Overlapping Stokes smoothings: survival of the error function and canonical catastrophe integrals

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We derive doubly uniform approximations for the remainder in the optimally truncated saddle-point expansion for an integral containing a large parameter. Double uniformity means that the formulae remain valid while distant saddles responsible for the divergence of the expansion coalesce and separate (as described by catastrophe theory) and while the subdominant exponentials they contribute switch on and off (as described by the error-function smoothing of the Stokes phenomenon). Two sorts of asymptotic singularity are thereby united in a common framework. The formula for the remainder incorporates both the Stokes error function and the canonical catastrophe integrals. A numerical illustration is given, in which the distant cluster contains two saddles; the asymptotic theory gives an accurate description of the details of the fractional remainder, even when this is of order $\exp(-36)$.

1. Introduction

An important general feature of the asymptotics of functions with a large parameter $|k|$ is the birth and disappearance of subdominant exponentials ('beyond all orders' in $|k|$) as parameters vary. This is the Stokes phenomenon (Stokes 1864). It is intimately related to the divergence of the asymptotic series that represents the function in descending powers of $|k|$ (Dingle 1973). Recently it was found (Berry 1989*a*) that the switching-on of the subdominant exponentials occurs smoothly, rather than discontinuously, and in a universal manner described by an error function. The universality class is very wide, and includes functions defined by integrals (Berry 1989*b*; Jones 1990; Olver 1990, 1991*a*, *b*; Boyd 1990; Paris 1992*a*), second-order differential equations (Berry 1990*a*, *b*; McLeod 1992), higher-order differential equations (Paris 1992*b*), difference equations (Berry 1991*a*) and series diverging faster than factorially (Berry 1991*b*).

Another important asymptotic phenomenon is the coalescence and separation of exponential contributions as parameters vary. The geometry of coalescence is described by catastrophe theory (Poston & Stewart 1978; Arnold 1975, 1986; Arnold *et al.* 1984). Each elementary singularity contributes a canonical integral approximating the function during the coalescence (in optics these are the 'diffraction catastrophes' (see Berry & Upstill 1980)).

Our aim here is to explore a situation where both phenomena occur simultaneously: there are several small exponentials, which can be made to switch on and off and coalesce as parameters vary. This occurs in the large-$|k|$ asymptotics of integrals of the form

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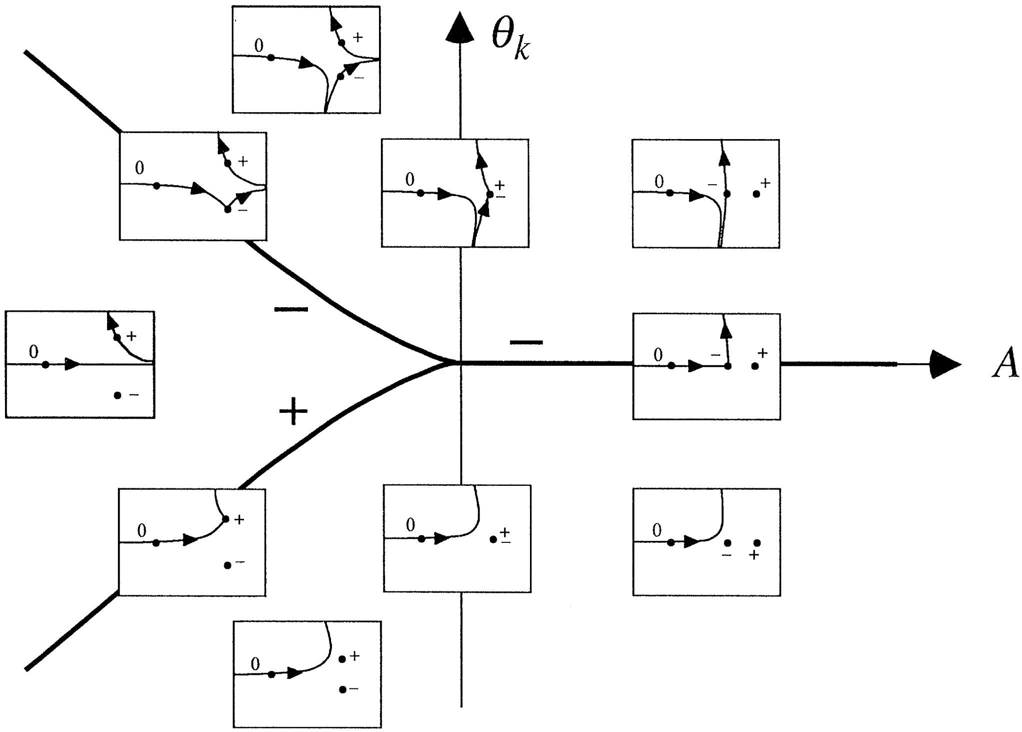

Figure 1. Stokes line (bold) for a distant cluster consisting of two saddles z_+ and z_-, in the plane A, θ_k, labelled by the saddles (+ or −) to which they refer. The insets show how $z_\pm$ coalesce and separate as the single catastrophe parameter A varies, and how the topology of the steepest-descent contour (linking z_0 with valleys at $-\infty$ to $+\mathrm{i}\infty$) changes as θ_k crosses the Stokes line.

$$I_0(k, \boldsymbol{A}) = \int_{C_0} dz\, g(z) \exp\{-kf(z, \boldsymbol{A})\} \tag{1}$$

involving parameters

$$\theta_k \equiv \arg k \tag{2}$$

and

$$\boldsymbol{A} = \{A_1, A_2, \ldots\}, \tag{3}$$

which are varied. The function f has several saddles (defined as zeros of $f' \equiv \mathrm{d}f(z, \boldsymbol{A})/\mathrm{d}z$). In particular, there is an isolated simple saddle at z_0, and a distant cluster of close-lying saddles $z_n(\boldsymbol{A})$. The infinite contour C_0 passes through z_0 and links specified distant valleys of the exponential; a valley is defined as an asymptotic direction where $\mathrm{Re}\, kf(z) \to +\infty$. In a region including C_0 and the cluster, f and g are analytic functions of z. I_0 is a smooth function of k and $\boldsymbol{A}$.

The parameters play different roles, illustrated in figure 1. θ_k is the Stokes parameter; as it varies, the steepest-descent contour through z_0, into which C_0 can be deformed, can sweep through saddles z_n in the distant cluster, and so change topology. Each such change is a Stokes phenomenon, in which I_0 gains a contribution from the one of the z_n. $\boldsymbol{A}$ are the catastrophe parameters; as they vary, the saddles $z_n(\boldsymbol{A})$ coalesce (that is, become degenerate), and unfold into a cluster of simple saddles.

