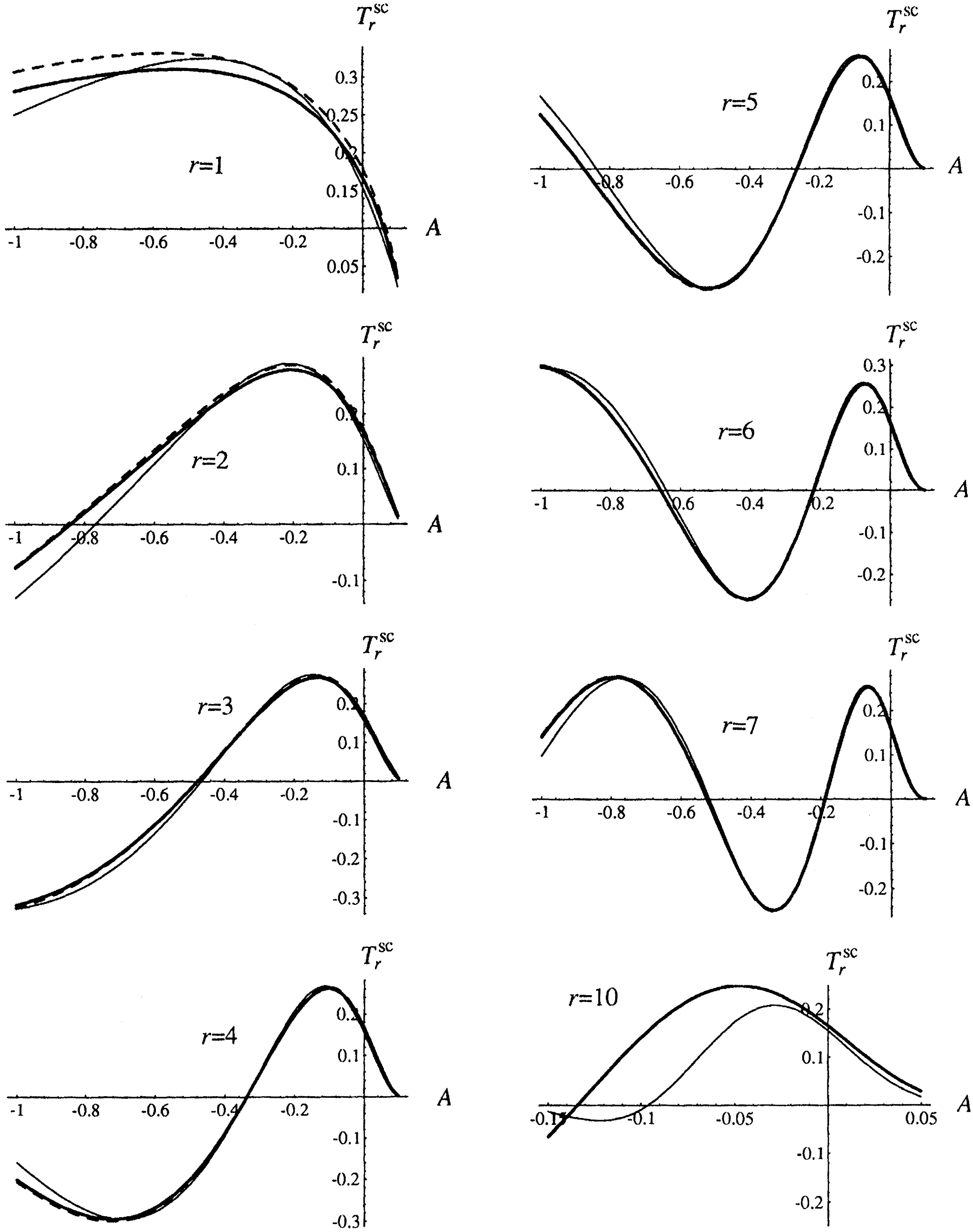

Figure 2. Single-saddle asymptotic coefficients $T_r(A)$, scaled according to (46), for the exponent function (39) where the saddle is at $z = 0$ and the expansion diverges because of the pair of distant saddles (40), for $r = 1$ through 7 and $r = 10$. Thick line: exact coefficients calculated from (45); dashed line: uniform approximation (43); thin line ($r = 1$ through 7): (43) with the term in Ai′ omitted. For $r = 10$, we compare the exact coefficients (45) (thick line) with the transitional approximation (44) (thin line).

defined. As expected, (43) gets more accurate as the asymptotic parameter r increases. The term involving the derivative Ai′ is unimportant and can be dropped with little loss of accuracy. This shows that the function $\ln\{f(z,A)-f_0\}$, with f given by (39), is accurately antisymmetric about the cluster centre $z = 1$ over a wide range. However, the curve for $r = 10$ shows that the transitional approximation (45) is very poor away from the immediate neighbourhood of the coalescence. Thus $\ln\{f(z,A)-f_0\}$ is well approximated by a cubic only over a very small range near $z = 1$.

3. Cluster-to-saddle

(a) *Uniform approximation for cluster: exact formalism*

Now consider the contour C in (1) to pass through a cluster of N close-lying saddles (figure 3) which degenerate when $\boldsymbol{A} = 0$. For simplicity, and without loss of generality, we take C to recede from the cluster along neighbouring valleys; when $\boldsymbol{A}$ is not too large, so that the cluster remains well defined, this topology is determined by the valleys of the coalesced case $\boldsymbol{A} = 0$. In these circumstances, an approximation to (1) that is valid uniformly as the saddles separate can be obtained by the method of Chester *et al.* (1957) as generalized from two to many saddles by Bleistein (1967) and Ursell (1972) (see also Wong 1989). Our main aim here is to determine the high orders of such uniform approximations. This requires a new explicit exact formula for the terms (analogous to (13) for a simple saddle). We now derive this, by extending existing formulations.

