Pancharatnam, virtuoso of the Poincaré sphere: an appreciation

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SIVARAJ Ramaseshan graciously invited me to write an essay review of the collected works of S. Pancharatnam. As a partial response to this invitation, I am happy to show my admiration of Pancharatnam by providing the following comments on three of his papers.

Geometric phases in polarization optics

Early in 1987, I received from Rajaram Nityananda a reprint of his paper[1] with Ramaseshan, drawing attention to Pancharatnam's anticipation, in 1956, of the geometric phases that had been in fashion since 1983 (and to some extent still are). Regrettably, I did not read their paper properly, and missed its main point. But I was fortunate to visit Bangalore in July 1987, and Ramaseshan met me and I came to appreciate the relevance of Pancharatnam's paper[2]. Ramaseshan honoured me with what he thought was the last copy of Pancharatnam's collected works[3] (now, more have come to light). Slowly, I realized that Pancharatnam's phase was something I had to understand. This was not easy because his arguments made heavy use of the geometry of the Poincaré sphere, which I knew about but had little facility with. For the long flight home, I set myself the task of interpreting Pancharatnam's discovery in more modern and general terms.

That flight was a revelation. I learned that not only had this young fellow of twenty-two created the simplest example of the geometric phase, but that he had also pointed out a feature (the definition of phase difference described below) that had not, by 1987, been perceived in any of the many papers developing my work[4] of 1983. There and then I decided to follow the example of Nityananda and Ramaseshan, and write an exposition[5] of Pancharatnam's phase that would bring the full originality of its conception to a larger readership.

Pancharatnam was inspired by his mentor C. V. Raman to study the complicated interference figures produced by light beams traversing crystal plates (see page 232, box). For this he needed to compare the phases of waves in different states of polarization. A given state of polarization (given for example by the eccentricity, axes and sense of traversal of the ellipse described by the tip of the electric $\boldsymbol{E}$ or $\boldsymbol{D}$ vector) is represented by a point on the Poincaré sphere. However, this does not specify the phase of the vibration: all states related by a phase factor correspond to the same point on the sphere. Therefore phase is an additional quantity, to be attached to points on the sphere if light beams are to be described completely.

To understand interference, it is necessary to know not the absolute phase of a wave but the phase difference between light beams in different states of polarization. Apparently, nobody had asked this simple question. Pancharatnam did ask it, and gave a simple answer: the phase difference between two beams is that phase change which when applied to one of them maximizes the intensity of their superposition. It was instructive[5] to express this in terms of more general mathematics (as used for example in quantum mechanics). Each beam is represented by a spinor with two complex components (corresponding for example to the amplitudes and phases of the components of the $\boldsymbol{D}$ vector perpendicular to the propagation direction). For two beams $|A\rangle$ and $|B\rangle$, Pancharatnam's rule implies that the phase difference is the phase of their complex scalar product:

$$\text{phase difference between } |A\rangle \text{ and } |B\rangle = \text{phase of } \langle A|B\rangle. \quad (1)$$

In particular, the beams are in phase (intensity of overlap a maximum) if the scalar product is real and positive. Nowadays this is known as 'Pancharatnam's connection'. It can be derived from a more general rule, in which $|A\rangle$ is transported to a neighbouring state $|A + dA\rangle$ (for example by passing the beam through an appropriate polarizing crystal), with the phase defined by the requirement

$$\langle A|\,d\,|A\rangle = 0, \text{ where } d|A\rangle \equiv |A + dA\rangle - |A\rangle. \quad (2)$$

It can be shown[5] that if this is integrated from $|A\rangle$ to $|B\rangle$ along the shorter arc of the great circle connecting their representative points on the sphere, then $|B\rangle$ is in phase with $|A\rangle$. The rule (2) is called parallel transport because it corresponds to moving $|A\rangle$, regarded as a unit vector perpendicular to the radius of the sphere and attached to its representative point, without turning it about the radius.