The influence of these contour jumps and coalescences on the integral (1) is exponentially small, and therefore beyond all orders in the divergent asymptotic series generated formally by expanding the exponent about the dominant saddle z_0. This is

$$I_0(k, \boldsymbol{A}) = \frac{\exp\{-kf_0(\boldsymbol{A})\}}{k^{\frac{1}{2}}} \sum_{r=0}^{\infty} \frac{T_r(\boldsymbol{A})}{k^r}, \tag{4}$$

where the subscript 0 denotes evaluation at z_0 and the coefficients are

$$T_r(\boldsymbol{A}) = \frac{(r-\frac{1}{2})!}{2\pi \mathrm{i}} \oint_{z_0} \mathrm{d}z \frac{g(z)}{[f(z,\boldsymbol{A})-f_0(\boldsymbol{A})]^{r+\frac{1}{2}}} \tag{5}$$

(see Dingle 1973). Subdominant contributions arise from the distant saddles $z_n(\boldsymbol{A})$, which are responsible for the divergence of the series (4). Recently, we showed (Berry & Howls 1993) that the divergences are always predominantly of the 'factorial/power' form, with the details of the late terms $T_r(\boldsymbol{A})$ (r large) depending delicately on $\boldsymbol{A}$, that is on the deportment of saddles in the cluster.

The subdominant contribution from z_n is of order $\exp\{-F_{0n}\}$, where

$$F_{0n} \equiv k(f_n - f_0) \tag{6}$$

is the 'singulant' associated with z_n. It switches on when θ_k and $\boldsymbol{A}$ have values such that F_{0n} is real and positive. Loci of such points are the 'Stokes lines' (figure 1), or, more generally, 'Stokes surfaces' (Wright 1980; Berry & Howls 1990). As $\boldsymbol{A}$ varies and the saddles in the cluster coalesce in the z plane, the Stokes lines associated with them also approach and coalesce in the space $\{\theta_k, \boldsymbol{A}\}$. Then the question arises: what is the proper description of the Stokes phenomenon, that is, of the appearance of the various small exponentials? The somewhat surprising answer will be that the confluence of the two very different asymptotic phenomena adds little complication: both the error function, describing the Stokes smoothing, and the canonical integrals, describing the catastrophes, survive in the final approximate formula for I_0.

We will reach this conclusion by estimating the remainder $R_N(k, \boldsymbol{A})$ after truncating the series (4) optimally, that is at $r = N$ where N is near the least term:

$$I_0(k, \boldsymbol{A}) = \frac{\exp\{-kf_0(\boldsymbol{A})\}}{k^{\frac{1}{2}}} \sum_{r=0}^{N-1} \frac{T_r(\boldsymbol{A})}{k^r} + R_N(k, \boldsymbol{A}). \tag{7}$$

Two steps are involved. In the first (§2), R_N is expressed in terms of an integral of the form (1), but with the contour passing through the distant cluster rather than z_0, and this is manipulated so as to reveal the ubiquitous error function. In the second (§3), singularity theory and the technique of uniform approximation (Chester *et al.* 1957; Bleistein 1967; Wong 1989) are applied, to obtain an explicit expression for R_N in terms of error functions and the canonical catastrophe integral that characterizes the topology of the cluster. In §4 we illustrate the general formula with numerical calculations.

2. Appearance of the error function

Consider first the case where none of the subdominant integrals has switched on, that is where θ_k is such that the steepest-descent contour into which C_0 can be deformed is a single doubly infinite curve passing through z_0 and not including any

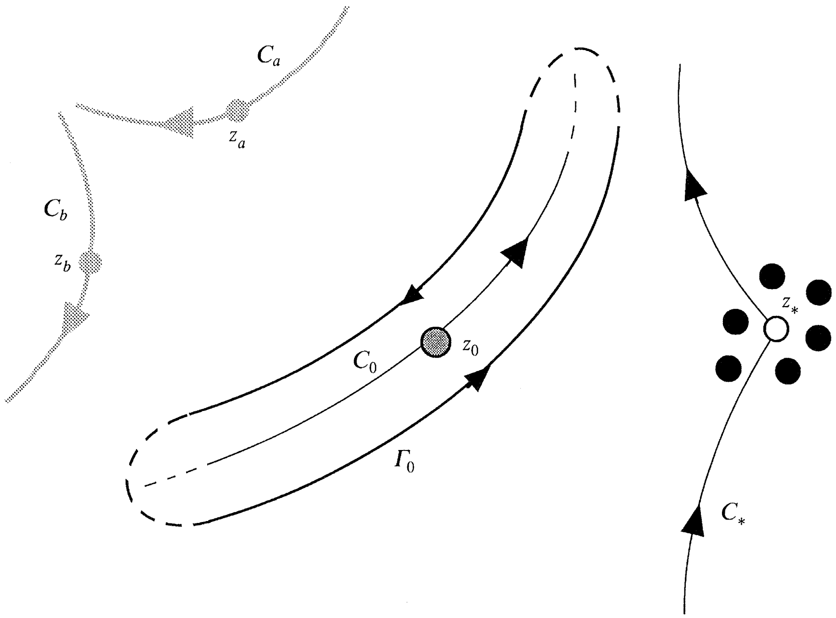

Figure 2. 'Sausage contour' Γ_0 (bold) surrounding integration contour C_0 (thin) through dominant saddle z_0, dilated to include contour C_* (thin) through distant cluster of subdominant saddles, and other arcs (grey) through sub-subdominant saddles z_a and z_b.

of the saddles z_n belonging to the distant cluster. In this case the remainder in (7) is given by the following exact formula (eqn (12) of Berry & Howls 1991):

$$R_N(k,\boldsymbol{A}) = \frac{\exp\{-kf_0(\boldsymbol{A})\}}{2\pi i k^{N+\frac{1}{2}}} \int_0^\infty du \exp\{-u\} u^{N-\frac{1}{2}} \times \oint_{\Gamma_0} dz \frac{g(z)}{[f(z,\boldsymbol{A})-f_0(\boldsymbol{A})]^{N+\frac{1}{2}}[1-u/k[f(z,\boldsymbol{A})-f_0(\boldsymbol{A})]]}. \tag{8}$$

Here Γ_0 is a 'sausage contour' surrounding C_0 (figure 2). Now we dilate Γ_0 into several arcs. The first, C_*, passes through the distant cluster and into two of its neighbouring asymptotic valleys (asymptotic valleys are the same as the valleys issuing from the cluster at parameters $\boldsymbol{A}$ for which it consists of a single degenerate saddle). The other arcs pass through any other distant saddles not belonging to the cluster. We assume that the contributions from these other saddles are exponentially small in comparison with those in the cluster (i.e. sub-subdominant), and henceforth retain only the contribution from C_*.

