The Levitron™: an adiabatic trap for spins

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A magnet in the form of a spinning-top can float stably above a repelling magnetic base. The principal mechanism of stability is static equilibrium in a potential energy field E, arising dynamically from the adiabatic coupling of the spin with the magnetic field $\boldsymbol{B}$ of the base and involving the magnitude B of this field. E is close to a harmonic potential, that is, one whose Laplacian is zero, for which Earnshaw's theorem would forbid stable equilibrium. Therefore its minimum is very shallow, and requires the mass of the top to be adjusted delicately so that it hangs within a small interval of height. The stability interval is increased by a post-adiabatic dynamic coupling of the velocity of the top to $\boldsymbol{B}$, through an effective 'geometric magnetic field' constructed from the spatial derivatives of $\boldsymbol{B}$; this effect gets stronger as the top is spun faster. The device is analogous to several traps for microscopic particles.

1. Introduction

An ingenious mechanical device, recently developed by Mr W. Hones and distributed commercially by him, is called the Levitron™. It consists of a magnet in the form of a spinning-top, that can be lifted so as to float in mid-air, gently bobbing and weaving for several minutes about an equilibrium point above a heavy base containing a magnetized ceramic slab (figure 1). My purpose here is to develop a theory for this device, which relies on, and is a fine illustration of, the application to mechanics of adiabatic theory and its first-order improvement, namely geometric magnetism (Berry & Robbins 1993*a*,*b*). The aim is not to give a quantitatively accurate simulation of the device, but rather to expose the mechanical principles underlying its operation.

The top is a rotationally symmetric rigid body with mass m and angular momentum $\boldsymbol{S}$, whose centre of mass is located at $\boldsymbol{r} = (x, y, z)$. It can be regarded as a magnetic dipole with vector moment $\boldsymbol{\mu}$ (fixed magnitude μ) located at $\boldsymbol{r}$ and directed along the axis of symmetry. The base provides a magnetic field $\boldsymbol{B}(\boldsymbol{r})$. The gradients of this field compensate the gravitational force mg by providing a repulsive force that acts on $\boldsymbol{\mu}$ in the presence of the spin $\boldsymbol{S}$ (whose gyroscopic effect prevents the top from overturning and falling) and must provide the mechanism for the top to spin stably above the base.

That this mechanism must be subtle is shown by the failure of the following naive argument. In the region where the top floats, both $\boldsymbol{B}(\boldsymbol{r})$ and $\boldsymbol{\mu}$ (parallel to the axis of the top) are approximately vertical, that is parallel to the z direction. Therefore the magnetic energy $-\boldsymbol{\mu} \cdot \boldsymbol{B}(\boldsymbol{r})$ (cf. equation (2.1) below) is approximately $-\mu B_z$, and the upward repulsive force, holding the top in equilibrium against gravity, is approximately $\mu \partial_z B_z$. At equilibrium, the total energy (magnetic + gravitational)

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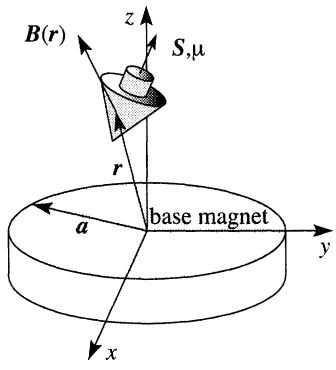

Figure 1. Diagram of the Levitron^{TM}.

has a critical point, that is its gradient vanishes. For the equilibrium to be stable, the critical point must be a minimum. But this is impossible, by the theorem of Earnshaw (1842; see also Page & Adams 1958): both gravity and B_z are harmonic potential fields (cf. equation (2.9) below), whose only critical points are saddles. This means that if the top were in stable equilibrium vertically, it would be unstable horizontally, and vice versa. (Edge (1995) describes a two-dimensional potential model for the Levitron^{TM}, that is stable in the plane, but it is unstable in the perpendicular direction.)