Pancharatnam showed[2] that this natural stipulation of phase has a remarkable property. It is non-transitive:

if $|A\rangle$ is in phase with $|B\rangle$, and $|B\rangle$ is in phase with a third state $|C\rangle$, then $|C\rangle$ need not be in phase with $|A\rangle$. Indeed, if $|C\rangle$ is in phase with a state $|A'\rangle$ represented by the same point on the sphere as $|A\rangle$, then the phase factor accumulated after this cycle of polarization states, embodying the phase difference between $|A\rangle$ and $|A'\rangle$, is

$$\langle A|A'\rangle = \exp\{-\tfrac{1}{2}\mathrm{i}\,\Omega_{ABC}\}, \qquad (3)$$

where Ω_{ABC} is the solid angle of the spherical triangle ABC. It can be shown that the same result holds for more general cycles (e.g. smooth loops) when the state is continued according to the connection (2). Then the solid angle is that of the loop, and the fact that the vector does not return to its original direction reflects the nonintegrability of this connection. What Pancharatnam actually stated was not exactly the result (3), but an unsymmetrical version equivalent to it: if two beams $|A\rangle$ and $|B\rangle$, which are in phase, are passed through an analyser bringing them to the state $|C\rangle$, their phase difference is $-\frac{1}{2}\Omega_{ABC}$.

In its original form, the geometric phase I found in 1983 concerned not polarization states of light beams, but quantum states of particles, that is, Hilbert-space vectors satisfying the Schrödinger equation. In the adiabatic limit where the environment of the system (that is, its Hamiltonian) changes slowly, the parallel transport law (2) applies, *mutatis mutandis*, and, being nonintegrable, generates a geometric phase factor when parameters on which the system depends are taken round a cycle. The generalization of the solid angle is the flux through the cycle (in parameter space) of a certain 'phase field' (mathematically, a 2-form)[4].

In special cases where the phase involves an actual solid angle, this is because the phase field is that of a monopole in parameter space. For spinning particles with angular momentum $n\hbar$, the strength of the monopole is $-n$, so the phase is $-n\Omega$. Therefore the Pancharatnam situation is analogous to the further specialization to spin 1/2, which, as is well known, is a model for any 2-state system. There was an apparent discordance here, between the fact that photons have spin 1, leading to geometric phases of $\pm\Omega$ for smooth cycles of their spin direction (e.g. in coiled optical fibres[6]), and Pancharatnam's $-1/2\Omega$. But it was clear[5] that there is no contradiction, because the two solid angles are different: the first is in the space of propagation directions $\boldsymbol{k}$ for light with a fixed state of polarization, whereas the second is on the Poincaré sphere of polarization states for light travelling in a fixed direction $\boldsymbol{k}$, so that, because of transversality, there are only two states (e.g. components of $\boldsymbol{D}$ perpendicular to $\boldsymbol{k}$).

Ignorant of Pancharatnam's work, I had also calculated geometric phases for the case he studied, that is in a light beam whose state of polarization is changed. I envisaged[5,7] a medium whose birefringence and gyrotropy were changed smoothly along the beam path – for example by a varying electric field (Cotton–Mouton effect) and a varying magnetic field (Faraday effect). I was astonished (and not a little humbled) to learn that Pancharatnam had had essentially the same idea (with discrete, rather than continuous polarization changes) thirty years earlier. This was one of several areas in which the geometric phase had been anticipated[8].

In spite of its startling originality, Pancharatnam's paper[2] was completely ignored until Ramaseshan and Nityananda[1] made us aware of it. Now his contribution is properly recognized, and his paper has been cited many times. I will not review subsequent work (Bhandari does so in his accompanying article), but will simply draw attention to two very different applications of the idea. In one, Schmitzer, Klein and Dultz[9] remark that the solid angle Ω_{ABC} can change much faster than the angle of rotation of an analyser (representing C with A and B fixed), and propose this as the basis of a new type of optical switch. In the other, Nye[10] employs Pancharatnam's connection in a study of phase and polarization singularities in electromagnetic waves that vary in space.

