High orders of the Weyl expansion for quantum billiards: resurgence of periodic orbits, and the Stokes phenomenon

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A formalism is developed for calculating high coefficients c_r of the Weyl (high energy) expansion for the trace of the resolvent of the Laplace operator in a domain $\mathscr{B}$ with smooth boundary $\partial\mathscr{B}$. The c_r are used to test the following conjectures. (*a*) The sequence of c_r diverges factorially, controlled by the shortest accessible real or complex periodic geodesic. (*b*) If this is a 2-bounce orbit, it corresponds to the saddle of the chord length function whose contour is first crossed when climbing from the diagonal of the Möbius strip which is the space of chords of $\mathscr{B}$. (*c*) This orbit gives an exponential contribution to the remainder when the Weyl series, truncated at its least term, is subtracted from the resolvent; the exponential switches on smoothly (according to an error function) where it is smallest, that is across the negative energy axis (Stokes line). These conjectures are motivated by recent results in asymptotics. They survive tests for the circle billiard, and for a family of curves with 2 and 3 bulges, where the dominant orbit is not always the shortest and is sometimes complex. For some systems which are not smooth billiards (e.g. a particle on a ring, or in a billiard where $\partial\mathscr{B}$ is a polygon), the Weyl series terminates and so no geodesics are accessible; for a particle on a compact surface of constant negative curvature, only the complex geodesics are accessible from the Weyl series.

1. Introduction

Our purpose is to present analytical arguments and numerical calculations supporting a conjecture in semiclassical asymptotics. It concerns the high orders of local expansions for a class of functions representing spectra. Although we believe our arguments are widely applicable, we will concentrate on the spectra of quantum billiards. A quantum billiard is a planar region $\mathscr{B}$ with area $\mathscr{A}$, where a quantum particle moves freely with specular reflection at the boundary $\partial\mathscr{B}$, which we stipulate to be a simple closed curve of length $\mathscr{L}$ that is smooth to all orders. The spectrum is the set of eigenvalues (energy levels) $\{E_n\}$ $(n = 1, 2, \ldots)$ of the Laplace operator with Dirichlet boundary conditions. This is the same as the spectrum of (squared) frequencies of a vibrating drumhead with boundary $\partial\mathscr{B}$. Thus

$$-\nabla^2\psi_n(\boldsymbol{r}) = E_n\psi_n(\boldsymbol{r})\,(\boldsymbol{r} \equiv (x, y) \text{ in } \mathscr{B}), \quad \psi_n = 0\ (\boldsymbol{r} \text{ on } \partial\mathscr{B}). \tag{1}$$

A good introduction to this problem is given by Baltes & Hilf (1976).

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The spectral function we choose to work with is the regularized resolvent

$$g(s) \equiv \lim_{N\to\infty}\left[\sum_{n=1}^{N}\frac{1}{E_n+s^2}-\frac{\mathscr{A}}{4\pi}\log\left\{\frac{E_N}{s^2}\right\}\right]. \tag{2}$$

Without the logarithmic subtraction, this would diverge because the asymptotic smoothed level density is constant, namely $\mathscr{A}/4\pi$ (see Voros (1987, 1992) for the general theory of such regularizations). As will become clear, $g(s)$ has a formal large-s asymptotic expansion

$$g(s) = \sum_{r=1}^{\infty}\frac{c_r}{s^r}. \tag{3}$$

This is known as the Weyl series.

We are interested in the behaviour of the coefficients c_r as $r\to\infty$. Our conjecture is that

$$c_r \to \alpha(r-\beta)!/l^{r-2}, \tag{4}$$

where α and β are dimensionless constants of order unity, and l is the length of the *shortest accessible periodic geodesic* in $\mathscr{B}$ (the meaning of 'accessible' will be explained later).

The conjecture will now be motivated by some recent results in asymptotics. Observe first that (4) would imply the divergence of the series (3). This can be interpreted by Borel-summing the divergent tail, assuming (4), after truncating at the least term

$$r = r^* \approx \mathrm{Int}\,[|ls|], \tag{5}$$

which is large when $|s|$ is large. In Berry (1989) this technique is explained in detail. It is based on replacing the factorial by its integral representation, summing the resulting geometric series, and evaluating the Borel integral for large $|ls|$ by exploiting the proximity of its pole to its saddle and the fact that $g(s)$ is real when s is real. The result is

$$g(s) = \sum_{r=1}^{r^*}\frac{c_r}{s^r}+R(s), \tag{6}$$

where the remainder is

$$R(s) \approx \exp\{-ls\}\frac{\mathrm{i}\pi\alpha l^2}{(ls)^{\beta-1}}\mathrm{erf}\{\sigma(s)\}, \tag{7}$$

in which

$$\sigma(s) = \frac{\mathrm{i}(r^*-ls)}{\surd(2r^*)} \approx \frac{\mathrm{Im}\,(ls)}{\surd(2\mathrm{Re}\,(ls))}. \tag{8}$$

What (7) describes is the smooth switching-on of the small exponential $\exp\{-ls\}$ across the Stokes line s real. When s is continued from the positive real axis to the positive imaginary axis, this exponential gets fully switched on (erf$\to 1$) and the exponential is oscillatory: $\exp\{-\mathrm{i}|ls|\}$). To see that this exponential should be present, observe first that for imaginary s (negative s^2) (2) shows that $g(s)$ has poles, at $s = \pm\mathrm{i}\surd E_n$. These are reflected in the singularities of the spectral density

$$d(E) = \sum_{n=1}^{\infty}\delta(E-E_n) = \frac{\mathscr{A}}{4\pi}-\frac{1}{\pi}\lim_{\epsilon\to 0}\mathrm{Im}\,g(\mathrm{i}\surd(E-\mathrm{i}\epsilon)). \tag{9}$$

From (3) this implies the formal asymptotic expansion

$$d(E) = \frac{\mathscr{A}}{4\pi} + \frac{c_1}{\pi\sqrt{E}} + \frac{1}{\pi\sqrt{E}} \sum_{r=1}^{\infty} \frac{(-1)^r}{E^r} c_{2r+1}. \tag{10}$$

If this series converged absolutely, it could not represent the singularities of $d(E)$. The customary procedure (Gutzwiller 1971; Balian & Bloch 1972, reviewed by Gutzwiller 1990; Berry 1983, 1991*a*) is to truncate (10) after a few terms and regard this as representing the smoothed spectral density $\bar{d}(E)$, and consider the singularities as originating in a further, separate, contribution d_{osc}, consisting of a sum over all *periodic geodesics* in the billiard. Thus

$$d(E) = \bar{d}(E) + d_{\text{osc}}(E), \tag{11}$$

where $$\bar{d}(E) \approx \frac{\mathscr{A}}{4\pi} \quad \text{and} \quad d_{\text{osc}}(E) \approx \operatorname{Re} \sum_j A_j(E) \exp\{-\mathrm{i} l_j \sqrt{E}\}, \tag{12}$$

in which j labels periodic orbits, with lengths l_j, and the amplitudes Aj depend on the stabilities of the orbits. The elementary model for this behaviour is the Poisson formula

$$\sum_{n=-\infty}^{\infty} \delta(E-n) = 1 + 2\sum_{j=1}^{\infty} \cos\{2\pi j E\}. \tag{13}$$

A less elementary model is the Selberg trace formula (see §2*c* and Balazs & Voros 1986).

Our conjecture (4) would imply that the oscillatory contributions in (12) are not separate at all, but are encoded in the divergent tail of the large-E series (10), and can be decoded by the Borel procedure embodied in (6) and (7). Only the shortest orbit appears in the leading approximation (4) to the late terms of (3), but there would also be exponentially smaller contributions from the longer periodic geodesics. The energy-dependence of the A_j is known to depend on whether the orbit is isolated or forms part of a continuous family, and this enables the constant β in (4) to be identified via (7). If the orbit forms a continuous family (e.g. because of rotation symmetry as in the circle billiard (§3.6)), then $\beta = \frac{3}{2}$; in the generic case where the orbit is isolated, $\beta = 2$.

The argument just given is an example of what has recently emerged as a general mathematical phenomenon called resurgence, originated by Dingle (1973) and extended by Écalle (1981, 1984), in which the divergent tails of asymptotic expansions are related to additional small exponential contributions to the function being expanded. Perhaps the most elaborate exploration of resurgence has been to contour integrals involving exponentials in which a large parameter multiplies a function with saddles (Berry & Howls 1991). Then the divergence of the saddle-point expansion for an individual saddle depends on the distances to certain other saddles (selected by a topological criterion called adjacency, related to the concept of accessibility mentioned after (4) – see also §5*a*). Here the generation of global from local information has become explicit. Understanding the full asymptotics of the high orders (that is, the 'asymptotics of the asymptotics') of the saddle-point expansion leads to very accurate 'hyperasymptotic' resummation schemes (see also Olde Daalhuis 1992, 1993). In quantum field theory the relation between high orders of perturbation theory and small exponentials has been studied extensively (Le Guillou & Zinn-Justin 1990; West 1990). In nonlinear mechanics the divergence of perturbation series has been connected directly to the long-period trajectories

(Bogomolny 1984*a*, *b*). In the quantum theory of adiabatic processes, the divergence of series (in powers of a parameter controlling the slowness with which the environment of a system is varied) is related directly to transitions (Berry 1990; Lim & Berry 1991; Berry & Lim 1993).

From the conjecture (4), certain conclusions follow. If the high orders of the Weyl series are available, the length of the shortest accessible periodic orbit can be reconstructed as

$$l = \exp\left\{-\lim_{r\to\infty}\frac{\mathrm{d}}{\mathrm{d}r}\log\left\{\frac{c_r}{(r-\beta)!}\right\}\right\}. \tag{14}$$

If in addition sufficiently many eigenvalues are available to enable the function $g(s)$ to be calculated accurately near the Stokes line from its definition (2), the switching-on of the small exponential can be monitored by employing the Stokes smoothing (7), which implies

$$\lim_{\mathrm{Re}\, s\to\infty}\left(\left[g(s)-\sum_{r=1}^{r^*}\frac{c_r}{s^r}\right]\exp\{ls\}\frac{(ls)^{\beta-1}}{\mathrm{i}\pi\alpha l^2}\right) = \mathrm{erf}\{\sigma(s)\}. \tag{15}$$

The plan of this paper is as follows. In §2 we give three preliminary examples, not involving planar billiards, to illustrate spectral resurgence and introduce several features we shall meet in the generic cases to be considered later. The harmonic oscillator (§2.1) shows resurgence of the series for $g(s)$ in its simplest form, as described above; the generation of periodic orbits can be described explicitly and illustrates some recent results from the asymptotics of the gamma function (Berry 1991*a*). The particle on a ring (§2*b*) is a cautionary example, where the Weyl series (3) consists of a single term and so cannot exhibit resurgence; there is no Stokes phenomenon, so that the periodic orbits cannot switch on but simply add inertly to $g(s)$ – in a terminology we shall develop later, they are not accessible via the Weyl series. Billiards on a compact surface with constant negative curvature (§2*c*) do have a divergent Weyl series, but its resurgence generates only complex geodesics (Cartier & Voros 1988), corresponding to motion on the ordinary sphere – the real geodesics, familiar as the skeleton of the chaotic motion, are inaccessible from the Weyl series.