The first step is to map the exponent in (1) onto the cuspoid catastrophe (4) near the cluster, changing variables from z to u by

$$f(z,\boldsymbol{A}) = f_0(\boldsymbol{A}) + F_N(u,\boldsymbol{X}) \tag{47}$$

and demanding that the N saddles $z_n(\boldsymbol{A})$ of the cluster correspond with the saddles $u_n(\boldsymbol{X})$ of the catastrophe. This determines the parameters $\boldsymbol{X}(\boldsymbol{A})$ and also the 'average height' f_0 of the cluster, defined as the height at the 'average position' z_0, which is in turn defined as the point corresponding to the centroid $u = 0$ of the mapped cluster. Thus (1) becomes

$$I_C(k,\boldsymbol{A}) = \exp\{-kf_0(\boldsymbol{A})\}\int_{C_N} \mathrm{d}u\, G_0(u)\exp\{-kF_N(u,\boldsymbol{X}(\boldsymbol{A}))\}, \tag{48}$$

where

$$G_0(u) \equiv g(z(u))\,\mathrm{d}z(u)/\mathrm{d}u \tag{49}$$

and C_N is one of the contours in (8). Henceforth we make the standard choice (6); this can be justified (cf. (9) and (51) below) by replacing the phase of k by a multiple of 2π.

Next, we decompose G_0 into a contribution from the saddles and a contribution vanishing at the saddles:

$$G_0(u) = \sum_{m=0}^{N-1} a_{0m}\,u^m + F'_N(u)\,H_0(u). \tag{50}$$

Substitution into (48) generates the canonical integrals W_N and their derivatives, each associated with one of the a_{0m}, and, after an integration by parts, an integral like (48) with G_0 replaced by H'_0/k. Thus the procedure can be iterated, and we obtain, after using (26) and (27), the uniform asymptotic expansion

$$I_C(k) = \exp\{-kf_0\}\sum_{m=0}^{N-1}\frac{1}{k^{(m+1)/(N+1)}}\left(\delta_{m,0} - m\frac{\partial}{\partial\xi_m}\right)W_N(\boldsymbol{\xi})\sum_{r=0}^{\infty}\frac{a_{rm}}{k^r}, \tag{51}$$

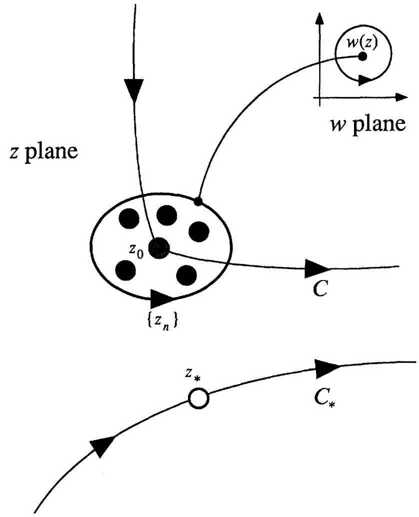

Figure 3. Contours for cluster of saddles near z_0, dominated by a simple distant saddle at z_*. The original contour (equation (1)) is C, and the z and w loops defining the coefficients a_{rm} (equation (58)) encircle, respectively, the saddles z_n in the cluster and the image $w(z)$ of each point on the z loop. The dominant arc C_* of the expanded z loop passes through the distant saddle.

where $\boldsymbol{\xi} = \{k^{(N+1-s)/(N+1)}X_s\}$. The coefficients $a_{rm}(\boldsymbol{X})$ depend on the parameters. Now we derive an explicit formula for them; later we shall use it to determine the high-order behaviour.

The a_{rm} are determined by the recursion scheme

$$G_r(u) = \sum_{m=0}^{N-1} a_{rm}\, u^m + F'_N(u)\, H_r(u) = H'_{r-1}(u) \tag{52}$$

with G_0 given by (49). By evaluating each function $G_r(u)$ at the saddles, the coefficients can be obtained by solving

$$G_r(u_n) = \sum_{m=0}^{N-1} a_{rm}\, u_n^m \quad (1 \leqslant n \leqslant N). \tag{53}$$

To get an explicit solution, we first make use of the observation by Soni & Soni (1990) and Olde Daalhuis & Temme (1992) that (52) has the solution

$$G_r(u) = \frac{1}{2\pi \mathrm{i}} \oint_{\{u_n,\, u\}} \mathrm{d}v\, G_0(v)\, R_r(u,v), \quad \text{where} \quad R_r(u,v) = \left[-\frac{1}{F'_N(v)}\frac{\partial}{\partial v}\right]^r \frac{1}{v-u}. \tag{54}$$

For convenience, we reproduce their proof in Appendix A.