Mirages

Although the young Pancharatnam and the elderly Raman were intellectually (as well as consanguinously) close, they wrote only one paper[11] together, about the mirage. When I read it (on the 1987 flight) I realized that it was based on a misunderstanding of the connection between wave optics and geometrical optics. I had suspected confusion on this point since a conversation in 1976 in which Ramaseshan described the Raman–Pancharatnam ideas to me.

Their claim was that the mirage cannot be explained in terms of rays but requires the wave theory. By considering light propagating in the air above a hot surface, considered as a stack of slabs with slightly different refractive indices, they convinced themselves that refraction could never make a downward-sloping ray turn upwards, because a ray once horizontal would remain so. In other words, refraction could never simulate reflection. This was a misunderstanding of the law of refraction in a continuously-varying medium, based on an incorrect limiting process. When applied to mechanics, the same argument (upside-down) would predict that an obliquely-fired projectile would never fall but would continue horizontally on reaching its greatest height. The correct slab limit shows that transverse gradients of refractive index cause rays to curve, just as transverse forces cause particle paths to curve.

Why dwell on a mistake, in what is supposed to be an appreciation? To make the point that the errors of first-rate scientists can be both instructive and productive. After concluding, wrongly, that the mirage cannot be a refraction effect, they set about constructing a wave theory for the neighbourhood of the layer where, according to them, the rays ought to turn but do not. This is the level of the caustic, where the air acts like a mirror. By using the mathematical analogy between the light wave equation with linearly varying refractive index and Schrödinger's equation for quantum particles in a region of constant force, they expressed the wave in terms of Bessel functions of order ± 1/3. Although this wave theory was constructed on the basis of a confusion, there is no doubt that it is correct. Indeed they successfully carried out experiments with heated plates to test several of its consequences.

One irony remains. In contrast to the startling originality and prescience of the polarization phase, the Raman–Pancharatnam theory of waves near a caustic turned out to be not a discovery but a rediscovery. Airy[12] had formulated essentially the same theory in 1838.

Propagation along singular crystal axes

A treasure among Pancharatnam's papers, that, if anything, pleased me more than his phase, is a surprising special case[13] of wave propagation in absorbing anisotropic crystals. In transparent crystals, the two polarization states that travel unchanged through the crystal in any direction are orthogonal (and so represented by antipodes on the Poincaré sphere). With absorption, the eigenpolarizations are no longer orthogonal, and there are even particular crystals and propagation directions (along the 'singular axes') for which they coincide, so that only one polarization can propagate.

Pancharatnam concentrated on this case, and studied what happens when the orthogonal polarization, which cannot propagate, is introduced into the crystal. Earlier authors were, he claimed, wrong: they had thought such a wave must be reflected. He argued that, on the contrary, the polarization would change gradually into the state that does propagate, and moreover would grow stronger (by a factor increasing linearly with distance) than a wave of the same intensity introduced with the correct polarization. I found this conclusion hard to believe, and his arguments (again based on Poincaré sphere geometry, see page 233, Ranganath, G. S., this issue) difficult to follow, and embarked on the task of reconstructing the theory in my own way (as I have already mentioned, it was a long flight). The results vindicated Pancharatnam completely, and moreover produced an analytical expression for the wave which appears nowhere in his papers (although I believe he must have known it). The expression is worth presenting.