In §3 we discuss in detail the unit circle billiard, whose shortest periodic orbit (diametral 2-bounce) has $l = 4$. Using a formalism (§3*a*) for $g(s)$ specific to the circle, devised by Stewartson & Waechter (1971), we compute coefficients up to c_{31}, and confirm the prediction (14) with $\beta = \frac{3}{2}$. Asymptotic analysis of the Stewartson–Waechter formula (§3*b*) enables us to extract explicitly all the periodic orbits from $g(s)$; in particular we can identify the coefficient α in (7). Using the fact that the circle eigenvalues are (squares of) zeros of Bessel functions, we are able to perform the subtraction in (15) and reveal the switching-on of the shortest orbit across the Stokes line.

In §4 we begin the main work of the paper, with a computation of c_r for a general smooth boundary. As is well known, the coefficients are integrals round $\partial\mathscr{B}$ of powers of derivatives of the curvature $\kappa(\sigma)$, where σ is arc distance. The algebra gets very complicated, so it is important to use an economical formalism. Our contribution here is to develop the general theory of Stewartson & Waechter (1971) so as to give an explicit algebraic recurrence for the coefficients. Previously, Smith (1981), with a different theory, had calculated the coefficients through c_6. We are able to reach c_{13}. This enables us to begin to see the large-order behaviour predicted by (4).

In §5*a* we study the geometry of 2-bounce orbits. This enables us to formulate a further conjecture, defining the shortest *accessible* periodic orbit: it is the orbit corresponding to the lowest saddle – as opposed to extremum – on the Möbius strip which is the space of chords of $\mathscr{B}$ (the distinction between saddles and extrema is not the same as that between stable and unstable orbits). Thus the divergence of the Weyl series is not governed by the absolutely shortest orbit if this is not a saddle. In §5*b* we study a class of billiard shapes (deformations of a circle) for which the formulae of §4 enable the c_r (through $r = 13$) to be calculated analytically as functions of a shape parameter a. Thus we are able to extend the testing of (4), via (14), to the generic case of periodic orbits that are not isolated. The results confirm both (4) and also the accessibility conjecture. We also describe a bifurcation where a pair of complex periodic orbits gives birth to the shortest accessible orbit, together with an inaccessible extremum orbit, as a increases; before the bifurcation, the divergence of the Weyl series is governed by the complex orbits.

This work raises many mathematical questions, in the concluding §6 we list some of them.

It is worth remarking that the resolvent is related to several other spectral functions. The counting function (spectral staircase), defined as

$$N(E) \equiv \sum_{n=1}^{\infty} \Theta(E - E_n), \tag{16}$$

where Θ denotes the unit step, is the integral of the spectral density (9) and so has the Weyl series (corresponding to (10))

$$N(E) = \frac{\mathscr{A}E}{4\pi} + 2\frac{c_1}{\pi}\sqrt{E} + c_2 - \frac{\sqrt{E}}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r}{(r-1/2)E^r}c_{2r+1}. \tag{17}$$

The heat kernel is defined by

$$K(t) \equiv \sum_{n=1}^{\infty} \exp\{-E_n t\} \tag{18}$$

(for imaginary t this is the trace of the evolution operator for quantum states). It has the short-time asymptotic expansion

$$K(t) = \sum_{r=0}^{\infty} a_r t^{r/2-1} \quad \left(a_0 = \frac{\mathscr{A}}{4\pi}\right), \tag{19}$$

$K(t)$ is related to $g(s)$ by the regularised Laplace transform

$$g(s) = \int_0^{\infty} \mathrm{d}t \exp\{-s^2 t\}\left(K(t) - \frac{a_0}{t}\right). \tag{20}$$

This yields the relation between the two sets of asymptotic coefficients:

$$c_r = a_r \Gamma(\tfrac{1}{2}r). \tag{21}$$

Voros (1987, 1992) discusses a variety of spectral functions, including some not considered here, and their interrelations.

Before proceeding further, we wish to remove two potential sources of confusion. The first involves a difference in the way mathematicians and physicists view the Weyl series. In the mathematical literature, one aim is to bound the remainders when the Weyl series (e.g. for the counting function $N(E)$) is truncated. Thus any oscillatory terms will be incorporated into remainders with the same order of magnitude – what matters is the size of the contribution, not its analytic structure.

Our emphasis is different: we treat the oscillatory and Weyl terms as qualitatively different (albeit connected by resurgence) because of their analytic behaviour as $s \to \infty$, i.e. exponential essential singularities (periodic orbits) rather than powers (Weyl).

The second potential confusion concerns the fact that the Weyl series (3), and its counterparts (10), (17) and (19), are not the only series that diverge in spectral functions. The sum over periodic orbits in d_{osc} (equation 12) cannot converge, because it has to generate the singularities in $d(E)$. Usually it diverges. The difficulties are most severe for classically chaotic systems. Here we do not consider this type of divergence, which arises from the long periodic orbits. It has been studied a great deal elsewhere (Berry 1985, 1991*a*; Sieber & Steiner 1991), and, in a quite different application of resurgence from that used here, essentially eliminated (Berry & Keating 1992).

All numerical and algebraic computations were carried out with *Mathematica* (Wolfram 1991).

2. Simple examples of spectral resurgence

(*a*) *Harmonic oscillator*

The energy levels are

$$E_n = (n - \tfrac{1}{2})\,\omega, \quad (n = 1, 2, \ldots), \tag{22}$$

where ω is the classical frequency (we set Planck's constant $\hbar = 1$). From (18), the heat kernel is immediately found to be

$$K(t) = \frac{1}{2\sinh(\tfrac{1}{2}\omega t)}. \tag{23}$$

Expanding in powers of t gives

$$K(t) = \frac{2}{\omega t} \sum_{k=0}^{\infty} (-1)^k \left(\frac{\omega t}{2\pi}\right)^{2k} \zeta(2k)\,(1 - 2^{1-2k}). \tag{24}$$

Here ζ denotes Riemann's zeta function (Abramowitz & Stegun 1972), and we have made use of its connection with the Bernoulli numbers:

$$\zeta(2k) = \frac{(2\pi)^{2k}\,(-1)^{k+1}}{2(2k)!} B_{2k}. \tag{25}$$

The series (24) has the form (19) with $a_{4k+1} = a_{4k+2} = a_{4k+3} = 0$. It converges if $t < 2\pi/\omega$, that is until the first classical return of the oscillator (which in this case is independent of starting conditions).

Our main interest is in the resolvent (3). For this, the Laplace transform (20) leads to (Voros 1986)

$$g(s) = \frac{1}{\omega}\left[\log\left\{\frac{s^2}{\omega}\right\} - \psi\left\{\frac{s^2}{\omega} + \frac{1}{2}\right\}\right], \tag{26}$$

where ψ is the logarithmic derivative of the gamma function (Abramowitz & Stegun 1972). Using the duplication formula and Stirling's series, or, more simply, the relation (21) we find the large-s expansion. It is convenient to express this using

$$-(1 - 2^{1-x})\,\zeta(x) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m^x}. \tag{27}$$

Thus we find the formal series

$$g(s) = -\frac{2}{\omega}\sum_{m=1}^{\infty}(-1)^m \sum_{k=1}^{\infty}(-1)^k (2k-1)!\left(\frac{\omega}{2\pi s^2 m}\right)^{2k}. \tag{28}$$

Each of the series over k diverges factorially, but can be summed using a slight variant of the results (4)–(7), applied across the Stokes line (where all terms in the series have the same phase) which here is $\arg s \equiv \theta = \frac{1}{4}\pi$. Noting that $g(s)$ is real for real s, we find

$$g(s) = -\frac{2}{\omega}\sum_{m=1}^{\infty}(-1)^m \left(\sum_{k=1}^{k_m^*-1}(-1)^k (2k-1)!\left(\frac{\omega}{2\pi s^2 m}\right)^{2k} + R_m(s)\right). \tag{29}$$

Here
$$k_m^* \approx \mathrm{Int}\{\pi m |s|^2/\omega\} \tag{30}$$
and
$$R_m(s) \approx \mathrm{i}\pi \exp\{2\pi \mathrm{i} s^2 m/\omega\}\tfrac{1}{2}[1+\mathrm{erf}\{\sigma_m(s)\}], \tag{31}$$
where the switching variable is

$$\sigma_m(s) = -|s|\cos(2\theta)\bigg/\sqrt{\left(\frac{\pi m}{\omega \sin(2\theta)}\right)}. \tag{32}$$

On the Stokes line across which they switch on, the exponentials in (31) take their smallest values. For s imaginary, that is $s = \mathrm{i}\surd E$, where $g(\mathrm{i}\surd E)$ has its poles (at $E = E_n$), the phases are
$$2\pi s^2 m/\omega = -2\pi E m/\omega. \tag{33}$$
These are precisely the actions of the classical periodic orbits of the harmonic oscillator (m corresponds to repetitions of the primitive orbit). Summing the remainders in (29) for $s = \mathrm{i}\surd E$ gives

$$-\frac{2}{\omega}\sum_{m=1}^{\infty}(-1)^m R_m(\mathrm{i}\surd E) \approx \frac{\pi}{\omega}\left(\mathrm{i}+\tan\left\{\frac{\pi E}{\omega}\right\}\right), \tag{34}$$

which indeed has poles at the eigenvalues (22).

In this example we can see explicitly how the resurgence of the Weyl series for $g(s)$ yields the periodic orbits, whose sum in turn generates the poles for imaginary s. Thus the harmonic oscillator provides the physical interpretation of recent asymptotics of $\Gamma(z)$ (Berry 1991*b*): $\arg z = \pm\frac{1}{2}\pi$ are Stokes lines across which infinitely many small exponentials appear simultaneously. In the context of special functions, infinitely many coincident Stokes lines seems special, but in semiclassical asymptotics it is obvious that all periodic orbits will appear simultaneously whenever (as in the cases considered in this paper) their actions are all proportional to the same function of the asymptotic parameter (in this case s).

An interesting feature of this example is that the Weyl series (24) for $K(t)$ converges for small t, behaviour we do not expect in general. We conjecture that the reason this case is special is that the propagator (whose trace is $K(t)$) is given exactly by its semiclassical approximation (see for example Schulman 1981). On the other hand, the energy Green function (whose trace is $g(s)$) is a combination of parabolic cylinder functions, whose leading-order asymptotics gives the semiclassical approximation, which is not exact. This conjecture will be supported by our next example.

(b) *Free particle on a ring*

The energy levels are

$$E_n = \pi n^2 \quad (n = \dots -2, -1, 0, 1, 2, \dots), \tag{35}$$

so the heat kernel is the theta function

$$K(t) = \sum_{n=-\infty}^{\infty} \exp\{-n^2\pi t\} = \frac{1}{\sqrt{t}}\left(1 + 2\sum_{m=1}^{\infty} \exp\{-m^2\pi/t\}\right), \tag{36}$$

where we have used the Poisson formula in the second equality. The exponentials involving m are the contributions from the periodic orbits, where particles return after m circuits of the ring in time t. The contribution $1/\sqrt{t}$ is the single term of the Weyl series.