Nevertheless, it is possible for the top to float stably in a static - that is, time-independent - potential field, whose origin (§2) is dynamical. The potential is the sum of gravity and the magnetic energy of the dipole $\boldsymbol{\mu}$ averaged over its precession around $\boldsymbol{B}$; the averaged energy involves the magnitude B rather than the component B_z. In mechanics this procedure of separating fast and slow variables is lowest-order adiabatic averaging (Arnold *et al.* 1988; Lochak & Meunier 1988); in chemistry it would be called the Born–Oppenheimer approximation (Messiah 1962). Stability requires the potential to possess a minimum, which, since it arises from the small difference between B and B_z (that is, deviations of $\boldsymbol{B}$ from vertical near the axis), exists only for a narrow range of the mass m (§3). The range depends sensitively on the form of $\boldsymbol{B}(\boldsymbol{r})$, which can change with temperature. This explains why the mass required to keep the top floating (which can be altered by addition of small washers – see §3) changes rapidly – sometimes over a few minutes.

The adiabatic averaging underlying static stability is not exact, and the first correction to it is an additional force depending on the velocity $\boldsymbol{v} = \dot{\boldsymbol{r}}$ with which the top is moving through the magnetic field. This force is geometric magnetism, that is, it has the form $\boldsymbol{v} \times \boldsymbol{B}_G(\boldsymbol{r})$, where $\boldsymbol{B}_G(\boldsymbol{r})$ is an effective field (§4) constructed in an interesting way from the derivatives of the components of $\boldsymbol{B}(\boldsymbol{r})$. The formula was previously obtained quantum-mechanically (Berry 1986), but for this case the classical limit is the same. When applied (§5) to the Levitron^{TM}, its effect is to slightly increase the vertical range of stability.

Stable levitation requires three speeds to be very different. Starting with the fastest, these are the spin angular velocity of the top, its precession angular velocity, and the rate at which the inhomogeneous field $\boldsymbol{B}$ changes as seen by the moving top. In §6 it is shown that there exists a range of spin angular velocities for which the assumed separation of time scales is a good approximation.

It is possible to regard the Levitron^{TM} as a macroscopic analogue of certain traps

for microscopic particles (Dehmelt 1990; Paul 1990). These analogies are explored in § 7.

The top slows down because of air resistance. After a few minutes it can no longer spin upright, and falls. If the base and the top were metal rather than ceramic, an additional source of dissipation would be induced eddy currents, and the top would fall faster.

2. Static stability

The potential energy of the top, including gravitational and magnetic contributions, is

$$E = mgz - \boldsymbol{\mu} \cdot \boldsymbol{B}(\boldsymbol{r}) \tag{2.1}$$

and the spin changes as the result of the magnetic torque $\boldsymbol{\mu} \times \boldsymbol{B}$, that is

$$\dot{\boldsymbol{S}}(t) = \boldsymbol{\mu}(t) \times \boldsymbol{B}(\boldsymbol{r}(t)). \tag{2.2}$$

Underlying (2.1) is the assumption that the top is *small*, in the sense that its distributed magnetism can be approximated by a point dipole, and higher multipoles neglected.

Now we make the approximation that the top is *fast*, in the sense that its angular momentum can be regarded as parallel to both its angular velocity vector and the symmetry axis. The condition for this is that the spin is much faster than the precession. Later (§ 6) we shall see that this condition holds. In the present application, its importance is that it ensures that $\boldsymbol{S}$ is parallel to $\boldsymbol{\mu}$. This enables the equation of motion (2.2) for the spin to be written in terms of the magnitude B and direction $\boldsymbol{b}$ of the field seen by the moving top, that is

$$\boldsymbol{B}(\boldsymbol{r}(t)) \equiv B(t)\boldsymbol{b}(t). \tag{2.3}$$

Thus the dynamics of the spin is determined by

$$\dot{\boldsymbol{S}}(t) = \Omega(t)\boldsymbol{b}(t) \times \boldsymbol{S}(t), \tag{2.4}$$