Let the beam travel in the z direction. A model crystal exhibiting Pancharatnam's phenomenon has dielectric tensor with transverse components

$$\mathsf{e} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{pmatrix} = \begin{pmatrix} \mathrm{i}a & \frac{1}{2}(b-a) \\ \frac{1}{2}(b-a) & \mathrm{i}b \end{pmatrix}. \tag{4}$$

This matrix has a single degenerate eigenvalue and a single eigenvector col(1, i)/√2, representing circularly polarized light. For the medium to be anisotropic, we must have $a \neq b$. For the medium to be absorbing, the rate of energy dissipation must be positive, which implies that the Hermitian matrix $\mathrm{i}(\mathsf{e}^\dagger - \mathsf{e})$ must have positive eigenvalues; when applied to (4) this gives the requirement

$$\mathrm{Re}\,(a+b) > |\,a-b\,|, \tag{5}$$

which is satisfied by a and b real. If the only other non-zero dielectric tensor component is ε_{zz}, then not only the electric $\boldsymbol{D}$ but also the $\boldsymbol{E}$ vector is transverse, and Maxwell's equations reduce to

$$\partial_z^2 \boldsymbol{D} + k^2 \mathsf{e} \cdot \boldsymbol{D} = 0, \tag{6}$$

where k is the free-space wave number.

The solution of (6) given (4) with the initial polarization

$$\boldsymbol{D}(0) = \begin{pmatrix} u \\ v \end{pmatrix} \tag{7}$$

is

$$\boldsymbol{D}(z) = \exp\{-\tfrac{1}{2}kz(1-\mathrm{i})\sqrt{a+b}\} \times \left[\begin{pmatrix} u \\ v \end{pmatrix} - \tfrac{1}{4}kz\frac{(a-b)}{\sqrt{a+b}}(1-\mathrm{i})(u+\mathrm{i}v)\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}\right], \tag{8}$$

as can be confirmed by substitution. For the eigenpolarization, $u = -\mathrm{i}v = 1/\sqrt{2}$, and the term involving z is zero. For any other initial polarization – in particular the orthogonal one $u = +\mathrm{i}v$ – the term involving z grows relative to the term not involving z, just as Pancharatnam discovered. (In spite of the increasing z factor, the energy flow into the medium must decay, because the medium is absorbing, and this is ensured by the exponential prefactor. A direct proof from (8) is lengthy, but the result follows from the fact that the dissipation rate (positive) is minus the divergence of the Poynting vector.)

Pancharatnam's law of singular axis propagation is an example of a general mathematical phenomenon: evolution driven by a non-Hermitian matrix **M** with a degenerate eigenvalue. The lack of Hermeticity is fundamental, because it drastically alters the nature of the degeneracy. If **M** is Hermitian (as in the quantum mechanics of spin), there are (in the simplest case) two orthogonal eigenvectors corresponding to a degenerate eigenvalue, and in the neighbourhood of the degeneracy (in the space of parameters on which **M** depends) the eigenvalues form a double cone, each sheet of which corresponds to an eigenvalue. If **M** is not Hermitian (as with (4)), there is only one eigenvector at the degeneracy, which is a branch point for the two eigenvalue sheets in its neighbourhood, and around which each eigenvector turns into the other. While writing this I found that the mirror property of a stack of transparent plates (such as glass microscope-slide cover slips or acetate overhead-projector sheets) depends on precisely this phenomenon of a non-Hermitian matrix with a degenerate eigenvalue (I thank S. Klein for posing this problem); so does the matrix governing waves incident at the critical angle on a slab of lower refractive index (I thank G. N. Borzdov for telling me this).

The phenomenon of propagation along singular axes (known as Voigt waves) is now well understood, as a result of extensive and definitive theoretical work by Fedorov and his colleagues[14]. In a crystal slab, the two waves travelling in each direction can be made to degenerate in several ways, leading to coordinate dependence that can be not only linear but quadratic[15] (for three-wave degeneracy) or cubic[16] (for four-wave degeneracy). However, Pancharatnam's pioneering paper is little known and infrequently cited.

Now, as we remember Pancharatnam's untimely death in his creative prime, and celebrate his youthful achievements, it is time to look again through all his work. Who knows what further delicious physics this will reveal?

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