The resolvent is defined by (2), without the subtraction (which in this case is unnecessary because the bare sum converges). From (20) (again without the subtraction) we find

$$g(s) \equiv \sum_{n=-\infty}^{\infty} \frac{1}{n^2\pi + s^2} = \frac{\sqrt{\pi}}{s}\coth\{s\sqrt{\pi}\} = \frac{\sqrt{\pi}}{s}\left(1 + 2\sum_{m=1}^{\infty} \exp\{-2sm\sqrt{\pi}\}\right). \tag{37}$$

In the last member, the exponentials involving m are again the contributions from the periodic orbits, where particles return after m circuits of the ring with energy $-s^2$; it is easy to confirm that for $s = \mathrm{i}\sqrt{E}$ the actions are multiples of $2\sqrt{(\pi E)}$. The contribution $\sqrt{\pi}/s$ is the single term of the Weyl series.

In this example the Weyl series for both $K(t)$ and $g(s)$ consist of single terms and so do not diverge. Therefore the periodic-orbit contributions do not arise from resurgence but simply add inertly. Concordant with our conjecture enunciated at the end of §2*a*, both the propagator and energy Green function are given exactly by their semiclassical approximations in this simple case.

The ring is the simplest example of a spectral problem whose Weyl series is finite, so that the periodic orbits are inaccessible from it. Billiards with polygonal boundaries (Van den Berg & Srisatkunarajah 1988) fall into this category: the Weyl series has three terms (this is easy to verify explicitly for the rectangle, with a double application of the Poisson formula employed in (36) above).

(c) *Negative-curvature billiards*

Let the eigenvalues λ_n of the Laplace–Beltrami operator on a compact surface with genus γ and curvature -1 (and hence area $\mathscr{A} = 4\pi(\gamma - 1)$) be written

$$\lambda_n \equiv E_n + \tfrac{1}{4}. \tag{38}$$

For this system the Selberg trace formula (Balazs & Voros 1986) provides an exact connection between spectral functions and the (chaotic) geodesics on the surface. In the case of the resolvent, Selberg's formula can be written (cf. equation (4.2) of Voros 1992)

$$g(s) = 2(\gamma - 1)\int_0^{\infty} \mathrm{d}\rho \frac{\rho}{\rho^2 + s^2}[\tanh(\pi\rho) - 1] + \sum_j A_j(s)\exp\{-l_j s\}, \tag{39}$$

where as in (12) j labels the periodic geodesics (the -1 accompanying the tanh corresponds to the subtraction in (2)). This is another case where the periodic orbits do not arise from resurgence but are superposed inertly onto the Weyl contribution, represented by the integral.

To get the Weyl expansion in the form we need, the simplest procedure is to expand the integrand in powers of $1/s^2$ and the tanh in exponentials. This gives the formal series

$$g_{\text{Weyl}}(s) = -4(\gamma-1) \sum_{m=1}^{\infty} (-1)^m \sum_{k=1}^{\infty} \frac{(-1)^k (2k-1)!}{(2\pi s m)^{2k}}, \tag{40}$$

which is identical in form with the Weyl series (28) for the harmonic oscillator, apart from a crucial difference: powers of s^2 replace powers of s^4. This implies that the Stokes line is $\arg s \equiv \theta = \frac{1}{2}\pi$ (rather than $\frac{1}{4}\pi$). Across this line, which corresponds to real energy $E = -s^2$, small exponentials, originating in the divergence of the Weyl series, switch on according to a slight variant of (4)–(7) (cf. (29)–(32)). Thus

$$g_{\text{Weyl}}(s) = -4(\gamma-1) \sum_{m=1}^{\infty} (-1)^m \left(\sum_{k=1}^{k_m^*-1} \frac{(2k-1)!}{(2\pi i s m)^{2k}} + R_m(s) \right), \tag{41}$$

where
$$k_m^* \approx \operatorname{Int}\{\pi m |s|\} \tag{42}$$
and
$$R_m(s) \approx i\pi \exp\{2\pi i s m\} \tfrac{1}{2}[1 + \operatorname{erf}\{\sigma_m(s)\}], \tag{43}$$
with the Stokes variable

$$\sigma_m(s) = -\cos\theta \sqrt{\left(\frac{\pi m |s|}{\sin\theta}\right)}. \tag{44}$$

The exponentials in (43) are quite different from the periodic orbit contributions in the Selberg formula (39). Cartier & Voros (1988) discovered that they correspond to *complex periodic orbits*, which for this system are the geodesics on a sphere of curvature $+1$, that is repetitions of great circles, with lengths $2\pi i m$ and actions $2\pi i m \sqrt{E}$. For real energy the sum of the remainders in (41) is

$$\sum_{m=1}^{\infty} (-1)^m R_m(i\sqrt{E}) \approx \tfrac{1}{2} i\pi \sum_{m=1}^{\infty} (-1)^m \exp\{-2\pi m \sqrt{E}\}$$
$$= -i \frac{\pi \exp\{-2\pi \sqrt{E}\}}{2(1+\exp\{-2\pi \sqrt{E}\})}. \tag{45}$$

This sum is imaginary for real E, whereas all terms in the Weyl series (41) are real. Therefore the small exponentials dominate the contribution from the integral in (39) to the spectral density (9), giving the exact smoothed spectral density (cf. 12)

$$\bar{d}(E) = (\gamma-1)\left(1 - \frac{2\exp\{-2\pi\sqrt{E}\}}{1+\exp\{-2\pi\sqrt{E}\}}\right) = (\gamma-1)\tanh(\pi\sqrt{E}). \tag{46}$$

(This result can also be obtained directly from the integral in (39), or its expression in terms of a ψ function.) In this case, therefore resurgence of the Weyl series generates small exponential contributions to $\bar{d}(E)$, rather than oscillatory exponentials in $d_{\text{osc}}(E)$.

3. Circle billiard

(*a*) *Weyl coefficients for the circle*

If $\partial\mathscr{B}$ is the unit circle, the exact solutions of (1) are Bessel functions, and the eigenfunctions and eigenvalues are labelled by two integers:

$$\psi_{mn}(r,\theta) = A_{mn} J_m(j_{mn} r) \exp\{im\theta\}, \quad E_{mn} = j_{mn}^2, \tag{47}$$

where $\boldsymbol{r} = \{r, \theta\}$, j_{mn} is the nth zero of the Bessel function J_m, and A_{mn} are normalization constants. To find the resolvent the easiest procedure involves not the definition (2) but the Green function $G(\boldsymbol{r}_0, \boldsymbol{r}, s)$ satisfying

$$-\nabla_{\boldsymbol{r}}^2 G(\boldsymbol{r}_0, \boldsymbol{r}, s) + s^2 G(\boldsymbol{r}_0, \boldsymbol{r}, s) = \delta(\boldsymbol{r} - \boldsymbol{r}_0), \quad G = 0 \text{ for } \boldsymbol{r} \text{ or } \boldsymbol{r}_0 \text{ on } \partial\mathscr{B}. \tag{48}$$

The resolvent is the trace of G minus the free Green function G_0, which satisfies the equation (48) but not the boundary conditions. With

$$G_0(\boldsymbol{r}_0, \boldsymbol{r}, s) = \frac{1}{2\pi} K_0(s|\boldsymbol{r} - \boldsymbol{r}_0|), \tag{49}$$

where K_0 is the modified Bessel function of the second kind (Abramowitz & Stegun 1972), we have

$$g(s) = \iint_{\mathscr{B}} \mathrm{d}^2\boldsymbol{r}_0 \lim_{\boldsymbol{r} \to \boldsymbol{r}_0} [G(\boldsymbol{r}_0, \boldsymbol{r}, s) - G_0(\boldsymbol{r}_0, \boldsymbol{r}, s)]. \tag{50}$$

To find G, and hence $g(s)$, we must add to G_0 a solution of the homogeneous equation corresponding to (48) (that is, without the δ-function), which makes the sum satisfy the boundary conditions. Following Stewartson & Waechter (1971) we accomplish this by expanding K_0 into its angular-momentum components using

$$K_0(s|\boldsymbol{r} - \boldsymbol{r}_0|) = \sum_{m=-\infty}^{\infty} I_m(sr) K_m(sr_0) \exp\{\mathrm{i}m(\theta - \theta_0)\} \quad (r < r_0), \tag{51}$$

where I_m is the modified Bessel function of the first kind, and requiring that the boundary condition be satisfied for each m. This gives

$$\lim_{\boldsymbol{r}_0 \to \boldsymbol{r}} [G(\boldsymbol{r}_0, \boldsymbol{r}, s) - G_0(\boldsymbol{r}_0, \boldsymbol{r}, s)] = -\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \frac{K_m(s)}{I_m(s)} I_m^2(sr). \tag{52}$$

The resolvent then follows from (50) after some straightforward manipulations of Bessel functions:

$$g(s) = -\frac{1}{2} \sum_{m=-\infty}^{\infty} f_m(s) = -\frac{1}{2} \sum_{\mu=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}m \exp\{-2\pi \mathrm{i} m\mu\} f_m(s), \tag{53}$$

where in the second equality we have used the Poisson sum formula, and

$$f_m(s) = (1 + m^2/s^2) I_m(s) K_m(s) - I_m'(s) K_m'(s) - I_m'(s)/s I_m(s). \tag{54}$$

Stewartson & Waechter (1971) obtain the Weyl series by replacing the first sum in (53) by an integral (equivalent to setting $\mu = 0$ in the second sum), and expanding the Bessel functions in their large-order formal asymptotic series. These are

$$\left.\begin{aligned}
I_m(s) &= \frac{1}{\sqrt{2\pi}} \frac{1}{(m^2+s^2)^{\frac{1}{4}}} \exp\{\tfrac{1}{2}F\} \left[1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{m^k}\right], \\
I_m'(s) &= \frac{(m^2+s^2)^{\frac{1}{4}}}{s\sqrt{2\pi}} \exp\{\tfrac{1}{2}F\} \left[1 + \sum_{k=1}^{\infty} \frac{v_k(t)}{m^k}\right], \\
K_m(s) &= \sqrt{(\tfrac{1}{2}\pi)} \frac{1}{(m^2+s^2)^{\frac{1}{4}}} \exp\{-\tfrac{1}{2}F\} \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k u_k(t)}{m^k}\right], \\
K_m'(s) &= -\sqrt{(\tfrac{1}{2}\pi)} \frac{(m^2+s^2)^{\frac{1}{4}}}{s} \exp\{-\tfrac{1}{2}F\} \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k v_k(t)}{m^k}\right],
\end{aligned}\right\} \tag{55}$$

where

$$\left.\begin{aligned} F = F(s,m) &\equiv 2(m^2+s^2)^{\frac{1}{2}} + 2m\log\left\{\frac{s}{m+(m^2+s^2)^{\frac{1}{2}}}\right\}, \\ t = t(s,m) &\equiv \frac{m}{(m^2+s^2)^{\frac{1}{2}}}, \end{aligned}\right\} \tag{56}$$

and $u_k(t)$ and $v_k(t)$ are polynomials satisfying recurrence relations given in §9.3 of Abramowitz & Stegun (1972).