We can write $R_r(u,v)$ as a contour integral by temporarily transforming variables from v to F_N:

$$\begin{aligned} R_r(u,v) &= \left[\left(-\frac{\partial}{\partial F}\right)^r \frac{1}{v(F)-u}\right]_{F=F_N(v)} \\ &= \frac{(-1)^r r!}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}\Phi}{(\Phi - F_N(v))^{r+1}(v(\Phi)-u)} \\ &= \frac{(-1)^r r!}{2\pi \mathrm{i}} \oint_v \frac{\mathrm{d}w\, F'_N(w)}{(F_N(w)-F_N(v))^{r+1}(w-u)}. \end{aligned} \tag{55}$$

Now (54) gives

$$G_r(u) = \frac{(-1)^r r!}{(2\pi i)^2} \oint_{\{u,\,u_n\}} dv\, G_0(v) \oint_v \frac{dw\, F'_N(w)}{[F_N(w)-F_N(v)]^{r+1}(w-u)}. \tag{56}$$

According to (53), the coefficients a_{rm} that we require are determined by this function at the saddles. We can identify the a_{rm} directly by using the following relation, derived in Appendix B:

$$\frac{F'_N(w)}{w-u_n} = \sum_{m=0}^{N-1} u_n^m L_m(w), \quad \text{where} \quad L_m(w) = \frac{1}{w^{m+1}}\left(w^N + \sum_{s=m+1}^{N-2} X_{s+1} w^s\right). \tag{57}$$

Substituting this relation into (56) after setting $u = u_n$, making the identification from (53), and changing the v variable back to the original z, we finally obtain

$$a_{rm} = \frac{(-1)^r r!}{(2\pi i)^2} \oint_{\{z_n\}} dz\, g(z) \oint_{w(z)} \frac{dw\, L_m(w)}{[F_N(w)-(f(z)-f_0)]^{r+1}}. \tag{58}$$

The contours are illustrated in figure 3. Equation (58) is the desired exact general formula for the coefficients in the uniform expansion (51). It is the expression we shall use in §3*b* to calculate the late terms $r \gg 1$. It is, however, possible to express the a_{rm} in terms of the derivatives of f and g at the saddles in the cluster, either by evaluating the integrals in (58) or directly from (49), (52) and (53) (see Clarisse 1992). Very complicated formulae are thus generated, as is illustrated in Appendix C with the first few coefficients for a cluster of two saddles.

The coefficients depend on $\boldsymbol{X}$, and it is interesting to study the coalesced case $\boldsymbol{X} = 0$. In Appendix D we show that

$$a_{rm}(\boldsymbol{X}=0) = \frac{(r-(N-m)/(N+1))!}{2\pi i (N+1)^{(m+1)/(N+1)}(-(N-m)/(N+1))!} \oint_{\{z_0\}} dz \frac{g(z)}{[f(z)-f_0]^{r+(m+1)/(N+1)}}. \tag{59}$$

(We note in passing that in the particular case $N = 1$, corresponding to a 'cluster' consisting of a single simple saddle, this reduces to

$$a_{r0} = \frac{(r-\frac{1}{2})!}{2\pi i \sqrt{(2\pi)}} \oint_{\{z_0\}} dz \frac{g(z)}{[f(z)-f_0]^{r+\frac{1}{2}}}, \tag{60}$$

which since $W_1 = \sqrt{(2\pi)}$ exactly reproduces the single-saddle series (12) and (13) when inserted into the general series (51).) In this coalesced case the structure of the general series (51) simplifies. To see this, note first that the canonical integrals and their derivatives can be evaluated explicitly as

$$\delta_{m,0} W_N(0) - m \frac{\partial}{\partial \xi_m} W_N(0) = \frac{-2i(-(N-m)/(N+1))!}{(N+1)^{(N-m)/(N+1)}} \sin\left\{\pi \frac{m+1}{N+1}\right\} \exp\left\{i\pi \frac{m+1}{N+1}\right\}. \tag{61}$$

On combining this with (59), the summands in (51) depend only on the single index $l = r(N+1)+m$, so that the two sums collapse into one, namely

$$I_C(k, \boldsymbol{X}=0) = \frac{-\exp\{-kf_0\}}{\pi(N+1)\, k^{1/(N+1)}} \sum_{l=0}^{\infty} \frac{((l-N)/(N+1))!}{k^{l/(N+1)}} \sin\left\{\pi \frac{l+1}{N+1}\right\} \exp\left\{i\pi \frac{l+1}{N+1}\right\} \times \oint_{\{z_0\}} dz \frac{g(z)}{[f(z)-f_0]^{(l+1)/(N+1)}}. \tag{62}$$

The same formula can be obtained more directly by applying the usual steepest-descent method to a degenerate saddle (cf. Dingle (1973) for the $N=2$ case).

(b) *High orders in the uniform approximation*

In determining the leading behaviour of the cluster-to-saddle coefficients (58) when r is large, the first step is to imitate the procedure for saddle-to-cluster of §2*a*, and expand the z contour into a series of steepest-descent arcs (figure 3) through the distant saddles adjacent to the cluster. As the z contour is expanded, it drags with it the w loop surrounding the image of each point under the map (47). Let z_* be the nearest distant saddle (i.e. that with the least $|f_* - f_0|$) and assume it is simple. For large r the dominant contribution comes from the neighbourhood of z_*, and the method of steepest descent can be applied to evaluate the z integral in (58). The result is

$$a_{rm} \approx \frac{(-1)^{r+\gamma}(r-\frac{1}{2})!\,g_*}{2\pi \mathrm{i}(2\pi f_*'')^{\frac{1}{2}}} \oint_{w(z_*)} \mathrm{d}w \frac{L_m(w)}{[F_N(w)-(f_*-f_0)]^{r+\frac{1}{2}}}, \tag{63}$$

where, as with saddle-to-cluster, $\gamma(=0$ or $1)$ is a contour orientation anomaly.