where

$$\Omega = -\mu B/S. \tag{2.5}$$

Equation (2.4) describes precession of the axis of the top around the instantaneous field direction $\boldsymbol{b}(t)$, with the magnitude S conserved. If the precession is fast, that is if $|\Omega| \gg |\dot{\boldsymbol{b}}|$, then $\boldsymbol{S}$ is slaved to $\boldsymbol{b}$ in the sense that the component $\boldsymbol{S} \cdot \boldsymbol{b}$ – the *adiabatic invariant* – is approximately conserved (Arnold *et al.* 1988; Lochak & Meunier 1988). Then the component

$$\mu_B \equiv \boldsymbol{\mu}(t) \cdot \boldsymbol{b}(t) \tag{2.6}$$

is also an adiabatic invariant, and the energy (2.1) becomes

$$E = E(r) = mgz - \mu_B B(\boldsymbol{r}). \tag{2.7}$$

This approximation will be examined in § 4; there, it will be shown that the adiabatic slaving of $\boldsymbol{S}$ to $\boldsymbol{b}$ cannot be exact if $\boldsymbol{b}$ is not constant, and this fact will be used to obtain the first-order (in $\dot{\boldsymbol{b}}$) correction to adiabaticity. It is important to note that the adiabatic assumption, that the *precession* is fast, is distinct from the assumption of the preceding paragraph that the *top* is fast. Indeed, it might appear that these assumptions conflict, since the fast top requires S to be large while adiabaticity

requires Ω to be large, which from (2.5) seems to require S to be small. In fact there is no conflict, as will be shown in § 6.

With the above assumptions, the top will float stably above the base if $E(\boldsymbol{r})$ has a minimum there. This requires

$$\left.\begin{array}{ll} \nabla E(\boldsymbol{r}) = 0 \quad \text{i.e. equilibrium} & (a) \\ \partial_z^2 E(\boldsymbol{r}) > 0 \quad \text{i.e. vertical stability} & (b) \\ \partial_x^2 E(\boldsymbol{r}) > 0 \quad \text{and} \quad \partial_y^2 E(\boldsymbol{r}) > 0, \quad \text{i.e. horizontal stability} & (c) \end{array}\right\} \tag{2.8}$$

Here the existence of a minimum will be investigated only on the vertical symmetry axis of the base; I have not explored the possibility that there could be minima off this axis.

Because $\boldsymbol{r}$ is outside the base, there are no currents contributing to the field $\boldsymbol{B}(\boldsymbol{r})$ from the base, so $\boldsymbol{B}$ is curl-free. Since $\nabla \cdot \boldsymbol{B} = 0$, the field can be written

$$\boldsymbol{B}(\boldsymbol{r}) = -\nabla \Phi(\boldsymbol{r}), \quad \text{where } \nabla^2 \Phi(\boldsymbol{r}) = 0. \tag{2.9}$$

In horizontal planes z, the potential Φ is stationary at $x = y = 0$, and to second order has circular symmetry in x and y; this is true whenever the base has the symmetry of an equilateral triangle or higher polygon, and applies to the Levitron$^{\text{TM}}$ where the base is square. Defining

$$\boldsymbol{R} \equiv (x, y) \quad \text{and} \quad \boldsymbol{R} \equiv |\boldsymbol{R}| \tag{2.10}$$

the potential near the axis can now be written

$$\Phi(\boldsymbol{r}) = \Phi(0, 0, z) + \tfrac{1}{2}\partial_x^2 \Phi(0, 0, z) R^2 + \ldots. \tag{2.11}$$

A convenient notation is

$$\phi_n(z) \equiv \partial_z^n \Phi(0, 0, z). \tag{2.12}$$

The requirement that (2.11) satisfies Laplace's equation in (2.9) now gives

$$\Phi(\boldsymbol{r}) = \phi_0(z) - \tfrac{1}{4}\phi_2(z) R^2 + \ldots. \tag{2.13}$$

The adiabatic energy (2.7) involves the magnitude of $\boldsymbol{B}$, which to second order in $\boldsymbol{R}$ is, from (2.9) and (2.13),

$$B(\boldsymbol{r}) = \phi_1 \operatorname{sgn}\phi_1 \left[1 + \frac{R^2}{8}\left(\frac{\phi_2^2}{\phi_1^2} - 2\frac{\phi_3}{\phi_1}\right)\right] \ldots, \tag{2.14}$$

where the z dependence of the ϕ_n has not been written explicitly.