Thus the coefficients in the Weyl series (3) can be obtained as

$$c_r = \int_0^\infty \mathrm{d}x \frac{\surd(1+x^2)}{x^r} B_r\left(\frac{x}{\surd(1+x^2)}\right), \tag{57}$$

where

$$\sum_{r=1}^{\infty} \frac{B_r(t)}{m^r} \equiv \left(1+\sum_{k=1}^{\infty}\frac{v_k(t)}{m^k}\right)\Bigg/\left(1+\sum_{k=1}^{\infty}\frac{u_k(t)}{m^k}\right) - \frac{1}{2}\left(1+\sum_{k=1}^{\infty}\frac{u_k(t)}{m^k}\right)\left(1+\sum_{k=1}^{\infty}\frac{(-1)^k u_k(t)}{m^k}\right)$$
$$-\frac{1}{2}\left(1+\sum_{k=1}^{\infty}\frac{v_k(t)}{m^k}\right)\left(1+\sum_{k=1}^{\infty}\frac{(-1)^k v_k(t)}{m^k}\right). \tag{58}$$

It follows from the recurrence relations for $u_k(t)$ and $v_k(t)$ that all the integrals (57) converge.

In table 1 we show the coefficients c_1 through c_{31}. Now we are in a position to test the main conjecture. For the unit circle billiard, the shortest periodic orbit is a diametral 2-bounce, with length $l = 4$. This should dominate the divergence of the sequence c_r according to (4) with $\beta = \frac{3}{2}$ (see §3*b* for the derivation of this value), and from (14) we should be able to reconstruct $l = 4$. Figure 1 shows how this limit is attained. The best value of the quantity in (14), with the limit approximated by 31 coefficients, is 3.98, in good agreement with the theoretical prediction $l = 4$ (the value is not very sensitive to β).

(*b*) *Resurgence of the periodic orbits for the circle*

Next we show how to obtain the exponential contributions to $g(s)$ explicitly. This will enable us to test how the shortest one (which dominates) switches on across the Stokes line $\arg s \equiv \theta = 0$. The first step is to note that in (55) the formal series in u_k and v_k for the Bessel functions $I_m(s)$ and $I'_m(s)$ generate small exponentials by resurgence of their divergent tails. This happens across the Stokes line where the large exponential prefactors in (55) maximally dominate. In the first quadrant of the s plane this line issues from the singularity $s = \mathrm{i}m$ and satisfies the condition that

$$F(s,m) - F(m\exp\{\tfrac{1}{2}\mathrm{i}\pi\}, m) = 2(m^2+s^2)^{\frac{1}{2}} + 2m\log\left\{\frac{s}{m+(s^2+m^2)^{\frac{1}{2}}}\right\} - \mathrm{i}m\pi \tag{59}$$

is positive real. The line reaches an asymptote at $\operatorname{Im} s = \frac{1}{2}m\pi$. Stokes resurgence (cf. (6)–(8) and Berry 1989) enables the formal series to be interpreted as

$$\left.\begin{aligned} \sum_{k=1}^{\infty}\frac{u_k}{m^k} &\approx \sum_{k=1}^{k^*-1}\frac{u_k}{m^k} + \mathrm{i}S(\sigma)\exp\{-(F(s,m)-\mathrm{i}m\pi)\}, \\ \sum_{k=1}^{\infty}\frac{v_k}{m^k} &\approx \sum_{k=1}^{k^*-1}\frac{v_k}{m^k} - \mathrm{i}S(\sigma)\exp\{-(F(s,m)-\mathrm{i}m\pi)\}, \end{aligned}\right\} \tag{60}$$

Table 1. *Coefficients of c_r of the Weyl series (3) for circle billiard*

r	c_r
1	$-\frac{1}{4}\pi$
2	$\frac{1}{6}$
3	$\frac{\pi}{256}$
4	$\frac{2}{315}$
5	$\frac{111\pi}{65536}$
6	$\frac{272}{45045}$
7	$\frac{5705\pi}{2097152}$
8	$\frac{69824}{4849845}$
9	$\frac{38306807\pi}{4294967296}$
10	$\frac{3644416}{59053995}$
11	$\frac{13256868717\pi}{274877906944}$
12	$\frac{33630171136}{82047473235}$
13	$\frac{27119184063411\pi}{70368744177664}$
14	$\frac{215714709864448}{55655536011075}$
15	$\frac{38317254577890717\pi}{9007199254740992}$
16	$\frac{24428054038052864}{494726268844665}$
17	$\frac{114260506848990741 8295\pi}{18446744073709551616}$
18	$\frac{7891118388906803226 2144}{96845140757687397075}$
19	$\frac{1357624703013056272008385\pi}{1180591620717411303424}$
20	$\frac{2931021236527588546969 6}{1733465876446278975}$

Table 1. (*cont.*)

r	c_r
21	$\dfrac{801457882948373302820667695 9\pi}{302231454903657293676544}$
22	$\dfrac{5833386006716072056569462784}{1353327477223926588 2685}$
23	$\dfrac{17984164513689437386989244901 07\pi}{2417851639229258349412352}$
24	$\dfrac{843455615434798985349322244096}{6367678367521438808 9775}$
25	$\dfrac{494067530821680026772248078670772559\pi}{19807040628566084398385987584}$
26	$\dfrac{3442851452821809442295969308555670978 56}{7127694104868579226358731607 25}$
27	$\dfrac{1249679514356466768319146981238602294925\pi}{1267650600228229401496703205376}$
28	$\dfrac{24122001593385764216774956160927147522785 28}{116949011736035996075562880909725}$
29	$\dfrac{1471868230415755282498855321504729955530359 3\pi}{32451855365842672678315602057625 6}$
30	$\dfrac{744229313262606956349747349482992358989496 32}{72983217850308178002043000868475}$
31	$\dfrac{1996843071532133130652878687325069383099054285 63\pi}{830767497365572420564879412675215 36}$

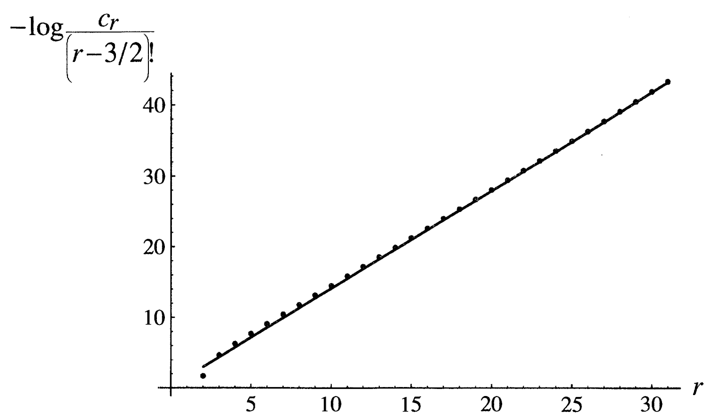

Figure 1. Graph of $-\log\{c_r/(r-\frac{3}{2})!\}$ against r for the unit circle billiard, using the Weyl coefficients c_r from table 1. The limiting slope gives $l = 3.98$. The straight line shows the predicted slope $l = 4$.

where k^* is the least term of the series and $S(\sigma)$ switches from 0 to 1 across the Stokes line. Exploratory computations, which we do not give here, confirms that (60) does indeed describe correctly the birth of the small exponentials, which must exist in order for the Bessel functions I and I' to oscillate along the imaginary s axis in accordance with $I_m(\mathrm{i}s) = \mathrm{i}^m J_m(s)$. The series for the Bessel functions K and K' do not exhibit resurgence, since their exponential prefactors are subdominant on the Stokes line.

Above the Stokes lines, where all the multipliers S in (60) are unity, the exponential contribution to the quantity $f_m(s)$ defined by (54) can be found by substitution. To lowest order the only contributions come from the term $-I'/sI$ in (54), and we obtain

$$f_m^{\exp}(s) \approx \frac{2\mathrm{i}\sqrt{(m^2+s^2)}}{s^2(1+\mathrm{i}\exp\{-(F-\mathrm{i}m\pi)\})}\exp\{-(F-\mathrm{i}m\pi)\}$$
$$= \frac{-2\sqrt{(m^2+s^2)}}{s^2}\sum_{p=1}^{\infty}(-\mathrm{i})^p(-1)^{mp}\exp\{-pF\}. \tag{61}$$

Thus from (53) the exponential contributions to $g(s)$ arise from

$$g^{\exp}(s) = \sum_{p=1}^{\infty}(-\mathrm{i})^p \sum_{\mu=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}m\frac{\sqrt{(m^2+s^2)}}{s^2}\exp\{\pi\mathrm{i}m(p-2\mu)-pF(s,m)\}. \tag{62}$$

To get these contributions explicitly, the integrals are evaluated by the method of steepest descent. The saddle condition giving the contributing values of m is easiest to interpret for $s = \mathrm{i}\sqrt{E}$, and can be written with the aid of (56) as

$$\chi \equiv \arccos\{m/\sqrt{E}\} = \pi\mu/p. \tag{63}$$

Semiclassically, the quantity $m/\sqrt{E}$ (angular momentum, measured in units of the wavenumber $\sqrt{E}$) corresponds to impact parameter, that is the distance from the centre of the closest approach of a straight trajectory (see Berry & Mount 1972; Nussenzveig 1992). Therefore χ is the angle between a chord segment of an orbit and the tangent to the circle. What (63) asserts is that this angle is a rational multiple of π, which is precisely the condition for a periodic orbit. The orbit labelled μ, p closes after p chords and μ circuits of the origin. If μ and p have a common factor n, the orbit consists of n repetitions of the primitive orbit corresponding to $n = 1$.

It is not difficult to obtain all the contributions explicitly. However, we are interested only in the shortest orbit (diametral 2-bounce) which has $p = 2$ and $\mu = 1$ and which (cf. (63)) has $m = 0$. For this case it is simplest to evaluate (62) directly, expanding the integrand about $m = 0$. For $s = \mathrm{i}\sqrt{E}$ this gives

$$g^{\exp}(\mathrm{i}\sqrt{E}) = -\frac{\exp\{-4\mathrm{i}\sqrt{E}\}}{s}\int_{-\infty}^{\infty}\mathrm{d}m\exp\{-2\mathrm{i}m^2/\sqrt{E}\}$$
$$= \frac{\sqrt{2\pi}}{E^{\frac{1}{4}}}\exp\{-4\mathrm{i}\sqrt{E}+\tfrac{1}{4}\mathrm{i}\pi\}. \tag{64}$$

Reinstating s, and inserting the canonical Stokes smoothing (7) and (8), gives the switching-on of the dominant small exponential across the Stokes line $\arg s \equiv \theta = 0$ as

$$g^{\exp}(s) = \mathrm{i}\sqrt{(\pi/2s)}\exp\{-4s\}\operatorname{erf}\{\sigma(s)\}, \tag{65}$$

where (cf. (8))

$$\sigma(s) = \sqrt{(2|s|)}\sin\theta/\sqrt{\cos\theta}. \tag{66}$$

From (7) we can now identify $\beta = \frac{3}{2}$ as previously asserted, and also the prefactor in (4) as $\alpha = (8\sqrt{2\pi})^{-1}$, in good agreement with the intercept in figure 1.