In (63), the w contour is a loop surrounding the image of z_*. This point is far from the saddles of $F_N(w)$, which are distributed around their centroid $w=0$. The next step is to expand this w loop so that its dominant arc passes near $w=0$. Now we note that because the saddle at z_* is distant from the cluster, we have, near the saddles of F,

$$|F_N(w)| \ll |f_* - f_0| \tag{64}$$

so that in (63) we can approximate

$$\begin{aligned}\frac{1}{[F_N(w)-(f_*-f_0)]^{r+\frac{1}{2}}} &= \frac{1}{(f_0-f_*)^{r+\frac{1}{2}}}\exp\left\{-(r+\tfrac{1}{2})\ln\left[1+\frac{F_N(w)}{f_0-f_*}\right]\right\} \\ &\approx \frac{1}{(f_0-f_*)^{r+\frac{1}{2}}}\exp\left\{-(r+\tfrac{1}{2})\frac{F_N(w)}{f_0-f_*}\right\}.\end{aligned} \tag{65}$$

We choose the expanded w contour to recede from $w=0$ along that pair of neighbouring valleys of $F_N/(f_0-f_*)$ which encloses $w(z_*)$.

It is possible not to make the approximation (64), and instead map the exponent of the exact second member of (65) onto $F_N(y)$, where $y=y(w)$ is a new variable, but the 'super-uniform' formulae thus generated are complicated and in the general case impenetrable. We have, however, carried out this procedure for the case $N=2$, and will give the details elsewhere in a study which also explores corrections to the leading-order late-term formulae obtained here.

Since L_m is a polynomial (equation (57)), the integral in (63) can now be evaluated (cf. (26) and (27)) in terms of the canonical integrals W_N and their derivatives (this choice of contour can be justified by careful choice of the phase of f_0-f_*). Thus we obtain the main result of this section:

$$a_{rm} \approx \frac{(-1)^{r+\gamma} g_*}{2\pi \mathrm{i}(2\pi f_*'')^{\frac{1}{2}}} \sum_{p=0}^{N-m-1} \frac{X_{m+p+2}(r-\frac{1}{2}-(p+1)/(N+1))!}{(f_0-f_*)^{r+\frac{1}{2}-(p+1)/(N+1)}}\left(\delta_{p,0}-p\frac{\partial}{\partial \eta_p}\right) W_N(\boldsymbol{\eta}), \tag{66}$$

where $\quad \boldsymbol{\eta} \equiv \{\eta_s\} \equiv \left\{\left(\frac{r+\frac{1}{2}}{f_0-f_*}\right)^{(N+1-s)/(N+1)} X_s\right\}$ and $X_N \equiv 0, \quad X_{N+1} \equiv 1.$