Next, we reverse the order of integrations in (8), and transform the integration variable u to v, defined by

$$v \equiv -1 + u/k[f(z,\boldsymbol{A})-f_0(\boldsymbol{A})]. \tag{9}$$

Thus

$$R_N \approx \frac{-1}{2\pi i}\int_{C_*} dz\, g(z)\exp\{-kf(z)\}\int_{-1,+i\epsilon}^{\infty}\frac{dv}{v}(1+v)^{N-\frac{1}{2}}\exp\{-vk[f(z)-f_0]\}, \tag{10}$$

where the only approximation is the neglect of sub-subdominant contributions (i.e. those not associated with C_*). Here the v contour passes above the pole at $v=0$. This follows from the requirement that none of the subdominant exponentials has switched on, which implies that $\mathrm{Im}\,[k(f(z)-f_0)] < 0$ for z on C_*, thereby determining that the u contour in (8) passes above the pole after dilation of Γ_0. Now we stipulate that the v contour is dragged by the pole, that is it continues to lie above the pole even when $\mathrm{Im}\,[k(f(z)-f_0)] > 0$, when one or more of the small exponentials have switched on. With this stipulation, (10) gives the remainder as a smooth function of k and $\boldsymbol{A}$, valid throughout the parameter range we are interested in. In (10) the z integral converges because we have chosen C_* to descend into the valleys of the exponential, and the v integral converges because $\mathrm{Re}\,[k(f(z)-f_0)] > 0$ for z on C_*.

The v integral is dominated by a pole at $v=0$ and a saddle at

$$v_{\mathrm{c}} = N/k[f(z)-f_0]-1. \tag{11}$$

As we shall see in §3, the z integral is dominated by the saddles in the cluster. For these, the singulants F_{0n} are close to the positive real axis (because we are considering Stokes jumps) and to each other (because we are considering saddle coalescence). Moreover, as will be explained in §3 we shall choose the truncation N near the least term, that is near $\mathrm{Int}\,[k(f_*-f_0)]$, where f_* is a typical value of $f(z)$ in the cluster. It follows that $v_{\mathrm{c}} \approx 0$, so that the saddle is close to the pole. In Appendix A we show that in this situation the v integral can be approximated for large k and N by

$$\int_{-1,+\mathrm{i}\epsilon}^{\infty} \frac{\mathrm{d}v}{v}(1+v)^{N-\frac{1}{2}}\exp\{-vk[f(z)-f_0]\} \approx -2\pi\mathrm{i}S_N\{k(f(z)-f_0)\}, \tag{12}$$

where

$$\left.\begin{aligned} S_N(F) &= \tfrac{1}{2}[1+\mathrm{erf}\,(\sigma)] + \frac{\mathrm{i}}{2\sqrt{\pi}}\left(\frac{\sqrt{(2F)}}{N-F}-\frac{\mathrm{i}}{\sigma}\right)\exp\{-\sigma^2\}, \\ \sigma &= \sigma_N(F) \equiv [N\ln F/N+N-F]^{\frac{1}{2}}. \end{aligned}\right\} \tag{13}$$

Here erf denotes the error function (Abramowitz & Stegun 1964, §7.1).

The function $S_N(F)$ is a generalization (see also Olver 1991*a*) of the Stokes multiplier familiar in the smoothing (Berry 1989*a*) of the ordinary Stokes phenomenon (e.g. for an integral with an isolated distant saddle). The generalization allows (12) to remain valid even when $|N-F|$ is large, that is far from the Stokes line. In many applications this refinement is unnecessary, and it suffices to use the close approximations

$$\left.\begin{aligned} S_N(F) &\approx S(\sigma) \equiv \tfrac{1}{2}[1+\mathrm{erf}\,(\sigma)], \\ \sigma_N(F) &\approx \sigma(F) \equiv \mathrm{i}(N-F)/\sqrt{(2N)} \approx \mathrm{Im}\,F/\sqrt{(2\,\mathrm{Re}\,F)} \quad (|N-F| \ll N). \end{aligned}\right\} \tag{14}$$

This Stokes multiplier is the same as that in the original smoothing theory.

Substituting into (10) we obtain the remainder in (7) as

$$R_N(k,\boldsymbol{A}) \approx \int_{C_*} \mathrm{d}z\, g(z)\, S_N\{k[f(z,\boldsymbol{A})-f_0(\boldsymbol{A})]\}\exp\{-kf(z,\boldsymbol{A})\}. \tag{15}$$

In this remarkably simple formula, all the complications associated with the changes of steepest contour as θ_k varies are incorporated in the Stokes multiplier function S.

3. Uniform approximation for coalescence

Let the distant cluster consist of M saddles $z_1, \ldots, z_M$. Then we can invoke singularity theory (Arnold 1986; Poston & Stewart 1978) to map the integration variable in (15) from z to a new variable ζ by introducing the polynomial normal form of the cuspoid catastrophe with co-dimension $M-1$, namely

$$f(z,\boldsymbol{A}) = f_* + \Phi_M(\zeta,\boldsymbol{X}), \tag{16}$$

where

$$\boldsymbol{X} \equiv \{X_m\} \equiv \{X_1,\dots,X_{M-1}\} \quad \text{and} \quad \Phi_M(\zeta,\boldsymbol{X}) = \frac{\zeta^{M+1}}{M+1} + \sum_{m=1}^{M-1} X_m \frac{\zeta^m}{m}. \tag{17}$$

At the origin of the canonical parameters $\boldsymbol{X}$,M saddles coalesce; neighbourhoods of the origin contain all possible stable unfoldings into combinations of simple saddles. The mapping

$$z \to \zeta(z,\boldsymbol{A}) \tag{18}$$

is guaranteed locally one-to-one by identifying the saddles z_n of f and ζ_n of Φ. This gives M equations determining the M unknowns $f_*(\boldsymbol{A})$, $\boldsymbol{X}(\boldsymbol{A})$. f_* can be regarded as the average height of the cluster, defined as the value of $f(z)$ at the point z_* corresponding to the centroid $\zeta = 0$ of the mapped saddles.

Thus (15) becomes

$$R_N(k,\boldsymbol{A}) \approx \exp\{-kf_*(\boldsymbol{A})\} \int_{\infty\exp\{2\mathrm{i}\pi/(M+1)\}}^{\infty} \mathrm{d}\zeta\, G(\zeta,k,\boldsymbol{X}) \exp\{-k\Phi(\zeta,\boldsymbol{X})\}, \tag{19}$$

where

$$G(\zeta,k,\boldsymbol{X}) \equiv \frac{\mathrm{d}z(\zeta,\boldsymbol{X})}{\mathrm{d}\zeta} g(z(\zeta,\boldsymbol{X}))\, S_N\{k[f(z(\zeta,\boldsymbol{X}),\boldsymbol{A}(\boldsymbol{X})) - f_0(\boldsymbol{A}(\boldsymbol{X}))]\}. \tag{20}$$

The contour in (19), like that in (15), links a particular pair of adjacent asymptotic valleys of the exponent. If we choose any other pair, the result differs by a phase factor, but this can be eliminated by careful choice of phases in subsequent expressions (see Berry & Howls 1993).

To evaluate the integral, we use the technique of uniform approximation (Chester *et al.* 1957; Bleistein 1967). This exploits the rapid variation of the exponential (because of the factor k in the exponent) and regards the multiplier G (equation (20)) as a slowly varying function of ζ by comparison. Certainly the factors $\mathrm{d}z/\mathrm{d}\zeta$ and g, which do not involve k, vary slowly, but the Stokes multiplier does involve k through its variable σ (equation (14)). However, this dependence is on $\surd k$ rather than k (since N is of order k for optimal truncation), so that S does vary slower than the exponential; this corresponds to the fact that the 'width' of the Stokes line is $O(1/\surd k)$ rather than $O(1/k)$ (Berry 1989*a*).