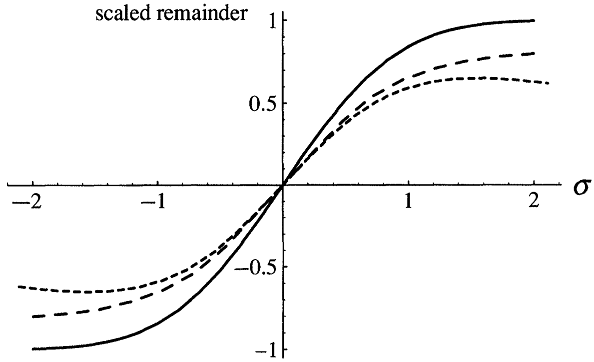

Figure 2. Graph of scaled remainder of truncated Weyl series for unit circle billiard (l.h.s of (68)) against the Stokes line crossing variable $\sigma(|s|\exp(i\theta)$ (equation (66)), for $|s| = 2$ (dotted line) and $|s| = 2\sqrt{2}$ (dashed line). The full line shows the theoretical prediction $\operatorname{erf}\sigma$.

We can investigate numerically whether this smoothing actually occurs by calculating the quantity on the l.h.s. of (15), provided we can compute $g(s)$ from its definition (2) and the eigenvalues (47). It is easier to calculate the derivative

$$\frac{\mathrm{d}}{\mathrm{d}s}g(s) = -2s\sum_{m=0}^{\infty}\epsilon_m\sum_{n=1}^{\infty}\frac{1}{(s^2+j_{mn}^2)^2}+\frac{1}{2s}, \quad \epsilon_0 = 1, \quad \epsilon_{m>0} = 2 \tag{67}$$

(the factors ϵ_m accommodate the double degeneracy of the eigenstates with $m \neq 0$). Instead of (15) we therefore test the prediction

$$\lim_{\mathrm{Re}\, s\to\infty}\left(\left[-2s\sum_{m=0}^{\infty}\epsilon_m\sum_{n=1}^{\infty}\frac{1}{(s^2+j_{mn}^2)^2}+\frac{1}{2s}+\sum_{r=1}^{r^*}\frac{rc_r}{s^{r+1}}\right]\exp\{4s\}\frac{\mathrm{i}}{2}\sqrt{\left(\frac{s}{2\pi}\right)}\right) = \operatorname{erf}\{\sigma(s)\}. \tag{68}$$

Computing the sum over Bessel zeros is not trivial, because it is necessary to achieve accuracy $\exp\{-s\}$ in order for the subtraction of the truncated Weyl series to reveal the error function. We found the roots $j_{mn} < s^*$ numerically, using as starting values the zeros of the lowest-order Debye approximations for J_m (Abramowitz & Stegun 1972), and replaced the tail $j_{mn} > s^*$ by an integral, using the first two terms of the Weyl series (10) for the spectral density; our choice of s^* corresponded to including the first 8000 Bessel zeros in the eigenvalue sum. The value of s is limited by the exponential accuracy required. We chose $|s| = 2$ and $|s| = 2\sqrt{2}$, which from (5) meant that the Weyl series had to be truncated at $r^* = 8$ and 11 respectively. On the imaginary s axis these values correspond to the energies $E = -s^2$ of the first one or two states, and so are hardly in the asymptotic regime of large s. Therefore, we do not expect to be close to the limit (68). Nevertheless, as figure 2 shows, the error function does give a reasonably accurate description of the smooth birth of the small exponential representing the shortest orbit – especially since each curve is the difference between two exponentially large quantities ($\mathrm{O}(\exp 8)$ and $\mathrm{O}(\exp 11)$ in the two cases shown).

4. Weyl coefficients for general smooth billiard

Several methods have been used to calculate the coefficients c_r in the Weyl series (3). Balian & Bloch (1970) convert the equation (48) satisfied by the Green function in $\mathscr{B}$ into an integral equation on $\partial\mathscr{B}$. Smith (1981) uses techniques based on pseudodifferential operators to derive a set of ordinary differential equations whose

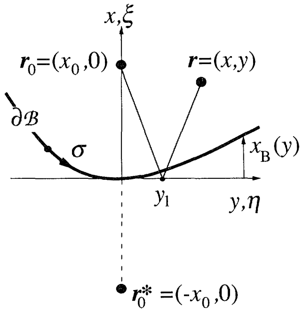

Figure 3. Coordinates (x, y) and scaled coordinates (ξ, η) near billiard boundary $\partial\mathscr{B}$.

solutions generate the c_r. We develop the technique devised by Stewartson & Waechter (1971) for the systematic calculation of the Green function $G(\boldsymbol{r}_0, \boldsymbol{r}, s)$ for large s and thence obtain explicit recursion relations for the c_r.

Set up rectangular coordinates (x, y) whose origin is the foot of the shortest perpendicular from the 'source' point $\boldsymbol{r}_0$ to $\partial\mathscr{B}$ (figure 3), with the x axis along this perpendicular and directed into $\mathscr{B}$, and the y axis along the tangent to $\partial\mathscr{B}$. Thus $\boldsymbol{r}_0 = (x_0, 0)$. We write G as the sum of the free Green function G_0, given by (49) which satisfies the equation (48), and a correction, satisfying the homogeneous counterpart of (48), which makes G vanish on $\partial\mathscr{B}$. The correction is itself the sum of two terms. First is an image wave, which is minus the free Green function at the image $\boldsymbol{r}_0^* = (-x_0, 0)$ of the source point. Second, and characteristic of Stewartson and Waechter's method, is the integral of a double layer potential $f(y_1, x_0, s)$ along the tangent to $\partial\mathscr{B}$ (rather than $\partial\mathscr{B}$ itself). Thus

$$G(\boldsymbol{r}_0, \boldsymbol{r}) = \frac{1}{2\pi}(K_0\{s|\boldsymbol{r}-\boldsymbol{r}_0|\} - K_0\{s|\boldsymbol{r}-\boldsymbol{r}_0^*|\}) + \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \mathrm{d}y_1 f(y_1, x_0, s)\,\partial_x K_0\{s\sqrt{[x^2+(y-y_1)^2]}\}. \tag{69}$$

The derivative in the last term ensures that the integral equation is non-singular.

The unknown function f is determined by requiring G to vanish on $\partial\mathscr{B}$. The boundary is specified by its local cartesian equation

$$x = x_B(y) = \sum_{n=2}^{\infty} \alpha_n(\sigma)\, y^n, \tag{70}$$

in which the coefficients α_n depend on the position of the origin of the rectangular coordinates, specified by arc length σ round $\partial\mathscr{B}$, measured from an origin independent of $\boldsymbol{r}_0$ (figure 3) (σ should not be confused with the Stokes variable (8), which will not occur again). The α_n can be expressed in terms of the curvature $\kappa(\sigma)$ of $\partial\mathscr{B}$, and its derivatives; table 2 shows the first few of these coefficients. The equation to be solved is therefore

$$-(K_0\{s\sqrt{[(x_0-x_B(y))^2+y^2]}\} - K_0\{s\sqrt{[(x_0+x_B(y))^2+y^2]}\}) = +\frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}y_1 f(y_1, x_0, s)\,\partial_{x_B} K_0\{s\sqrt{[x_B^2+(y-y_1)^2]}\}. \tag{71}$$

To solve this for large s, it is convenient to change to scaled coordinates

$$\xi \equiv sx, \quad \eta \equiv sy, \quad \eta_1 \equiv sy_1, \tag{72}$$

Table 2. *Coefficients α_n specifying the boundary $\partial\mathscr{B}$ in terms of the curvature $\kappa(\sigma)$ and its derivatives*

n	α_n
2	$\dfrac{\kappa}{2}$
3	$\dfrac{\kappa'}{6}$
4	$\dfrac{3\kappa^3+\kappa''}{24}$
5	$\dfrac{19\kappa^2\kappa'+\kappa'''}{120}$
6	$\dfrac{45\kappa^5+48\kappa\kappa'^2+34\kappa^2\kappa''+\kappa^{(iv)}}{720}$
7	$\dfrac{729\kappa^4\kappa'+48\kappa'^3+199\kappa\kappa'\kappa''+55\kappa^2\kappa'''+\kappa^{(v)}}{5040}$
8	$\dfrac{1575\kappa^7+5324\kappa^3\kappa'^2+1891\kappa^4\kappa''+343\kappa'^2\kappa''+234\kappa\kappa''^2+365\kappa\kappa'\kappa'''+83\kappa^2\kappa^{(iv)}+\kappa^{(vi)}}{40320}$

scale the unknown function f and expand it in powers of $1/s$:

$$f(y_1, x_0, s) \equiv \phi(\eta_1, \xi_0, s) = \sum_{n=1}^{\infty} \frac{\phi_n(\eta_1, \xi_0)}{s^n}. \tag{73}$$

We shall also need the expansions of powers of the scaled boundary ξ_B:

$$\xi_B^m(\eta, s) = \sum_{k=2m}^{\infty} \frac{\alpha_{mk}(\sigma)}{s^{k-m}} \eta^k \quad (\alpha_{0k} = \delta_{k0}, \alpha_{1k} = \alpha_k), \tag{74}$$

where α_k are the coefficients in (70). Subsequent algebra will be simplified by taking double integral transforms of the unknowns ϕ_n: Fourier along the tangent η and Laplace along the inward normal ξ. Thus we define

$$\bar{\phi}_n(q, w) \equiv \int_{-\infty}^{\infty} \mathrm{d}\eta \exp\{-\mathrm{i}\eta q\} \int_0^{\infty} \mathrm{d}\xi \exp\{-\xi w\} \phi_n(\eta, \xi). \tag{75}$$

Once the $\bar{\phi}_n$ are found, the expansion (3) of the resolvent $g(s)$ can be found from the trace (50) with (69) and the scalings (73). The integration variables transform as

$$\iint_{\mathscr{B}} \mathrm{d}^2\boldsymbol{r}_0 \ldots = \frac{1}{s} \oint \mathrm{d}\sigma \int_0^{\infty} \mathrm{d}\xi_0 \left(1 - \frac{\kappa(\sigma)\,\xi_0}{s}\right) \ldots, \tag{76}$$

where the loop integral is round $\partial\mathscr{B}$:

$$\oint \mathrm{d}\sigma \equiv \int_0^{\mathscr{L}} \mathrm{d}\sigma. \tag{77}$$