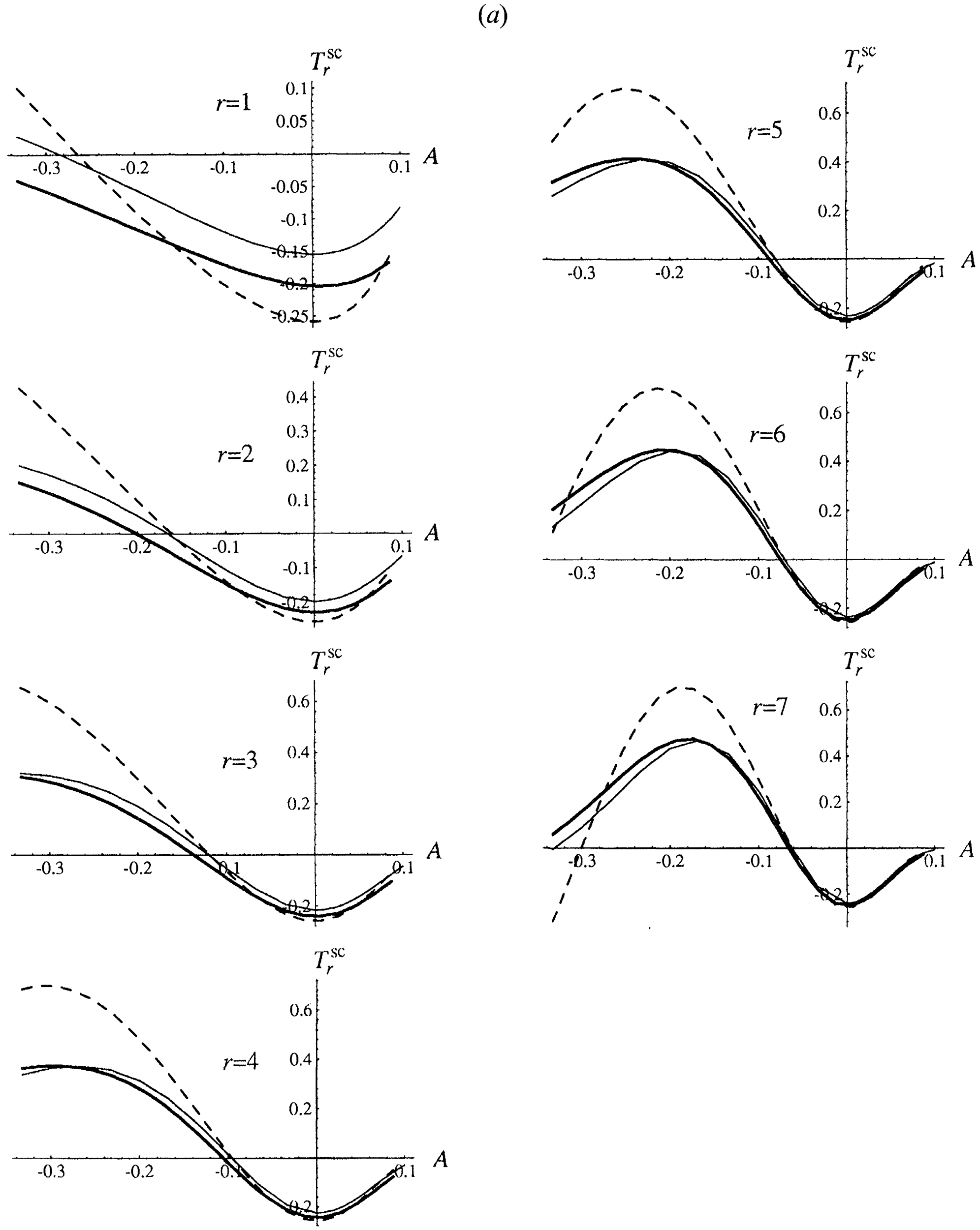

Figure 4. (a) Cluster asymptotic coefficients $a_{r0}(A)$, scaled according to (72), for the exponent function (39) where the cluster is a pair of saddles near $z = 1$ and the expansion diverges because of the distant saddle at $z = 0$, for $1 \leqslant r \leqslant 7$. Thick line: exact coefficients calculated with the formalism of §3a; dashed line: late-term approximation (71); thin line: 'superuniform' approximation described at the end of §3d.

We see a remarkable duality: in the high-order coefficients there appear the same canonical integrals and their derivatives as these coefficients themselves multiply in the uniform expansion (51) for the original integrals. Of course the arguments are different: instead of the original large parameter k whose powers multiply the

(b)

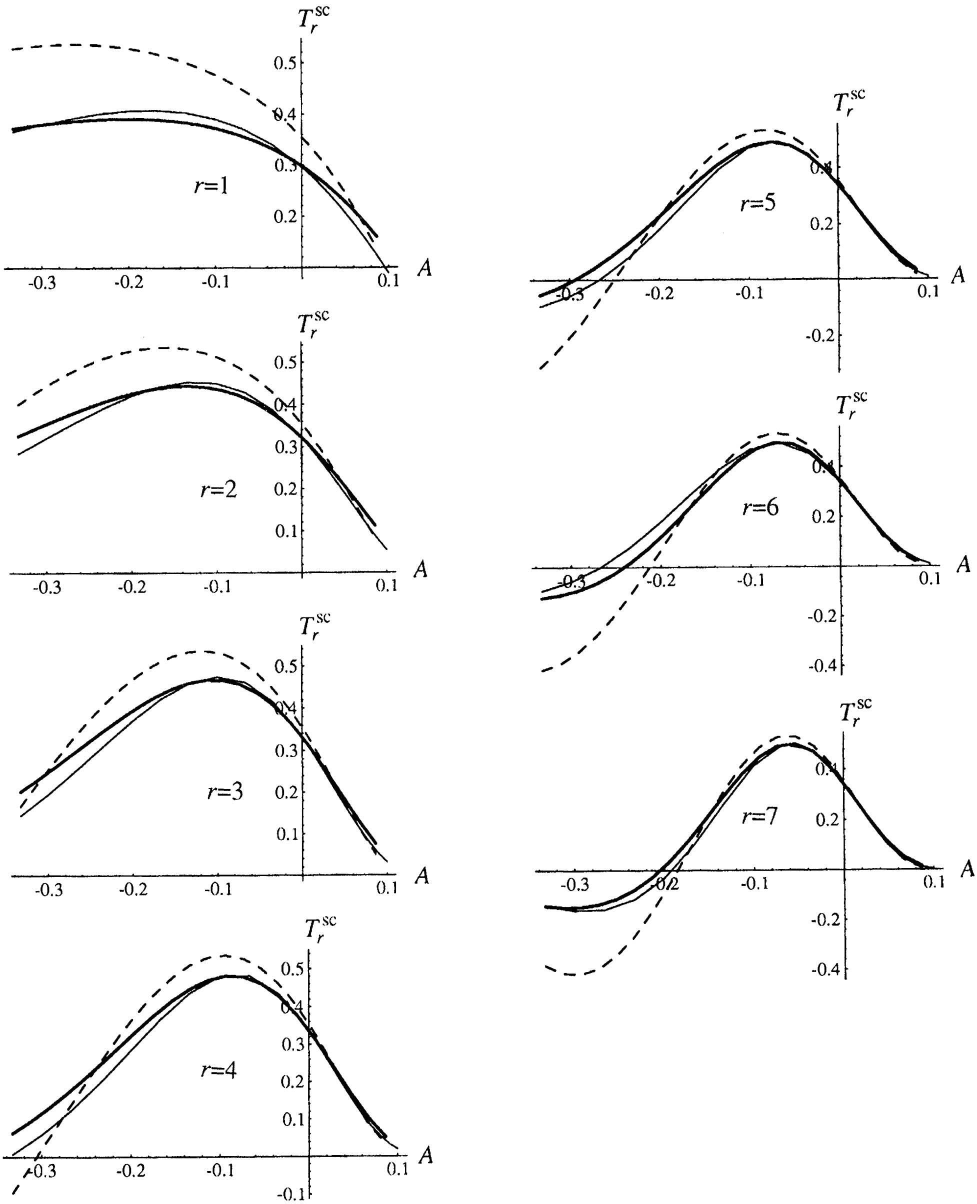

Figure 4. (b) Cluster asymptotic coefficients $a_{r1}(A)$, scaled according to (72), for the exponent function (39) where the cluster is a pair of saddles near $z = 1$ and the expansion diverges because of the distant saddle at $z = 0$, for $1 \leqslant r \leqslant 7$. Thick line: exact coefficients calculated with the formalism of §3a; dashed line: late-term approximation (71); thin line: 'superuniform' approximation described at the end of §3d.

parameters $\boldsymbol{X}$ in the argument ξ of (51), there appears in the new argument η the large order r, divided by the distance $f_0 - f_*$ between the cluster and the distant saddle. The crossover described by (66) takes place over the same parameter range (29) as for the saddle-to-cluster; it gets smaller as r increases.