On the axis, horizontal equilibrium in (2.8a) is guaranteed by symmetry. Vertical equilibrium requires that gravity is balanced by an upward magnetic force determined by the gradient of the magnitude of the field, that is

$$mg = \mu_B \partial_z B = \mu_B \phi_2 \operatorname{sgn}\phi_1. \tag{2.15}$$

Of course mg must be positive, and this, together with the stability conditions (2.8b, 2.8c), gives the conditions

$$\left.\begin{array}{ll} \mu_B \phi_2 \operatorname{sgn}\phi_1 > 0 \text{ for equilibrium} & (a) \\ \mu_B \phi_3 \operatorname{sgn}\phi_1 < 0 \text{ for vertical stability} & (b) \\ \mu_B \operatorname{sgn}\phi_1 \,(2\phi_3 - \phi_2^2/\phi_1) > 0 \text{ for horizontal stability} & (c) \end{array}\right\} \tag{2.16}$$

It is impossible to satisfy these conditions for $\mu_B > 0$, because then (b) implies that

ϕ_1 and ϕ_3 have opposite signs and (c) is violated because both terms on its left-hand side are negative. Therefore $\mu_B < 0$, that is, the projection of $\boldsymbol{\mu}$ along $\boldsymbol{B}$ is antiparallel to $\boldsymbol{B}$, and the conditions for stable equilibrium become

$$\left.\begin{array}{lll} \phi_1 \text{ and } \phi_2 \text{ have opposite signs} & \text{(equilibrium)} & (a) \\ \phi_1 \text{ and } \phi_3 \text{ have the same signs} & \text{(vertical stability)} & (b) \\ \phi_2^2 - 2\phi_3\phi_1 > 0 & \text{(horizontal stability)} & (c) \end{array}\right\} \tag{2.17}$$

Without the term ϕ_2^2 in (c) it would be impossible to satisfy conditions (b) and (c) simultaneously, and the fact that this term arises precisely from the difference between $\boldsymbol{B}$ and B confirms that the adiabatic potential (2.7) does indeed give the possibility for the top to float stably in spite of Earnshaw's theorem.

3. Magnetic field on the axis

The base can be regarded as a planar distribution of vertically oriented dipole sources of magnetic field, with density $\rho(\boldsymbol{R})$ (where $\boldsymbol{R} = 0$ is the centre of the hole), and the formula (Jackson 1975) for the potential of a dipole gives

$$\phi_0(z) = z \iint_{\text{base}} \mathrm{d}^2\boldsymbol{R} \frac{\rho(\boldsymbol{R})}{(R^2 + z^2)^{3/2}}. \tag{3.1}$$

Without loss of generality, ρ can be taken positive, that is, the dipoles in the base point up. Then the construction of the Levitron™ is such that the dipoles in the spinning top point down (i.e. $\mu_z < 0$) (this can be confirmed by observing that when the top is held upright and close to the material of the top of the base, it is repelled (unlike dipoles repel, unlike unlike poles).

In the Levitron™, the base is a square slab that is uniformly magnetized apart from a central unmagnetized hole, whose purpose is to provide an approximately field-free zone where the top can be spun up by hand before being lifted – on a plastic plate – to the position where it can float stably. The stability analysis is however the same for a circular disk (radius a, with ρ constant), and can be conveniently illustrated for this case because the analysis is simpler.