To obtain the leading-order large-k approximation to (19), G is first expanded as follows:

$$G(\zeta) = \sum_{s=0}^{M-1} a_s \zeta^s + \Phi_M'(\zeta) H(\zeta). \tag{21}$$

Since Φ' vanishes at the saddles, the coefficients a_s can be determined by solving

$$G(\zeta_n) = g(\zeta_n) \frac{\mathrm{d}z_n}{\mathrm{d}\zeta} S_N\{F_{0n}\} = \sum_{s=0}^{M-1} a_s \zeta_n^s, \tag{22}$$

where F_{0n} are the values of the singulants (6) at the saddles in the cluster. The required derivatives of the mapping at the saddles can be found by differentiating (16) twice:

$$f_n'' \left(\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right)_n^2 = M\zeta_n^{M-1} + \sum_{m=2}^{M-1} (m-1) X_m \zeta_n^{m-2}. \tag{23}$$

Since the mapping is one-to-one, $(\mathrm{d}z/\mathrm{d}\zeta)_n$ has a finite limit even when $f''_n = 0$, that is when some or all of the M saddles are degenerate.

The solution of (22) is

$$a_s = \sum_{n=1}^{M} (V^{-1})_{sn}\, G(\zeta_n), \tag{24}$$

where V is the Vandermonde matrix of powers ζ_n^s of the mapped saddles. Its explicit inverse is shown in Appendix B to be

$$(V^{-1})_{sn} = P_s(\zeta_n)/\zeta_n^{s+1} \prod_{k=1,\, k\neq n}^{M} (\zeta_n - \zeta_k) \quad (0 \leqslant s \leqslant M-1, 1 \leqslant n \leqslant M), \tag{25}$$

where P_s are the polynomials

$$P_s(\zeta) = \zeta^M + \sum_{k=s+1}^{M-2} X_{k+1}\, \zeta^k. \tag{26}$$

Next, the term involving H in (2) is neglected because it is of higher order in $1/k$ than the terms involving the a_s. When substituted into (19), this approximation for G is seen to depend on the canonical catastrophe integrals, defined by

$$W_M(\boldsymbol{X}) \equiv \int_{\infty \exp\{2\mathrm{i}\pi/(M+1)\}}^{\infty} \mathrm{d}\zeta \exp\{-\varPhi_M(\zeta, \boldsymbol{X})\}. \tag{27}$$

We require

$$\int_{\infty \exp\{2\mathrm{i}\pi/(M+1)\}}^{\infty} \mathrm{d}\zeta\, \zeta^s \exp\{-k\varPhi_M(\zeta, \boldsymbol{X})\} = \frac{1}{k^{(s+1)/(M+1)}} \left(\delta_{s,0} - s\frac{\partial}{\partial \xi_s}\right) W_M(\boldsymbol{\xi}), \tag{28}$$

where

$$\boldsymbol{\xi} \equiv \{\xi_s\} \equiv \{k^{(M+1-s)/(M+1)} X_s\}. \tag{29}$$

Thus the remainder (19) becomes

$$R_N(k, \boldsymbol{A}) \approx \frac{\exp\{-kf_*\}}{k^{1/(M+1)}} \left\{ a_0\, W_M(\boldsymbol{\xi}) - \sum_{s=1}^{M-1} \frac{a_s\, s}{k^{s/(M+1)}} \frac{\partial}{\partial \xi_s} W_M(\boldsymbol{\xi}) \right\}. \tag{30}$$

This is our main result. As an approximation for the remainder of the truncated asymptotic series (4) representing the integral (1), it is uniform in a double sense. First, by remaining valid as the catastrophe parameters $\boldsymbol{A}$ vary and the saddles coalesce; this behaviour is captured by the canonical catastrophe integrals $W_M(\boldsymbol{\xi})$ and their derivatives. Secondly, by remaining valid as the Stokes parameter θ_k varies and the steepest-descent contour changes topology; this behaviour is captured by the Stokes multiplier function $S_N(F)$, appearing in the coefficients a_s via (22) and (24).

It might seem that the terms involving the derivatives (i.e. $s > 0$) can be neglected in (30) as being of lower order in k, but in fact these are necessary to match smoothly onto the case where the saddles are well separated, when the W_M themselves can be represented by their lowest-order saddle-point approximations. (As we will discuss in §4, there are circumstances in which including these derivatives is still not sufficient to accomplish this smooth matching.)

An interesting special case is where all M saddles coalesce into a single degenerate saddle, at $z = z_*$, say. By (17), this corresponds to $\boldsymbol{X} = 0$, for which (27) gives

$$W_M(0) = -2\mathrm{i}\left(-\frac{M}{M+1}\right)! \exp\left\{\mathrm{i}\frac{\pi}{M+1}\right\} \sin\left\{\frac{\pi}{M+1}\right\} \bigg/ (M+1)^{M/(M+1)}. \tag{31}$$

To leading order in k, only the coefficient a_0 is required; from (16)–(21) this is

$$a_0 = (M!/f_*^{(M+1)})^{1/(M+1)} g_* S_N(F_{0*}), \tag{32}$$

where the singulant F_{0*} corresponds to the degenerate saddle. By choosing N as the least term of the series (4) for this case, that is

$$N = N_* \equiv \operatorname{Int}\{|F_{0*}|\} \tag{33}$$

we obtain, on substituting into (30) using (14)

$$R_{N_*} \approx g_* \exp\{-kf_*\}\left(\frac{M!}{kf_*^{(M+1)}}\right)^{1/(M+1)} W_M(0)\frac{1}{2}\left[1+\operatorname{erf}\left\{\frac{\operatorname{Im} F_{0*}}{\sqrt{(2\operatorname{Re} F_{0*})}}\right\}\right]. \tag{34}$$

This describes the Stokes smoothing as the contour sweeps through an isolated degenerate distant saddle. When $M = 1$ we recapture the known special case of an isolated non-degenerate distant saddle (Berry 1989*a*):

$$R_{N_*} \approx \mathrm{i} g_* \exp\{-kf_*\}\left(\frac{2\pi}{-kf_*''}\right)^{\frac{1}{2}}\frac{1}{2}\left[1+\operatorname{erf}\left\{\frac{\operatorname{Im} F_{0*}}{\sqrt{(2\operatorname{Re} F_{0*})}}\right\}\right]. \tag{35}$$

Now we return to the general formula (30), and discuss the choice of truncation N, which occurs in the coefficients a_s via the Stokes multiplier (13). Optimal truncation would give purely real Stokes variables σ and hence a maximally compact error-function smoothing. But this cannot be achieved, for two reasons. First, the more sophisticated σ of equation (13) (rather than the approximation (14)), has a small imaginary part on the Stokes line (F positive real). Second, we only have a single N at our disposal, but different optimal truncations $N = \operatorname{Int}|kF_{0n}|$ would be appropriate for the different saddles z_n in the cluster. A crude expedient would be to choose N as the singulant corresponding to the centroid of the mapped cluster, i.e.