(c) Simple cases

If $N=1$, so that the cluster consists of a simple saddle, $W_1=\surd(2\pi)$ and (66) gives

$$a_{r0}\approx\frac{(-1)^{\gamma}g_*}{2\pi(-f_*'')^{\frac{1}{2}}}\frac{(r-1)!}{(f_*-f_0)^r}. \tag{67}$$

Apart from a factor $\surd(2\pi)$, arising from the W_1 this coefficient multiplies in (51), this is identical with the $N=1$ saddle-to-cluster result (31), as it must be since these two cases are the same.

If N is arbitrary but the original parameters $\boldsymbol{A}$ are such that the cluster is completely degenerate, then all $\boldsymbol{X}$ are zero except $X_{N+1}\equiv 1$, and (66) gives

$$a_{rm}\approx\frac{(-1)^{r+\gamma}g_*}{2\pi\mathrm{i}(2\pi f_*'')^{\frac{1}{2}}}\frac{(r-\frac{1}{2}-(N-m)/(N+1))!}{(f_0-f_*)^{r+\frac{1}{2}-(N-m)/(N+1)}}(N-m-1)\frac{\partial}{\partial\eta_{N-m-1}}W_N(0). \tag{68}$$

The derivative of the canonical integral is given by (61), and the resulting a_{rm} give exactly the same high orders in the original integral as is obtained by applying the steepest-descent method to the exact coalesced-saddle formula (62).

If $N=2$, there is a single parameter X_1 that henceforth we denote by X, $m=0$ or 1, and the only contributions to (66) are from $p=1$ and $p=0$ respectively. For W_2 we take the formula (7), and thus obtain

$$\left.\begin{aligned}a_{r0}&\approx\frac{(-1)^{r+\gamma}g_*}{(2\pi f_*'')^{\frac{1}{2}}}\frac{(r-\frac{7}{6})!}{(f_0-f_*)^{r-\frac{1}{6}}}\mathrm{Ai}'\left\{-X\left(\frac{r+\frac{1}{2}}{f_0-f_*}\right)^{\frac{2}{3}}\right\},\\ A_{r1}&\approx\frac{(-1)^{r+\gamma}g_*}{(2\pi f_*'')^{\frac{1}{2}}}\frac{(r-\frac{5}{6})!}{(f_0-f_*)^{r+\frac{1}{2}}}\mathrm{Ai}\left\{-X\left(\frac{r+\frac{1}{2}}{f_0-f_*}\right)^{\frac{2}{3}}\right\}.\end{aligned}\right\} \tag{69}$$

(To get these simplest expressions, which will be useful later, we have changed the phase of f_0-f_* by 2π.)

(d) Numerical example

We choose exactly the same exponent function as in (39), but the roles of z_0 and z_* are reversed: now z_0 refers to the cluster of two saddles at $z=1\pm\surd A$ (cf. (40)), and z_* refers to the simple saddle at $z=0$. For the quantities in (69) we have

$$\left.\begin{aligned}g_*=1;\quad f_*''=1-A;\quad X=-(\tfrac{3}{4}(f_+-f_-))^{\frac{2}{3}}=-A;\\ f_0-f_*=\tfrac{1}{2}(f_++f_-)=\tfrac{1}{12}-\tfrac{1}{2}A-\tfrac{1}{4}A^2\equiv\tfrac{1}{12}\alpha(A)\end{aligned}\right\} \tag{70}$$

so that

$$\left.\begin{aligned}a_{r0}&\approx\frac{(-1)^{r+\gamma}}{\surd[2\pi(1-A)]}\frac{12^{r-\frac{1}{6}}(r-\frac{7}{6})!}{\alpha(A)^{r-\frac{1}{6}}}\mathrm{Ai}'\{\eta\},\\ a_{r1}&\approx\frac{(-1)^{r+\gamma}}{\surd[2\pi(1-A)]}\frac{12^{r+\frac{1}{6}}(r-\frac{5}{6})!}{\alpha(A)^{r+\frac{1}{6}}}\mathrm{Ai}\{\eta\},\end{aligned}\right\} \tag{71}$$

where $\eta\equiv A(12(r+\frac{1}{2})/\alpha(A))^{\frac{2}{3}}$.

These expressions are analogous to the saddle-to-cluster result (43). To assess their accuracy, it is necessary to compare them with a_{r0} and a_{r1} calculated exactly using the formalism of §3*a*. This is much more difficult than for saddle-to-cluster, because we have not been able to find counterparts of the explicit formula (45). Applying the general formalism requires very heavy algebra (cf. Appendix C). We used *Mathematica* and stopped at $r = 7$ (where some of the expressions in this special case contained 44 terms, involving 16-digit integers and the calculation of many more intermediate terms). In the comparison, we plot

$$\left.\begin{aligned} a_{r0}^{\mathrm{sc}} &\equiv \frac{(-1)^{r+\gamma}\sqrt{[2\pi(1-A)]}\,\alpha(A)^{r-\frac{1}{6}}}{12^{r-\frac{1}{6}}(r-\frac{7}{6})\,!}\,a_{r0}, \\ a_{r1}^{\mathrm{sc}} &\equiv \frac{(-1)^{r+\gamma}\sqrt{[2\pi(1-A)]}\,\alpha(A)^{r+\frac{1}{6}}}{12^{r+\frac{1}{6}}(r-\frac{5}{6})\,!}\,a_{r1}, \end{aligned}\right\} \tag{72}$$

versus A, in the range for which the cluster is well defined. The theory (71) predicts $\mathrm{Ai}'\{\eta(A)\}$ and $\mathrm{Ai}\{\eta(A)\}$ respectively.