(Choosing the upper limit in (76) as $\xi_0 = \infty$, rather than the local diameter of $\mathscr{B}$, has no effect on the Weyl expansion.) Thus we obtain

$$g(s) = \oint \mathrm{d}\sigma \left[-\frac{1}{2\pi s} \int_0^{\infty} \mathrm{d}\xi_0 K_0\{2\xi_0\} \left(1 - \frac{\kappa(\sigma)\,\xi_0}{s}\right) + T(\sigma, s) \right], \tag{78}$$

where

$$T(\sigma,s)=\frac{1}{2\pi^2 s}\int_0^\infty \mathrm{d}\xi_0\left(1-\frac{\kappa(\sigma)\,\xi_0}{s}\right)\int_{-\infty}^{\infty}\mathrm{d}\eta\,\phi(\eta,\xi_0,s)\,\partial_{\xi_0}K_0\{\surd(\xi_0^2+\eta^2)\}. \tag{79}$$

Substituting the series (73) and incorporating the transform (75), we can represent (78) as the desired expansion in $1/s$. A useful integral is

$$\int_{-\infty}^{\infty}\mathrm{d}\eta\exp\{-\mathrm{i}\eta q\}K_0\{\surd(\xi_0^2+\eta^2)\}=\frac{\pi}{\surd(1+q^2)}\exp\{-\xi_0\surd(1+q^2)\}. \tag{80}$$

This can be inverted to replace K_0 in (79), and leads to

$$g(s)=-\frac{\mathscr{L}}{8s}+\frac{1}{4s^2}+\sum_{r=2}^{\infty}\frac{1}{s^r}\oint\mathrm{d}\sigma\,T_{r-1}(\sigma), \tag{81}$$

where $$T_n(\sigma)=-\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}q\{\bar\phi_n(q,w)+\kappa(\sigma)\,\partial_w\bar\phi_{n-1}(q,w)\}_{(w=\surd[1+q^2])}. \tag{82}$$

The coefficients in the Weyl series (3) are therefore given by the boundary integrals

$$\left.\begin{aligned} c_r&=\oint\mathrm{d}\sigma\,d_r(\sigma) && (a),\\ d_1&=-\frac{1}{8},\quad d_2=\frac{1}{8\pi}+T_1,\quad d_r=T_{r-1}\,(r\geqslant 3) && (b).\end{aligned}\right\} \tag{83}$$

To determine the $\bar\phi_n$ we must solve the boundary-condition equation (71). After scaling, expanding the Bessel functions in powers of ξ_B and then expanding these powers in $1/s$ according to (74), we can write the l.h.s. of (71) as

$$\begin{aligned}&-(K_0\{\surd[(\xi_0-\xi_B(\eta,s))^2+\eta^2]\}-K_0\{\surd[(\xi_0+\xi_B(\eta,s))^2+\eta^2]\})\\ &\qquad=2\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\sum_{k=4m+2}^{\infty}\frac{\alpha_{2m+1,k}}{s^{k-2m-1}}\eta^k\,\partial_{\xi_0}^{2m+1}K_0(\surd[\xi_0^2+\eta^2]).\end{aligned} \tag{84}$$

Taking the Fourier and Laplace transforms (cf. (75)), using (80) and changing indices in the summations, we obtain

$$\begin{aligned}&\int_0^\infty\mathrm{d}\xi_0\exp\{-\xi_0 w\}\int_{-\infty}^{\infty}\mathrm{d}\eta\exp\{-\mathrm{i}\eta q\}\{\text{l.h.s. of (71)}\}\\ &\qquad=-2\pi\sum_{n=1}^{\infty}\frac{1}{s^n}\sum_{m=0}^{\mathrm{Int}[(n-1)/2]}\frac{\alpha_{2m+1,2m+n+1}}{(2m+1)!}(\mathrm{i}\partial_q)^{2m+n+1}\frac{(1+q^2)^m}{w+\surd[1+q^2]}.\end{aligned} \tag{85}$$

For the r.h.s. of (71) we first use

$$\begin{aligned}&\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\eta\exp\{-\mathrm{i}\eta q\}\,\partial_{\xi_B}K_0\{\surd[\xi_B^2(\eta,s)+(\eta-\eta_1)^2]\}\\ &\quad=\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\eta\exp\{-\mathrm{i}\eta q\}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=2m}^{\infty}\frac{\alpha_{mk}}{s^{k-m}}\eta^k[\partial_\epsilon^{m+1}K_0\{\surd[\epsilon^2+(\eta-\eta_1)^2]\}]_{\epsilon=0}\\ &\quad=-\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{k=2m}^{\infty}\frac{\alpha_{mk}}{s^{k-m}}(\mathrm{i}\partial_q)^k\,[\exp\{-\mathrm{i}\eta_1 q\}\,(q^2+1)^{m/2}].\end{aligned} \tag{86}$$

Table 3. *Coefficients $d_r(\sigma)$, whose integrals round the boundary give the Weyl coefficients c_r for a general smooth billiard*

r	$d_r(\sigma)$
2	$\frac{\kappa}{12\pi}$
3	$\frac{\kappa^2}{512}$
4	$\frac{\kappa^3}{315\pi}$
5	$\frac{111\kappa^4}{131072}-\frac{3\kappa'^2}{16384}$
6	$\frac{136\kappa^5}{45045\pi}-\frac{8\kappa\kappa'^2}{3465\pi}$
7	$\frac{5705\kappa^6}{4194304}-\frac{2455\kappa^2\kappa'^2}{1048576}+\frac{25\kappa''^2}{1048576}$
8	$\frac{34912\kappa^7}{4849845\pi}-\frac{384\kappa^3\kappa'^2}{17017\pi}+\frac{8\kappa\kappa''^2}{15015\pi}$
9	$\frac{38306807\kappa^8}{8589934592}-\frac{12054609\kappa^4\kappa'^2}{536870912}-\frac{46081\kappa'^4}{201326592}+\frac{226653\kappa^2\kappa''^2}{268435456}-\frac{245\kappa'''^2}{67108864}$
10	$\frac{1822208\kappa^9}{59053995\pi}-\frac{76813312\kappa^5\kappa'^2}{334639305\pi}-\frac{83456\kappa\kappa'^4}{8729721\pi}+\frac{18944\kappa^3\kappa''^2}{1616615\pi}+\frac{128\kappa''^3}{969969\pi}-\frac{256\kappa\kappa'''^2}{2078505\pi}$
11	$\frac{13256868717\kappa^{10}}{549755813888}-\frac{8579464911\kappa^6\kappa'^2}{34359738368}-\frac{111438783\kappa^2\kappa'^4}{4294967296}+\frac{544811643\kappa^4\kappa''^2}{34359738368}$
	$+\frac{1295991\kappa'^2\kappa''^2}{2147483648}+\frac{2527335\kappa\kappa''^3}{4294967296}-\frac{1181205\kappa^2\kappa'''^2}{4294967296}+\frac{1323{\kappa^{(iv)}}^2}{2147483648}$
12	$\frac{16815085568\kappa^{11}}{82047473235\pi}-\frac{82248300544\kappa^7\kappa'^2}{29113619535\pi}-\frac{1740560384\kappa^3\kappa'^4}{3011753745\pi}+\frac{212171776\kappa^5\kappa''^2}{1003917915\pi}$
	$+\frac{1028224\kappa\kappa'^2\kappa''^2}{30421755\pi}+\frac{5486464\kappa^2\kappa''^3}{334639305\pi}-\frac{342400\kappa^3\kappa'''^2}{66927861\pi}-\frac{50348413\kappa''\kappa'''^2}{102965940\pi}$
	$+\frac{19828793\kappa\kappa''\kappa'''^2}{40562340\pi}+\frac{640\kappa{\kappa^{(iv)}}^2}{22309287\pi}$
13	$\frac{27119184063411\kappa^{12}}{140737488355328}-\frac{60094134185481\kappa^8\kappa'^2}{17592186044416}-\frac{3936714637823\kappa^4\kappa'^4}{3298534883328}-\frac{4436857161\kappa'^6}{687194767360}$
	$+\frac{1276582035597\kappa^6\kappa''^2}{4398046511104}+\frac{32558130605\kappa^2\kappa'^2\kappa''^2}{274877906944}+\frac{20959388131\kappa^3\kappa''^3}{549755813888}+\frac{868659385\kappa''^4}{3298534883328}$
	$-\frac{9824224223\kappa^4\kappa'''^2}{1099511627776}-\frac{31924959\kappa'^2\kappa'''^2}{137438953472}-\frac{54127535\kappa\kappa''\kappa'''^2}{68719476736}$
	$+\frac{23266133\kappa^2{\kappa^{(iv)}}^2}{274877906944}-\frac{7623{\kappa^{(v)}}^2}{68719476736}$

Then we introduce the expansion (73) for the unknown function to obtain, after a little reduction,

$$\int_{-\infty}^{\infty} \mathrm{d}\eta \exp\{-\mathrm{i}\eta q\} \int_0^{\infty} \mathrm{d}\xi_0 \exp\{-\xi_0 w\} \{\text{r.h.s. of (71)}\}$$

$$= -\sum_{n=1}^{\infty} \frac{1}{s^n} \sum_{p=0}^{n-1} \sum_{m=0}^{p} \frac{(-1)^m \alpha_{m,m+p}}{m!} (\mathrm{i}\partial_q)^{p+m} [\bar{\phi}_{n-p}(q,w)(1+q^2)^{m/2}] \quad (87)$$

(empty sums contribute zero). Equating coefficients of $1/s^n$ in (85) and (87), and separating the contributions from $m = 0$, we finally find the recurrence relation satisfied by the $\bar{\phi}_n$:

$$\bar{\phi}_n(q,w) = 2\pi \sum_{m=0}^{\mathrm{Int}[(n-1)/2]} \frac{\alpha_{2m+1,2m+n+1}}{(2m+1)!} (\mathrm{i}\partial_q)^{2m+n+1} \frac{(1+q^2)^m}{w+\surd[1+q^2]}$$

$$- \sum_{p=1}^{n-1} \sum_{m=1}^{p} \frac{(-1)^m \alpha_{m,m+p}}{m!} (\mathrm{i}\partial_q)^{p+m} [\bar{\phi}_{n-p}(q,w)(1+q^2)^{m/2}]. \quad (88)$$

Having determined the $\bar{\phi}_n$ from this relation, we can obtain the T_r in equation (82) (the integrations are all elementary). These in turn give the quantities d_r (equation (83*b*)) whose boundary integrals generate the Weyl coefficients c_r from (83*a*). The d_r are sums of monomials made from products of powers of the curvature $\kappa(\sigma)$ and its derivatives (which enter via the coefficients α_{mk} defined in (70) and (74)). As first obtained, there is considerable redundancy in the formulae for d_r, because some monomials vanish when integrated round $\partial\mathscr{B}$ (e.g. κ' and $\kappa''\kappa'^2$), and others can be integrated by parts to give monomials involving lower derivatives (e.g. $\kappa''\kappa \to -\kappa'^2$). By inspection we can determine the irreducible representation of each d_r, consisting of monomials whose boundary integral is not identically zero and which cannot be expressed in terms of lower derivatives of κ.

Table 3 shows d_r for $1 \leqslant r \leqslant 13$. From (83) we find the familiar results $c_1 = -\frac{1}{8}\mathscr{L}$, $c_2 = \frac{1}{6}$. The coefficients for $r \leqslant 6$ confirm results of previous workers for the heat kernel coefficients a_r, related to the c_r by (21), and in particular the correction by Smith (1981) of an error in a_5 as given by Stewartson & Waechter (1971), and also reveal a slight error in Smith's value of a_6 (101 should be 1001).