$$N = N_{\text{centroid}} \equiv \operatorname{Int}|k(f_* - f_0)|. \tag{36}$$

Other, more sophisticated, choices are possible. In the numerical computations of §4, we used the following. For any A, a certain subset of the saddles belonging to the cluster is relevant, in the sense that it contributes to the saddle-point approximation to (30). In the terminology of Berry & Howls (1991), these are the saddles 'adjacent' to z_0 (in figure 1, z_- is adjacent when $A > 0$, and both z_+ and z_- are adjacent when $A < 0$). Each relevant saddle z_n contributes a 'factorial/power' to the late terms T_r/k^r in (4), and this contribution is smallest (optimal truncation) when $r = \operatorname{Int}|F_{0n}|$. The dominant contribution is the one whose optimal truncation occurs soonest. Thus we choose

$$N = \min(\operatorname{Int}|F_{0n}|), \tag{37}$$

where the minimization is over the relevant saddles.

4. Numerical illustration

We study the same example as in Berry & Howls (1993), where the distant cluster has two saddles (i.e. $M = 2$):

$$g(z) = 1, \quad f(z, A) = \tfrac{1}{4}z^4 - \tfrac{2}{3}z^3 + \tfrac{1}{2}(1-A)z^2. \tag{38}$$

The isolated saddle z_0, the cluster saddles z_+, z_- and their heights are

$$z_0 = 0, \quad z_\pm = 1 \pm A^{\frac{1}{2}}; \quad f_0 = 0, \quad f_\pm = \tfrac{1}{12} - \tfrac{1}{2}A - \tfrac{1}{4}A^2 \mp \tfrac{2}{3}A^{\frac{3}{2}} \tag{39}$$

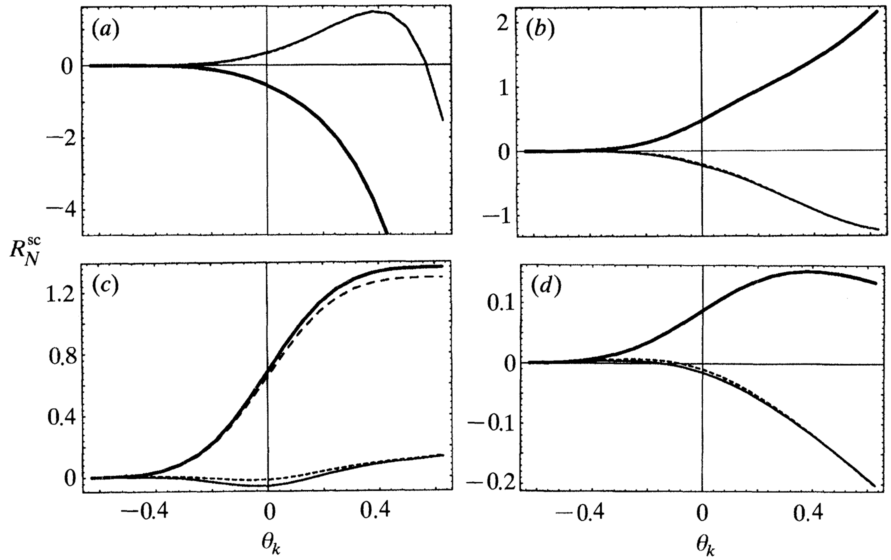

Figure 3. Scaled remainder R_N^{sc} defined by (43), as a function of the Stokes variable θ_k, for (*a*) $A = -2/25$; (*b*) $A = -1/24$; (*c*) $A = 0$; (*d*) $A = +1/24$. The graphs show $\mathrm{Re}\,[R_N^{sc}]$ (thick lines, exact; dashed lines, asymptotic theory) and $\mathrm{Im}\,[R_N^{sc}]$ (thin lines, exact; dotted lines, asymptotic theory). The truncations are (*a*) $N = 36$; (*b*) $N = 31$; (*c*) $N = 25$; (*d*) $N = 20$.

so that the cluster coalesces when the single catastrophe parameter $A = 0$. In the integral (1), the contour C_0 runs from $-\infty$ to $+\mathrm{i}\infty$. This case corresponds to figure 1: for $A > 0$, the saddles in the cluster are on the real axis, and for $A < 0$ they are complex conjugates of each other.

From (30) we can calculate the uniform approximation to the remainder, which in this case consists of two terms:

$$R_N(k,A) \approx \exp\{-kf_*\}\,\pi\mathrm{i}/k^{\frac{1}{2}}A^{\frac{1}{4}} \times\left\{\left(\frac{S_N(F_+)}{(1+A^{\frac{1}{2}})^{\frac{1}{2}}}+\frac{S_N(F_-)}{(1-A^{\frac{1}{2}})^{\frac{1}{2}}}\right)\xi^{\frac{1}{4}}\,\mathrm{Ai}\{\xi\}+\left(\frac{S_N(F_+)}{(1+A^{\frac{1}{2}})^{\frac{1}{2}}}-\frac{S_N(F_-)}{(1-A^{\frac{1}{2}})^{\frac{1}{2}}}\right)\xi^{-\frac{1}{4}}\,\mathrm{Ai}'\{\xi\}\right\}, \tag{40}$$

where Ai denotes the Airy function (Abramowitz & Stegun 1964, §10.4) and

$$f_* = \tfrac{1}{2}(f_+ + f_-) = \tfrac{1}{12} - \tfrac{1}{2}A - \tfrac{1}{4}A^2, \quad \xi = k^{\frac{2}{3}}A, \quad F_\pm = kf_\pm. \tag{41}$$

We expect (40) to apply when the cluster is well defined, in the sense that $|f_+ - f_-| \ll |f_* - f_0|$; this requires $-0.33 < A < 0.09$.

To compare the theoretical result (40) with the remainder defined exactly by (7), it is necessary to compute the integral I_0 (equation (1)) and the coefficients T_r in its asymptotic expansion. We computed the integral by numerical quadrature, after deforming the stipulated path C_0 to the following straight lines: from the valley $\infty\exp\{\mathrm{i}(\pi - \frac{1}{4}\theta_k)\}$ to $\frac{1}{2}$, and from $\frac{1}{2}$ to the valley $\infty\exp\{\mathrm{i}(\frac{1}{2}\pi - \frac{1}{4}\theta_k)\}$. (With f and g given by (38) there is a convergent double-series representation of (1), but numerical quadrature converged more quickly to the required accuracy.) We computed the T_r from their exact representations in terms of Gegenbauer polynomials (§22.9 of Abramowitz & Stegun 1964):

$$T_r(A) = \frac{\surd 2(r-\frac{1}{2})!}{(1-A)^{2r+\frac{1}{2}}}\,C_{2r}^{(r+\frac{1}{2})}\left\{\frac{2}{3}\Big/\sqrt{\left(\frac{2}{1-A}\right)}\right\}. \tag{42}$$

We chose the truncation N according to the prescription (37). It is convenient to divide by the dominant factors in (40), and display the scaled remainder defined by

$$R_N^{\mathrm{sc}}(k,A) \equiv \frac{k^{\frac{1}{2}}\exp\{+kf_*\}}{\pi\mathrm{i}\surd 2}R_N(k,A). \tag{43}$$

All computations were carried out on a Quadra 700 using *Mathematica*. Typical times for single evaluations were: 400 s for the integral I_0; 200 s for the optimally truncated asymptotic series; 3 s for the uniform approximation (40).