Figure 4 shows the results. They show that the theory captures the crossover from oscillatory to exponential behaviour as the pair of saddles coalesces and separates again, described, according to the duality we have discovered, by the same functions Ai and Ai′ as in the primary expansion (51). It is clear that the theory is better for small $|A|$, as expected because of the replacement of the logarithm in (65). Considerable improvement is evidently achieved with the much more complicated 'super-uniform' approximations obtained if this approximation is avoided, i.e. if (63) is evaluated by uniformly mapping the logarithm in the second member of (65).

4. Concluding remarks

The upshot of this study of the high orders r of asymptotic expansions involving coalescing saddles is that the coefficients are dominated by the 'factorial divided by power' (familiar from simple saddles) but this is multiplied by combinations of canonical integrals whose exponents are catastrophe polynomials describing the geometry of coalescence. These integrals describe the crossover between different forms of 'factorial divided by power' as the cluster coalesces. In the arguments of the canonical integrals, the original large parameter k is replaced by the large order r, together with a scaling depending on the distance between the cluster or saddle and the distant saddle or cluster that causes the divergence of the expansion in powers of $1/k$. This duality between k and r is remarkable.

One application of these ideas, to which a later paper will be devoted, is to the Stokes phenomenon. This describes the birth of a small exponential from the divergent tails of asymptotic series, as a parameter varies. In the method of steepest descent the small exponential typically arises from a distant saddle which makes the expansion diverge. But when the distant saddle is in fact a cluster, which can coalesce and separate as other parameters vary, there are several small exponentials and their births – again governed by the high orders of the expansion – are more complicated.

The methods reported here apply more generally. An example is the uniform expansion for an integral whose contour has an end-point which can be made to coalesce with a saddle (Wong 1989); the series diverges if there are other, distant, saddles. Using arguments analogous to those we used for cluster-to-saddle, we have found the form of the divergence, including crossover and duality which both depend on the canonical integral, in this case an error function. This will be reported elsewhere.

We have restricted ourselves to the leading behaviour of the coefficients for large r. It is, however, possible to imagine penetrating much more deeply into the asymptotics, and obtaining expansions for the high-order coefficients, in descending powers of r, analogous to the resurgence relations found by Dingle (1973) for isolated saddles, and further studied by us in I in connection with hyperasymptotics. Some progress has been made in achieving this for two coalescing saddles (the Airy case), and will be reported later.

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Appendix A. Proof of the relation (54)

By Cauchy's theorem, the relation is obviously true for $r = 0$. For $r > 0$ we have successively, using (52) and integrating by parts,

$$\begin{aligned} G_r(u) &= \frac{1}{2\pi \mathrm{i}} \oint_{\{u_n,\, u\}} \mathrm{d}v\, G_r(v)\, R_0(u,v) = \frac{1}{2\pi \mathrm{i}} \oint_{\{u_n,\, u\}} \mathrm{d}v\, H'_{r-1}(v)\, R_0(u,v) \\ &= -\frac{1}{2\pi \mathrm{i}} \oint_{\{u_n,\, u\}} \mathrm{d}v\, H_{r-1}(v) \frac{\partial}{\partial v} R_0(u,v) \\ &= -\frac{1}{2\pi \mathrm{i}} \oint_{\{u_n,\, u\}} \mathrm{d}v \frac{G_{r-1}(v) - \sum_{m=0}^{N-1} a_{r-1,\,m}\, v^m}{F'_N(v)} \frac{\partial}{\partial v} R_0(u,v) \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{\{u_n,\, u\}} \mathrm{d}v\, G_{r-1}(v) \left[\frac{-1}{F'_N(v)} \frac{\partial}{\partial v}\right] R_0(u,v), \end{aligned} \tag{A 1}$$

where neglect of the integrals involving the a_{rm} in the last equality is justified by expanding the contour to infinity and noting that the integrands are non-singular and decay at least as fast as $1/v^3$, so that these integrals vanish. Iterating this procedure, we obtain (54).

Appendix B. Proof of (57)

First we note that the $L_m(w)$ in (57) are polynomials, because F'_N is a product of factors, one of which cancels the denominator $w - u_n$. Let Q_{mp} denote the coefficient of w^p in $L_m(w)$, so that

$$\sum_{m=0}^{N-1} u_n^m Q_{mp} = \frac{1}{2\pi \mathrm{i}} \oint_0 \frac{\mathrm{d}w}{w^{p+1}} \frac{F'_N(w)}{w - u_n}. \tag{B 1}$$

The integrand has no singularity at $w = u_n$, so we can enlarge the contour and expand the denominator in powers of w/u_n, thereby obtaining

$$\begin{aligned} \sum_{m=0}^{N-1} u_n^m Q_{mp} &= \frac{1}{2\pi \mathrm{i}} \sum_{m=0}^{\infty} u_n^m \oint_0 \frac{\mathrm{d}w}{w^{p+2+m}} F'_N(w) \\ &= \sum_{m=0}^{N-p-3} u_n^m X_{p+2+m} + u_n^{N-p-1}. \end{aligned} \tag{B 2}$$

Thus

$$Q_{mp} = X_{p+2+m} + \delta_{m,\, N-p-1} \tag{B 3}$$

and

$$L_m(w) = \sum_p Q_{mp}\, w^p = \sum_{p=0}^{N-m-3} X_{p+2+m}\, w^p + w^{N-m-1} \tag{B 4}$$

from which (57) follows at once.