5. Examples of high-order Weyl coefficients

(*a*) *Geometry of 2-bounce periodic orbits, and the accessibility conjecture*

The availability of a large number of Weyl coefficients (table 3) makes it possible to test the conjecture (4) in generic cases where, in contrast to the circle, the periodic geodesics are isolated. According to the conjecture, c_r for large r will be dominated by one or more short periodic orbits. We expect that these will usually consist of a double traversal (back and forth) of a chord of $\mathscr{B}$, whose ends are perpendicular to $\partial\mathscr{B}$. Occasionally the shortest orbit will have three or more bounces, but we do not consider this case further. Now we give the general theory of 2-bounce orbits.

Chords are specified by the positions σ_1, σ_2 (arc lengths) of their ends on $\partial\mathscr{B}$. Since $\sigma + \mathscr{L}$ is the same point as σ, and (σ_1, σ_2) represents the same chord as (σ_2, σ_1), chord space is a Möbius strip; this is a 2-torus with identification of opposite sides of the diagonal $\sigma_1 = \sigma_2$. Chord length $\rho(\sigma_1, \sigma_2)$ is therefore a non-negative function on the Möbius strip, on whose edge (the diagonal) $\rho = 0$. The 2-bounce periodic orbits, with

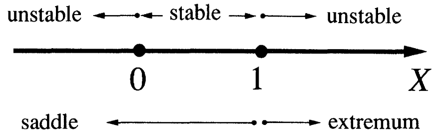

Figure 4. Types of 2-bounce orbit, in terms of X given by (92).

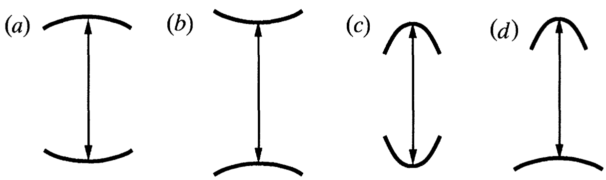

Figure 5. Examples of 2-bounce orbits. (*a*) saddle, stable; (*b*) minimum, unstable; (*c*) maximum, unstable; (*d*) saddle, unstable.

length 2ρ, encounter $\partial\mathscr{B}$ perpendicularly and so correspond to stationary points of ρ, whose nature (saddle, minimum or maximum) is connected with the stability of the orbit, as will now be explained.

Whether a stationary chord is a saddle or an extremum depends on the Gaussian curvature of $\rho(\sigma_1, \sigma_2)$, that is

$$\left.\begin{aligned} \rho_{11}\rho_{22}-\rho_{12}^2 &< 0 \quad \text{(saddle)},\\ &> 0 \quad \text{(maximum or minimum)},\end{aligned}\right\} \tag{89}$$

where subscripts denote derivatives with respect to σ_1 or σ_2. Stability depends on the monodromy matrix M, that is the 2×2 symplectic matrix of derivatives of phase-space variables $(\sigma, \cos\phi)$ at the end of the orbit with respect to those at the beginning, where ϕ is the angle between the chord and the tangent to $\partial\mathscr{B}$ (see Berry (1981) for an introduction to this aspect of billiard geometry). The stability criterion is

$$\left.\begin{aligned} |\mathrm{Tr}\, M| &< 2 \quad \text{(stable)},\\ &> 2 \quad \text{(unstable)}.\end{aligned}\right\} \tag{90}$$

For a 2-bounce orbit, calculation gives

$$|\mathrm{Tr}\, M| = |4\rho_{11}\rho_{22}/\rho_{12}^2 - 2|. \tag{91}$$

Thus the properties of the orbit depend on the quantity

$$X \equiv \rho_{11}\rho_{22}/\rho_{12}^2. \tag{92}$$

Figure 4 classifies the possibilities in terms of X, and figure 5 shows an example from each class. Elementary geometry enables X to be expressed in terms of the chord length and the curvatures at its ends:

$$X = (1-\kappa_1\rho)(1-\kappa_2\rho). \tag{93}$$

Now we are in a position to formulate the accessibility conjecture. Which short orbit dominates the divergence of the Weyl series? We are guided by the analogous case of one-dimensional saddle-point integration (Berry & Howls 1991), where the divergence of the Weyl series for the integral through one saddle (a local expansion about that saddle) is dominated by the nearest adjacent saddle. Adjacent saddles are those reached by certain lines (constant-phase contours of the exponent function)

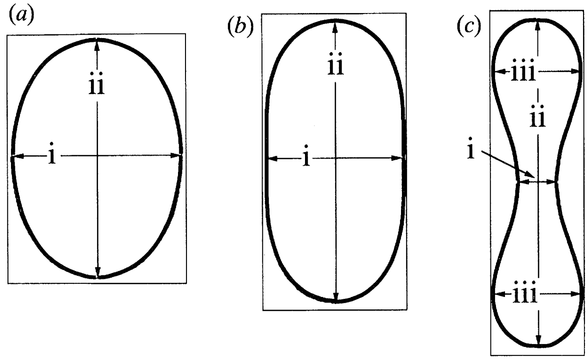

Figure 6. 2-bulge billiards (equation (96) with $n = 2$) for (a) $a = 0.5$, (b) $a = 1$, (c) $a = 2$. The 2-bounce periodic orbits are shown; (i) is a stable saddle in (a), marginal in (b), and an unstable minimum in (c); (ii) is an unstable maximum; (iii) is a stable saddle.

issuing from the original saddle. Not knowing the precise analogue of this construction for the infinite-dimensional complex Feynman integral representing $g(s)$, we guess the following equivalent. Start at the Möbius diagonal corresponding to the zero-length chords, regarded as representing the local (zero-length) orbits generating the Weyl series, and climb the ρ-contours until we first reach a contour through a stationary point. This will be a saddle rather than an extremum, and as stated previously it corresponds to a 2-bounce periodic geodesic. This, we conjecture, is the shortest accessible orbit. Sometimes, as we shall see at the end of §5*b*, this construction should be carried out in complexified chord space.

(*b*) *N-bulge billiards*

As a family of curves for which the curvature $\kappa(\sigma)$ is known and the integrals round $\partial\mathscr{B}$ of the coefficients $d_r(\sigma)$ in table 3 can be easily evaluated, we choose

$$\kappa(\sigma, a, n) \equiv 1 + a\cos n\sigma \, (n \geqslant 2). \tag{94}$$

Since $\kappa = \mathrm{d}\psi/\mathrm{d}\sigma$, where ψ is the angle between $\partial\mathscr{B}$ and a global x axis (not to be confused with the local cartesian axes used in §4), we have

$$\psi \equiv \sigma + (a/n)\sin n\sigma, \tag{95}$$

and the equation of the curve is

$$x(\sigma) + \mathrm{i}y(\sigma) = \int_0^\sigma \mathrm{d}\sigma \exp\left\{\mathrm{i}\left(\sigma + \frac{a}{n}\sin n\sigma\right)\right\}. \tag{96}$$

For $n \geqslant 2$ this closes when $\sigma = 2\pi$, which is therefore the length $\mathscr{L}$ of this family of curves. (In general the condition for closure is that $\kappa(\sigma)$ contains no Fourier component $\cos(2\pi\sigma/\mathscr{L})$ or $\sin(2\pi\sigma/\mathscr{L})$.)

Figures 6 and 7 show the curves for $n = 2$ and $n = 3$ respectively, for several values of the parameter a, and also the 2-bounce periodic orbits. For $a = 0$ the curves are circles. As a increases, they develop n bulges. For $a > 1$ the curves are no longer everywhere convex, and for sufficiently large a they cross themselves, that is $\partial\mathscr{B}$ is no longer simple. It is easy to substitute the curvature functions (94) into the $d_r(\sigma)$ of table 3 and calculate the Weyl coefficients c_r by integration according to (83). Tables 4 and 5 show these coefficients as functions of a for $n = 2$ and $n = 3$. For each

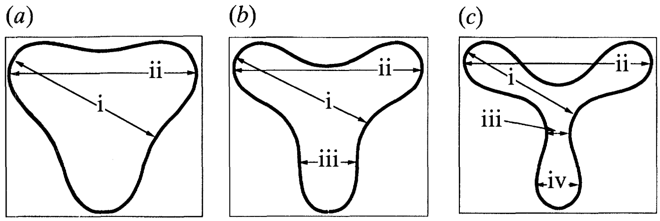

Figure 7. 3-bulge billiards (equation (96) with $n = 3$) for (*a*) $a = 2$, (*b*) $a = 2.9717$, (*c*) $a = 4$. One 2-bounce periodic orbit is shown in each symmetry class; (i) is an unstable saddle; (ii) is an unstable maximum (unphysical in (*c*) because it crosses $\partial\mathscr{B}$); (iii) is marginal in (*b*), and an unstable minimum in (*c*); (iv) is a stable saddle.

a we evaluated the coefficients and estimated the length of the shortest accessible periodic orbit from (14). The results were not very sensitive to the value of β, but we used the correct value as obtained from the s-dependence of the prefactor in (7) based on the fact that when $a > 0$ the orbits are isolated. This gave $\beta = 2$. (The quickest way to obtain this result is to note that for isolated orbits the prefactor in the oscillatory contributions to the counting function $N(E)$ is independent of E, and then exploit the connection (9) between the spectral density $\mathrm{d}N/\mathrm{d}E$ and the resolvent $g(E)$.)

The theoretical prediction of orbit length was compared with the actual lengths of the periodic orbits in $\mathscr{B}$. These were computed by solving (95) for pairs of σ-values with ψ differing by π, and then using (96) to calculate

$$l = 2\rho(\sigma_1, \sigma_2) = 2\surd[(x(\sigma_1)-x(\sigma_2))^2+(y(\sigma_1)-y(\sigma_2))^2]. \tag{97}$$

Figure 8 shows the predicted orbit length as a function of a for the 2-bulge billiard, and for comparison, the lengths of the 2-bounce orbits shown in figure 6. For $a < 1$, the predicted length agrees well with that of the shortest orbit (i) (the agreement for a close to zero is improved by using the value $\beta = \frac{3}{2}$ appropriate to the circle). For $a > 1$ the agreement is not with (i) but with the two symmetrically placed next shortest orbits (iii). These have been born from a bifurcation at $a = 1$, at which the shortest orbit splits into three. The reason for this behaviour can be seen from the Möbius contour plot in figure 9. For $a < 1$, the shortest orbit (i) is a saddle; at $a = 1$ it splits into a minimum (the shortest orbit, still called (i)) and two saddles (iii). According to our conjecture, it is the saddles that remain accessible after the bifurcation.