We display graphs of R_N^{sc} for $|k| = 300$. Figure 3 shows the variation with θ_k for four fixed values of A, corresponding to 'vertical' sections through figure 1; according to the theory, this behaviour should depend predominantly on the error function controlling the crossing of Stokes lines. Figure 4 shows the variation with A for three fixed values of θ_k, corresponding to 'horizontal' sections through figure 1; according to the theory, this behaviour should depend predominantly on the Airy function controlling the crossing of the saddle coalescence. (Some of the graphs in figure 4 are jagged, because the truncation N changes with A.)

It appears that the theory (40) gives an accurate description of the detailed behaviour of the remainder R_N over the whole range, which includes the confluence of the catastrophe and Stokes singularities. In some of the graphs, the theoretical and exact curves cannot be distinguished; fractional errors in these cases are typically 1%, and for $A < 0$ this agreement persists well beyond the ranges displayed. Note that in terms of the original integral (1) the remainder is 'beyond all orders' in $1/k$; in fact it is of order $\exp\{-N\}$, where N ranges from 17 to 36 in figures 3 and 4.

There is one régime where the apparent agreement between the theoretical and exact remainders dissolves under closer examination. Figure 5 shows the phase of the complex quantity R_N^{sc}. It is clear that the generally good agreement fails for large negative A and θ_k (figure 5*a*). This was not clear from figures 3 and 4 because it occurs where the scaled remainder is extremely small.

We begin to explain this failure by noting that in this régime – corresponding to the bottom left-hand corner of figure 1 – the two saddles on the 'bright' side of the Airy function should be well separated. Therefore their contributions (including Stokes smoothing) could be calculated by applying the ordinary method of steepest descents to the integral (15), or, equivalently, replacing Ai and Ai′ in (40) by their lowest-order asymptotic approximations. This gives (using $A^{\frac{1}{2}} = \exp\{\frac{1}{2}\mathrm{i}\pi\}\surd|A|$)

$$R_N^{\mathrm{sd}}(k,A) \approx \frac{\mathrm{i}\surd\pi}{k^{\frac{1}{2}}A^{\frac{1}{4}}}\left(\mathrm{i}\frac{S_N(F_+)}{(1+A^{\frac{1}{2}})^{\frac{1}{2}}}\exp\{-kf_+\}+\frac{S_N(F_-)}{(1-A^{\frac{1}{2}})^{\frac{1}{2}}}\exp\{-kf_-\}\right), \tag{44}$$

that is

$$R_N^{\mathrm{sc,\,sd}}(k,A) \approx \frac{\exp(-\frac{1}{4}\mathrm{i}\pi)}{|A|^{\frac{1}{4}}\surd(2\pi)} \times\left(\mathrm{i}\frac{S_N(F_+)}{(1+\mathrm{i}\surd|A|)^{\frac{1}{2}}}\exp\{-\tfrac{2}{3}\mathrm{i}k|A|^{\frac{3}{2}}\}+\frac{S_N(F_-)}{(1-\mathrm{i}\surd|A|)^{\frac{1}{2}}}\exp\{\tfrac{2}{3}\mathrm{i}k|A|^{\frac{3}{2}}\}\right). \tag{45}$$

This approximation is shown as the dotted liine in figure 5*a*; clearly, it follows the exact curve more closely than the uniform approximation (40). At first sight this is puzzling because (45) ought to emerge from (40) as a limiting case for large negative A.

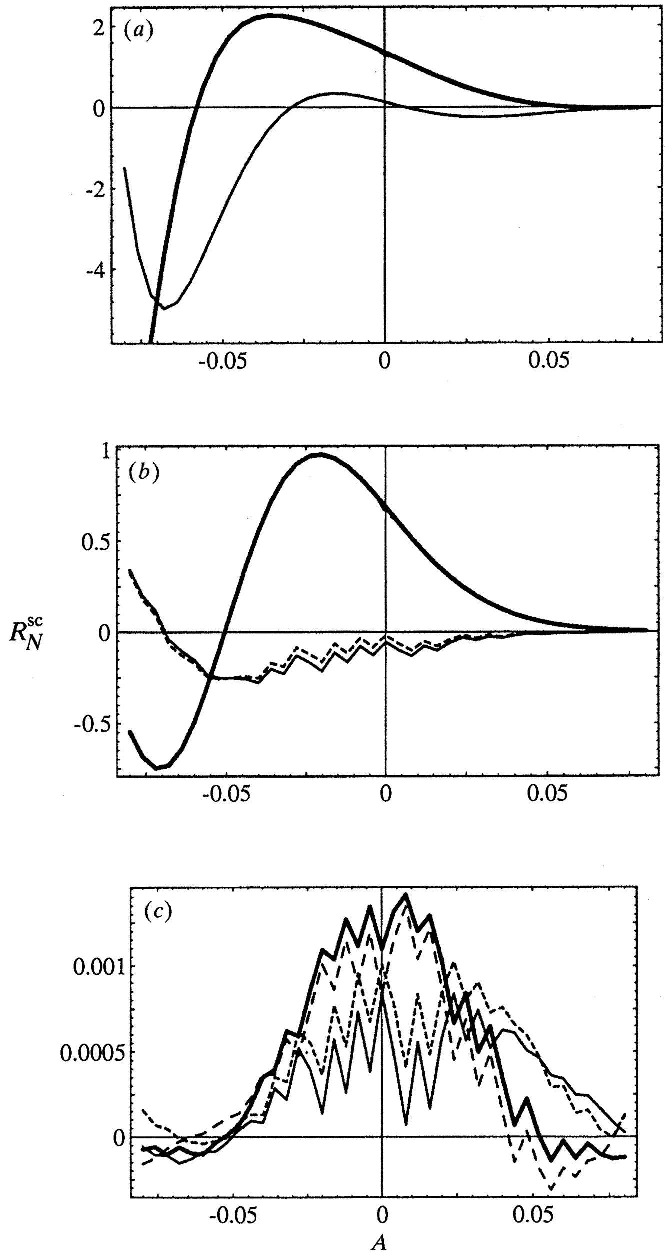

Figure 4. Scaled remainder R_N^{sc} defined by (43), as a function of the catastrophe variable A, for (*a*) $\theta_k = +\pi/5$; (*b*) $\theta_k = 0$; (*c*) $\theta_k = -\pi/5$. The graphs show $\mathrm{Re}[R_N^{sc}]$ (thick lines, exact; dashed lines, asymptotic theory) and $\mathrm{Im}[R_N^{sc}]$ (thin lines, exact; dotted lines, asymptotic theory). The truncations range from $N = 36$ to $N = 17$.