Appendix C. Expansion coefficients for a cluster of two saddles

Label the saddles + and −. In the expansion (51) there are two sets of coefficients: a_{r0}, multiplying the canonical integral Ai, and a_{r1} multiplying Ai′. From (53) and (4) we have (cf. (36))

$$a_{r0} = \tfrac{1}{2}[G_{r+}+G_{r-}], \quad a_{r1} = \tfrac{1}{2}(-X)^{-\frac{1}{2}}[G_{r+}-G_{r-}]. \tag{C 1}$$

The first few G_{r+} are

$$G_{0+} = \surd(2/f''_+)\, Y^{\frac{1}{4}} g_+,$$

$$G_{1+} = \frac{1}{\surd(1152)\, Y^{\frac{5}{4}} f''^{\frac{7}{2}}_+}\{g_+(f''^3_+ + 2Y^{\frac{3}{2}}(5f'''^2_+ - 3f''_+ f^{(4)}_+))$$

$$-24g'_+ Y^{\frac{3}{2}} f''_+ f'''_+ + 24 g''_+ Y^{\frac{3}{2}} f''^2_+\} - \frac{\mathrm{i}}{\surd(32 f''_-)\, Y^{\frac{5}{4}}} g_-,$$

$$G_{2+} = \frac{1}{\surd(10616832)\, Y^{\frac{11}{4}} f''^{\frac{13}{2}}_+}$$

$$\times\{g_+(1540 Y^3 f'''^4_+ + f''_+(-35 f''^5_+ + 4Y^{\frac{3}{2}} f''^2_+(5f'''^2_+ - 3f''_+ f^{(4)}_+)$$

$$-12Y^3(f''_+(8f''_+ f^{(6)}_+ - 56 f'''_+ f^{(5)}_+ - 35 f^{(4)^2}_+) + 210 f'''^2_+ f^{(4)}_+)))$$

$$+48 g'_+ f''_+ Y^{\frac{3}{2}}(2Y^{\frac{3}{2}}(35 f'''_+(f''_+ f^{(4)}_+ - f'''^2_+) - 6 f''^2_+ f^{(5)}_+) - f''^3_+ f'''_+)$$

$$+48 g''_+ f''^2_+ Y^{\frac{3}{2}}(f''^3_+ + 10 Y^{\frac{3}{2}}(7 f'''^2_+ - 3 f''_+ f^{(4)}_+)) - 1920 g'''_+ f''^3_+ f'''_+ Y^3$$

$$+576 g^{(4)}_+ f''^4_+ Y^3\} + \frac{\mathrm{i}}{\surd(73728)\, Y^{\frac{11}{4}} f''^{\frac{7}{2}}_-}\{g_-(2Y^{\frac{3}{2}}(3f''_- f^{(4)}_- - 5 f'''^2_-) - 35 f''^3_-)$$

$$+24 g'_- f''_- f'''_- Y^{\frac{3}{2}} - 24 g''_- f''^2_- Y^{\frac{3}{2}}\}, \tag{C 2}$$

where $Y \equiv -X = [\frac{3}{4}(f_+ - f_-)]^{\frac{2}{3}}$ (for G_{r-}, simply interchange subscripts + and −). The number of terms increases rapidly: G_{1+}: 6 terms; G_{2+}: 22; G_{3+}: 64; G_{4+}: 161. The intermediate algebra required to obtain these terms grows rapidly too.

Appendix D. Exact cluster expansion coefficients (58) for a coalesced saddle

In this special case, $X = 0$ and (4) and (57) give

$$F_N(w) = (N+1)^{-1} w^{N+1}, \quad L_m(w) = w^{(N-m-1)}. \tag{D 1}$$

Thus the coefficients are, from (58),

$$a_{rm} = \frac{(-1)^r r!}{(2\pi\mathrm{i})^2} \oint_{\{z_0\}} \mathrm{d}z\, g(z) \oint_{+[(N+1)(f(z)-f_0)]^{1/(N+1)}} \frac{\mathrm{d}w\, w^{N-w-1}}{[(N+1)^{-1} w^{N+1} - (f(z)-f_0)]^{r+1}}$$

$$= \frac{(-1)^r r!\,(N+1)^{(N-m)/(N+1)}}{2\pi\mathrm{i}} \oint_{\{z_0\}} \mathrm{d}z \frac{g(z)}{[f(z)-f_0]^{r+(m+1)/(N+1)}} \frac{1}{2\pi\mathrm{i}} \oint_1 \frac{\mathrm{d}t\, t^{N-m-1}}{(t^{N+1}-1)^{r+1}}, \tag{D 2}$$

where the w integration contour encircles the single point $w(z)$ determined by the one-to-one mapping. Thus we require

$$\frac{1}{2\pi\mathrm{i}}\oint_1 \frac{\mathrm{d}t\, t^p}{(t^{N+1}-1)^{r+1}} = \frac{1}{2\pi\mathrm{i}(N+1)}\oint_1 \frac{\mathrm{d}u\, u^{(p-N)/(N+1)}}{(u-1)^{r+1}}$$

$$= \frac{1}{(N+1)\,r\,!}\left[\frac{\mathrm{d}^r}{\mathrm{d}u^r}u^{(p-N)/(N+1)}\right]_{u=1} = \frac{(-1)^r(r-(p+1)/(N+1))\,!}{(N+1)\,r\,!\,(-(p+1)/(N+1))\,!}. \quad \text{(D 3)}$$

Combining (D 2) and (D 3), we obtain the formula (59).

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