Figure 10 shows the predicted orbit length for the 3-bulge billiard over the range $2 \leqslant a \leqslant 4$. The predicted length never agrees with that of the shortest orbit, which is (i) for $a \leqslant 2.9717$ and (iii) for $a \geqslant 2.9717$. The reason is that for $a \leqslant 2.9717$ the Weyl series is dominated by a complex orbit (a saddle), which at $a = 2.9717$ bifurcates into a pair of real orbits, whose dominant member is the stable saddle (iv) rather than the unstable minimum (iii). The lengths of these orbits are given from (97) by

$$l = 4x(\sigma(a)), \tag{98}$$

where $x(\sigma)$ is given by (96) with $\sigma(a)$ the smallest real or complex solution of (cf. (95))

$$\tfrac{1}{2}\pi = \sigma(a) + \tfrac{1}{3}a\sin\{3\sigma(a)\}. \tag{99}$$

The bifurcation occurs when $\sigma(a)$ corresponds to an inflection, that is (cf. (94))

$$\sigma(a) = \tfrac{1}{3}\arccos\{-1/a\}, \tag{100}$$

and this gives the value $a = 2.9717$.

Table 4. *Coefficients $c_r(a)$ of the Weyl series for the 2-bulge billiard*

r	c_r
2	$\frac{1}{6}$
3	$\frac{(2+a^2)\pi}{512}$
4	$\frac{2+3a^2}{315}$
5	$\frac{3\pi(296+760a^2+111a^4)}{524288}$
6	$\frac{2(136+472a^2+255a^4)}{45045}$
7	$\frac{5\pi(18256+76632a^2+86978a^4+5705a^6)}{33554432}$
8	$\frac{4(17456+84184a^2+147030a^4+38185a^6)}{4849845}$
9	$\frac{7\pi(700467328+3795519232a^2+8386799392a^4+5247380576a^6+191534035a^8)}{549755813888}$
10	$\frac{8(7744384+46718208a^2+116415072a^4+131277600a^6+19058445a^8)}{1003917915}$
11	$\frac{9\pi(377084265728+2522387787904a^2+6722838917568a^4+10937881091104a^6+4029808768390a^8+92798081019a^{10})}{70368744177664}$
12	$\frac{16(23120742656+171083723648a^2+481961997632a^4+949800390688a^6+694551451510a^8+62588572893a^{10})}{902522205585}$
13	$\frac{11\pi(2524549498266624+20620444561941504a^2+61812178600424064a^4+13122533608796032 0a^6+156677891816201768a^8+36109265919314616a^{10}+569502865331631a^{12})}{72057594037927936}$

Table 5. *Coefficients $c_r(a)$ of the Weyl series for the 3-bulge billiard*

r	c_r
2	$\frac{1}{6}$
3	$\frac{(2+a^2)\pi}{512}$
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6	$\frac{2(136+212a^2+255a^4)}{45045}$
7	$\frac{5\pi(18256+8472a^2+67338a^4+5705a^6)}{33554432}$
8	$\frac{4(17456-10626a^2+44430a^4+38185a^6)}{4849845}$
9	$\frac{7\pi(700467328-901942528a^2-2482438368a^4+4145244896a^6+191534035a^8)}{549755813888}$
10	$\frac{8(7744384-12000672a^2-87067008a^4+41262000a^6+19058445a^8)}{1003917915}$
11	$\frac{9\pi(377084265728-580358180736a^2-5377750903872a^4-3976799840736a^6+3267189665190a^8+92798081019a^{10})}{70368744177664}$
12	$\frac{16(23120742656-32507863712a^2-228134940408a^4-909185806392a^6+258812163960a^8+62588572893a^{10})}{902522205585}$
13	$\frac{33\pi(841516499422208-1047186097308672a^2-1152254681949312a^4-4398963236151168 0a^6-14766871221004104a^8+9996863479537032a^{10}+189834288443877a^{12})}{72057594037927936}$

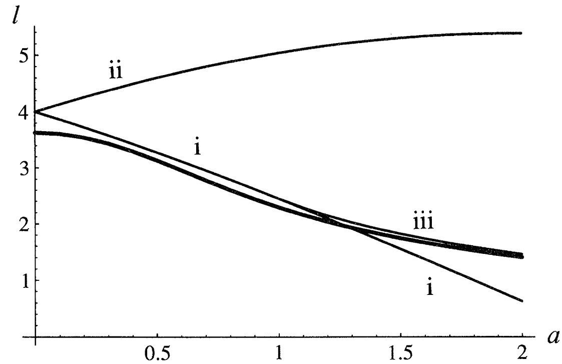

Figure 8. Thick line: predicted length of shortest accessible periodic orbit in the 2-bulge billiard, computed from the Weyl series using (14); thin lines: actual lengths of 2-bounce orbits (numbers label orbits as in figure 6).

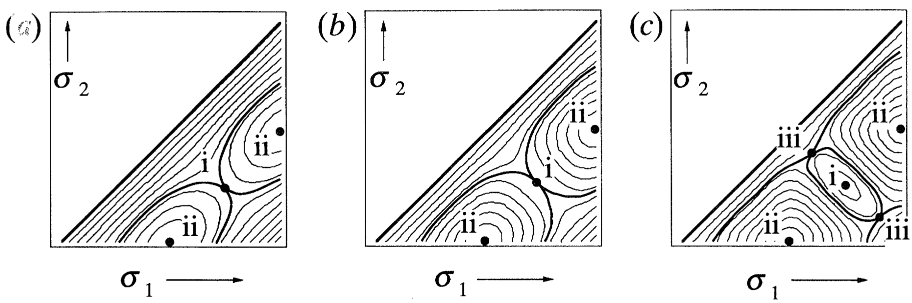

Figure 9. Contours of chord length function $\rho(\sigma_1, \sigma_2)$ on the Möbius strip ($0 \leqslant \sigma_2 \leqslant \sigma_1 \leqslant 2\pi$, with identification of $(\sigma, 0)$ with $(2\pi, \sigma)$), for the 2-bulge billiard with (*a*) $a = 0.5$, (*b*) $a = 1$, (*c*) $a = 2$ (numbers labelling the stationary points refer to orbits as in figure 6).

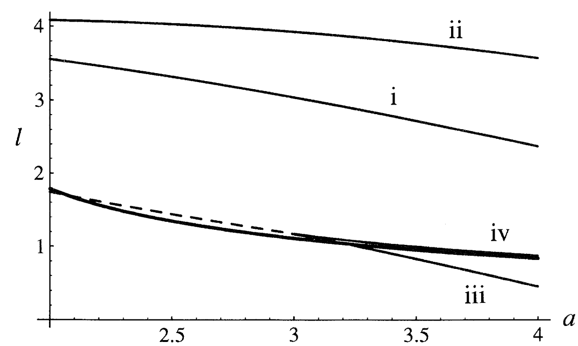

Figure 10. Thick line: predicted length of shortest accessible periodic orbit in the 3-bulge billiard, computed from the Weyl series using (14); thin lines: actual lengths of real 2-bounce orbits (numbers label orbits as in figure 7); dashed line: modulus of length of complex orbit for $a < 2.9717$.

The real orbits in the 3-bulge billiard are the stationary points of the Möbius contour plot in figure 11; after the bifurcation the dominant orbit is (according to the accessibility conjecture) the stable saddle (iv). It is clear from figure (10) that the length as predicted from the Weyl series agrees very well with $|l|$ for the complex orbit before the bifurcation, and with l for the real orbit (iv) afterwards.

The complex orbit before the bifurcation is one of a conjugate pair. We envisage

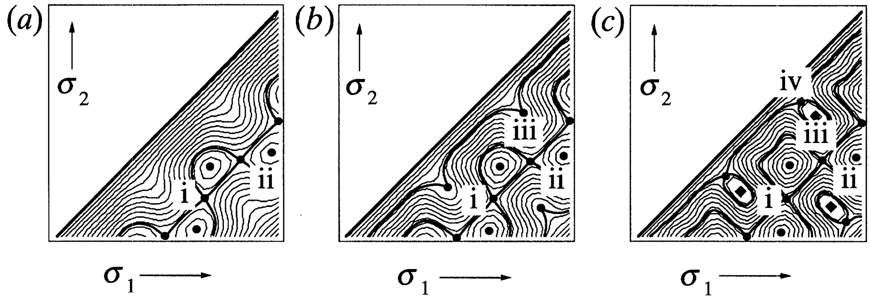

Figure 11. Contours of chord length function $\rho(\sigma_1, \sigma_2)$ on the Möbius strip ($0 \leqslant \sigma_2 \leqslant \sigma_1 \leqslant 2\pi$, with identification of $(\sigma, 0)$ with $(2\pi, \sigma)$), for the 3-bulge billiard with (*a*) $a = 2$, (*b*) $a = 2.9717$, (*c*) $a = 4$ (numbers labelling the stationary points refer to real orbits as in figure 7).

that the high c_r could be modulated by delicate interference and coalescence effects between these two complex orbits, analogous to those we have recently found in saddle-point expansions dominated by coalescing distant saddles (Berry & Howls 1993). Our billiard calculations were not sensitive enough to detect such effects.

6. Concluding remarks

The resurgence arguments of §1 and the numerical evidence of §5*b* strongly suggest the truth of our central conjecture (4): that Weyl expansions for spectral functions describing smooth billiards diverge factorially, in a manner determined by a short periodic geodesic. The accessibility conjecture of §5*a*, according to which the dominating orbit, if it is a 2-bounce, is the (real or complex) saddle of chord length first reached when climbing from the diagonal of the Möbius strip, is also supported by numerical evidence.

Now the need is to provide proofs of the conjectures, and precise statements of their domain of validity. One approach might be to analyse the large-order behaviour of the recurrence relation (88), which determines the Weyl coefficients via (82)–(83). This should lead to intriguing geometrical relations, still obscure, between the boundary integrals of powers of derivatives of the curvature and the lengths of short geodesics.

Proof of the condition for accessibility could provide a way to characterise those systems where (unlike smooth billiards) none of the orbits is accessible from the Weyl series. We have seen several examples of this (particle on a ring, polygonal billiards, heat kernel for the harmonic oscillator), but do not have a general understanding. One possibility (see the end of §2*a*) is that spectral resurgence does not occur when relevant wave functions are in some sense exact (rather than asymptotic).

Several extensions and generalizations can be envisaged. The easiest is to boundary conditions other than Dirichlet (for example Neumann). Next would be billiards with corners or cusps, for which the Weyl series is much more complicated (Stewartson & Waechter 1971).

For smooth billiards, the formalism of §4 opens up the possibility that many more Weyl coefficients could be calculated with high precision, so that refinements of the conjecture (4) could be explored. Analogous phenomena for integrals with saddles (Berry & Howls 1991) suggest two sorts of correction to (4). The first kind form a series in descending powers of s, involving the coefficients in the asymptotic series multiplying the exponential contribution to the resolvent from the shortest accessible orbit. (Alonso & Gaspard (1993) give a theory for this class of semiclassical corrections.) The second kind are smaller exponentials, associated with longer accessible orbits.

It is probable that the ideas explored here are not restricted to spectral functions for billiards. More general hamiltonian dynamical systems (for example particles influenced by electric and magnetic forces, or on riemannian manifolds) can be quantized to give a set of eigenvalues, which can be described in terms of a spectral function with a semiclassical (Weyl) expansion. Powerful general formalisms now exist (Molzahn *et al.* 1990; Molzahn & Osborn 1994) that could in principle enable the computation of sufficiently many terms to discern the large-order behaviour.

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