The origin of the disagreement lies in the failure of the approximation underlying (45), which is essentially

$$\mathrm{i}^{\frac{1}{2}}(S_+ + S_-)\,(k^{\frac{2}{3}}|A|)^{\frac{1}{4}}\,\mathrm{Ai}\{-k^{\frac{2}{3}}|A|\} + \mathrm{i}^{-\frac{1}{2}}(S_+ - S_-)\,\mathrm{Ai}'\{-k^{\frac{2}{3}}|A|\}/(k^{\frac{2}{3}}|A|)^{\frac{1}{4}}$$
$$\approx (1/\sqrt{\pi})\,(\mathrm{i}S_+ \exp\{-\tfrac{2}{3}\mathrm{i}k|A|^{\frac{3}{2}}\} + S_- \exp\{\tfrac{2}{3}\mathrm{i}k|A|^{\frac{3}{2}}\}). \quad (46)$$

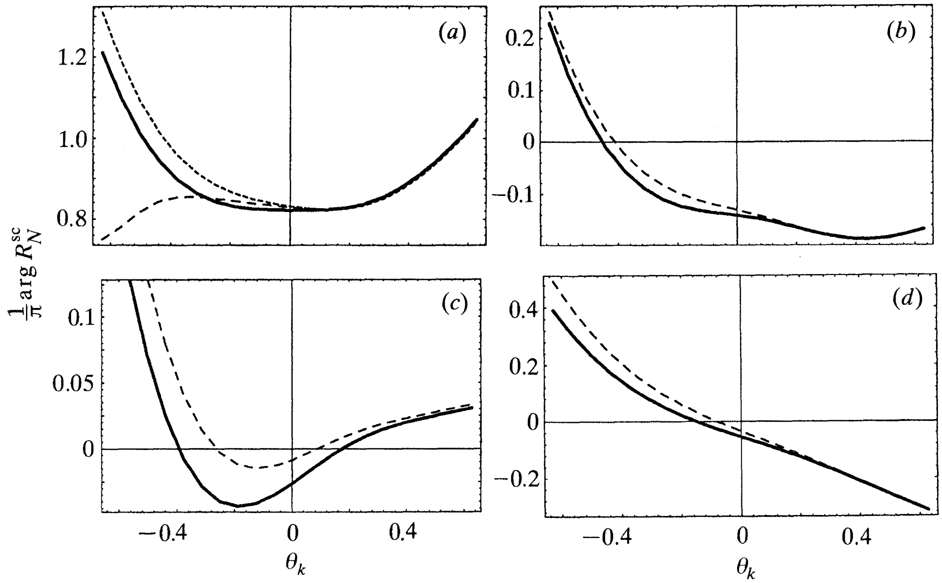

Figure 5. Phase $\arg\{R_N^{\rm sc}\}/\pi$ of the scaled remainder (43), as a function of the Stokes parameter θ_k, for (a) $A=-2/25$; (b) $A=-1/24$; (c) $A=0$; (d) $A=+1/24$ (thick lines, exact; dashed lines, asymptotic theory). In (a) the dotted line is the separated-saddle approximation (45).

This is obtained by substituting the lowest-order asymptotic approximations for Ai and Ai′. In the region of interest ($\theta_k<0$), the exponential associated with the saddle z_+ is switching on, and that associated with z_- has not begun to switch on, so $|S_-|\ll|S_+|$. On the other hand, for $\theta_k<0$ the second exponential in (46) dominates the first. In these circumstances – where a dominant exponential is multiplied by a very small coefficient – it is necessary to include the next term in the asymptotic series for Ai and Ai′. Then (46) becomes

$$\begin{aligned}&\mathrm{i}^{\frac{1}{2}}(S_++S_-)(k^{\frac{2}{3}}|A|)^{\frac{1}{4}}\,\mathrm{Ai}\{-k^{\frac{2}{3}}|A|\}+\mathrm{i}^{-\frac{1}{2}}(S_+-S_-)\,\mathrm{Ai}'\{-k^{\frac{2}{3}}|A|\}/(k^{\frac{2}{3}}|A|)^{\frac{1}{4}}\\&\qquad\approx(1/\sqrt{\pi})(\mathrm{i}S_+\exp\{-\tfrac{2}{3}\mathrm{i}k|A|^{\frac{3}{2}}\}+(S_--(\mathrm{i}/8k|A|^{\frac{3}{2}})S_+)\exp\{\tfrac{2}{3}\mathrm{i}k|A|^{\frac{3}{2}}\}),\qquad(47)\end{aligned}$$

which is radically different from (46) when $|kA^{\frac{3}{2}}S_-|\ll|S_+|$. What has failed is not the saddle-point approximation to (15) but our basic uniform Airy approximation. Putting it right would require – in this régime only – high-order corrections to the uniform approximation for the remainder (15) (obtained for example with the formalism given by Berry & Howls 1993) rather than the lowest-order approximation used here.

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Appendix A. Error function approximation for integral with a saddle near a pole

This is the derivation of (12) and (13). The v integral in (10) has the form

$$I_N(F)\equiv-\frac{1}{2\mathrm{i}\pi}\int_{-1,+\mathrm{i}\epsilon}^{\infty}\frac{\mathrm{d}v}{v}(1+v)^{N-\frac{1}{2}}\exp\{-vF\}.\qquad(\mathrm{A}\,1)$$

We wish to approximate this when N and $|F|$ are large and not too different. Then I is dominated by its pole at $v = 0$ and nearby saddle at

$$v = N/F - 1 \tag{A 2}$$

and we can approximate the integral by using the method of uniform approximation (Wong 1989) appropriate to this case. First we define a new variable u by mapping the exponent of the integrand as follows:

$$vF - N\ln(1+v) = u^2 + 2\mathrm{i}\sigma u \tag{A 3}$$

and fixing σ by demanding that saddle (A 2) corresponds with its counterpart $u = -\mathrm{i}\sigma$. Thus

$$\sigma = [N\ln F/N + N - F]^{\frac{1}{2}}. \tag{A 4}$$

The required integral (A 1) now becomes

$$I_N(F) = -\frac{1}{2\mathrm{i}\pi}\int_{-\infty,+\mathrm{i}\epsilon}^{\infty} \mathrm{d}u \frac{\mathrm{d}v}{\mathrm{d}u}\frac{1}{v(1+v)^{\frac{1}{2}}}\exp\{-u^2 - 2\mathrm{i}\sigma u\}. \tag{A 5}$$

To evaluate it, we write

$$\frac{1}{v(1+v)^{\frac{1}{2}}}\frac{\mathrm{d}v}{\mathrm{d}u} \equiv \gamma(u) = \frac{\alpha}{u} + \beta + u(u+\mathrm{i}\sigma)\,h(u). \tag{A 6}$$

Short calculations give

$$\alpha = \lim_{u\to 0} u\gamma(u) = 1, \quad \beta = \gamma(-\mathrm{i}\sigma) + \frac{\alpha}{\mathrm{i}\sigma} = -\frac{\mathrm{i}}{\sigma} + \frac{\sqrt{(2F)}}{N-F}. \tag{A 7}$$

In lowest order we can neglect h in (A 6), because the term involving it vanishes at the saddle and pole which dominate. Thus (A 5) becomes

$$I_N(F) \approx S_N(F) \equiv -\frac{1}{2\mathrm{i}\pi}\int_{-\infty,+\mathrm{i}\epsilon}^{\infty} \mathrm{d}u \left[\frac{1}{u} + \left(\frac{\sqrt{(2F)}}{N-F} - \frac{\mathrm{i}}{\sigma}\right)\right]\exp\{-u^2 - 2\mathrm{i}\sigma u\}. \tag{A 8}$$

The integral involving the second term in [...] is elementary. To evaluate the part involving the first term, we write

$$\frac{\exp\{-2\mathrm{i}\sigma u\}}{u} = -2i\int_{-\infty}^{\sigma} \mathrm{d}w \exp\{-2\mathrm{i}wu\}, \tag{A 9}$$

which converges at the lower limit if we take $\operatorname{Im} u > 0$ in (A 8). Thus we need

$$-2\mathrm{i}\int_{-\infty}^{\sigma}\mathrm{d}w\int_{-\infty}^{\infty}\mathrm{d}u \exp\{-2\mathrm{i}wu - u^2\}$$

$$= -2\mathrm{i}\sqrt{\pi}\int_{-\infty}^{\sigma}\mathrm{d}w \exp\{-w^2\} = -2\pi\mathrm{i}[\tfrac{1}{2}(1+\operatorname{erf}(\sigma))]. \tag{A 10}$$

When substituted into (A 8), this gives

$$S_N(F) = \tfrac{1}{2}[1+\operatorname{erf}(\sigma)] + \frac{\mathrm{i}}{2\sqrt{\pi}}\left(\frac{\sqrt{(2F)}}{N-F} - \frac{\mathrm{i}}{\sigma}\right)\exp\{-\sigma^2\} \tag{A 11}$$

as in (13).

The following asymptotic forms are instructive:

$$\left.\begin{aligned} S_N(F) &\approx 1+\frac{\mathrm{i}\surd(2F)}{2(N-F)\surd\pi}\exp\{-\sigma^2\} && (\operatorname{Re}\sigma \gg 1) \\ &\approx \tfrac{1}{2}-\tfrac{1}{12}\mathrm{i}\surd(2/\pi N) && (\sigma\approx 0) \\ &\approx \frac{\mathrm{i}\surd(2F)}{2(N-F)\surd\pi}\exp\{-\sigma^2\} && (\operatorname{Re}\sigma \ll -1). \end{aligned}\right\} \quad \text{(A 12)}$$

The common term for $\operatorname{Re}\sigma \gg 1$ and $\operatorname{Re}\sigma \ll 1$ is precisely the contribution from the saddle of (A 1), for arbitrary values of N and F; this wide range of validity is lost with the approximations (14), which apply only when F is close to N. The switching-on of the term 1 – the contribution from the pole in (A 1) – as $\operatorname{Re}\sigma$ increases through 0 is the essence of the Stokes phenomenon.

The multiplier (A 11) is a twofold generalization (cf. Olver 1991*a*) of the more familiar approximation (14), because the argument σ of the error function is complex, and the multiplier itself has an imaginary part (an approximation to which was given by Berry (1989*a*)).

Appendix B. Inverse of Vandermonde matrix (see also Traub 1966)

This is the derivation of the inversion formula (25) for the linear equations (22), that is

$$\sum_{s=0}^{M-1} a_s\,\zeta_n^s = G_n. \quad \text{(B 1)}$$

Here $\{\zeta_n\}$ and $\{G_n\}$ are sets of M arbitrary numbers, and the set of M numbers $\{a_s\}$ is to be determined.

If we knew the polynomial

$$\Gamma(\zeta) = \sum_{s=0}^{M-1} a_s\,\zeta^s \quad \text{(B 2)}$$

we could find the $\{a_s\}$ by

$$a_s = \frac{1}{2\mathrm{i}\pi}\oint \mathrm{d}\zeta\frac{\Gamma(\zeta)}{\zeta^{s+1}}, \quad \text{(B 3)}$$

where the contour encloses the origin. This gives a_s in terms of the derivatives of $\Gamma(\zeta)$ at the origin. We require a_s in terms of the values G_n of $\Gamma(\zeta)$ at the given numbers ζ_n.

To achieve this, we expand the contour to infinity, and multiply the integrand by a function $\Lambda_s(\zeta)$ with the following properties. Λ deviates from unity at infinity by less than $O(\zeta^{-(M-s-1)})$ (so as not to alter the value of the integral), has an $(s+1)$st order zero at $\zeta=0$ (to cancel the pole in (B 3)), and has simple poles at each of the ζ_n. Such a function can be constructed in terms of the characteristic polynomial of the $\{\zeta_n\}$, namely

$$P(\zeta) = \prod_{n=1}^{M}(\zeta-\zeta_n) \equiv \zeta^M + \sum_{k=0}^{M-1} X_{k+1}\,\zeta^k \quad \text{(B 4)}$$

(this is the derivative of the cuspoid catastrophe polynomial (17), with the slight generalization that since we are allowing the $\{\zeta_n\}$ to be arbitrary we must allow X_M to be non-zero). From $P(\zeta)$ we define the polynomial with the lowest $s+1$ terms subtracted, that is

$$P_s(\zeta) \equiv P(\zeta) - \sum_{k=0}^{s} X_{k+1}\,\zeta^k = \zeta^M + \sum_{k=s+1}^{M-1} X_{k+1}\,\zeta^k. \tag{B 5}$$

Then

$$A_s(\zeta) = P_s(\zeta)/P(\zeta). \tag{B 6}$$

It can be confirmed that this function has the three properties specified above.

Multiplying the integrand in (B 3) by $A_s(\zeta)$, we obtain

$$a_s = \frac{1}{2\mathrm{i}\pi}\oint \mathrm{d}\zeta \frac{\Gamma(\zeta)}{\zeta^{s+1}}\frac{P_s(\zeta)}{P(\zeta)}. \tag{B 7}$$

Now we can evaluate this in terms of the residues at the poles $\{\zeta_n\}$. The result is

$$a_s = \sum_{n=1}^{M} P_s(\zeta_n)\,G_n \bigg/ \zeta_n^{s+1} \prod_{k=1,\,k\neq n}^{M} (\zeta_n - \zeta_k). \tag{B 8}$$

This corresponds to the inversion formula (25) for the Vandermonde matrix ζ_n^s in (B 1).

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