Equation (3.1) gives, for a uniformly magnetized disk,

$$\phi_0(z) = 2\pi\rho\left(1 - \frac{z}{\sqrt{a^2 + z^2}}\right). \tag{3.2}$$

The derivatives ϕ_1 and ϕ_2 are respectively negative and positive, so that condition (2.17a) is always satisfied: gravity can be made to balance magnetic repulsion at any height by choosing the mass according to (2.15). ϕ_3 is positive for $z < \frac{1}{2}a$ and negative for $z > \frac{1}{2}a$ so that (2.17b) ensures vertical stability when $z > \frac{1}{2}a$. The function in (2.17c) is proportional to $2a^2 - 5z^2$, so that horizontal stability requires $z < a\sqrt{\frac{2}{5}} = a \times 0.6325\ldots$. Thus the zone of stable equilibrium is

$$\tfrac{1}{2} < z/a < \sqrt{\tfrac{2}{5}} \tag{3.3}$$

It is interesting to see how the minimum (stable equilibrium) of the adiabatic potential energy (2.7) appears and disappears as the mass is increased so that the equilibrium height varies through the rather narrow zone of stability. Defining di-

mensionless coordinates, energy and mass by

$$\xi \equiv \frac{x}{a}, \quad \eta \equiv \frac{y}{a}, \quad \zeta \equiv \frac{z}{a}; \quad E \equiv \frac{Ea}{2\pi|\mu_B|\rho}; \quad M \equiv \frac{mga^2}{2\pi|\mu_B|\rho}, \tag{3.4}$$

we find from (2.7), (2.14) and (3.2) that the potential energy near the axis is

$$E(\xi,\eta,\zeta) = M\zeta + \frac{1}{(1+\zeta^2)^{3/2}} + \frac{3(\xi^2+\eta^2)(2-5\zeta^2)}{8(1+\zeta^2)^{7/2}}. \tag{3.5}$$

The critical range of masses M satisfying (3.3) is determined by the equilibrium condition (2.15), which gives

$$M = \frac{3\zeta}{(1+\zeta^2)^{5/2}}. \tag{3.6}$$

This has a maximum at the lower stability limit $\zeta = \frac{1}{2}$, where the mass is

$$M_+ = \frac{48}{5^{5/2}} = 0.85865\ldots. \tag{3.7}$$

When $M > M_+$, no equilibrium is possible and the top will fall. At the upper stability limit $\zeta = \sqrt{\frac{2}{5}}$, the mass is

$$M_- = \frac{75\sqrt{2}}{7^{5/2}} = 0.818147\ldots. \tag{3.8}$$

When $M < M_-$, the top can be in vertically stable equilibrium but is horizontally unstable. The stable interval $M_- < M < M_+$ is only about 5% of the mean stable mass.

Figure 2 shows contours of potential energy (2.7) as a function of the position of the top, for a sequence of masses including the stable interval. Between figures 2*b* and 2*f*, corresponding to M_- and M_+, the potential has a minimum. This is created from an axial saddle (unstable equilibrium) as M increases through M_-, by a local bifurcation which also generates a ring of saddles that recedes from the axis. It is destroyed as M increases through M_+ by a different local bifurcation: annihilation with an axial saddle from below. In the stable range, a critical event is the saddle-connection (non-local bifurcation) at figure 2*d*, where the contours through the saddles change topology; this occurs at M_{SC} where the three saddles all have the same height, and a little analysis gives

$$M_{\mathrm{SC}} = 0.847837\ldots. \tag{3.9}$$

For each M in the zone of stability, the basin of stable motion of the top is the interior of the surface of revolution corresponding to the largest closed contour surrounding the potential minimum. This is widest at $M = M_{\mathrm{SC}}$, and a calculation based on (3.5) gives the extent of this 'most stable motion' as

$$\sqrt{\xi^2+\eta^2} < \sqrt{\frac{\sqrt{6588344}}{1875}(M_{\mathrm{SC}} - M_-)} = 0.20160\ldots. \tag{3.10}$$

In this case, the minimum is at $\zeta_{\mathrm{SC}} = 0.565373\ldots$.

For small deviations from the minimum, the potential (3.5) is quadratic in ξ, η and ζ, and the motion consists of harmonic oscillations with three frequencies (see § 5 for more discussion of this). For larger deviations, the vertical component of angular

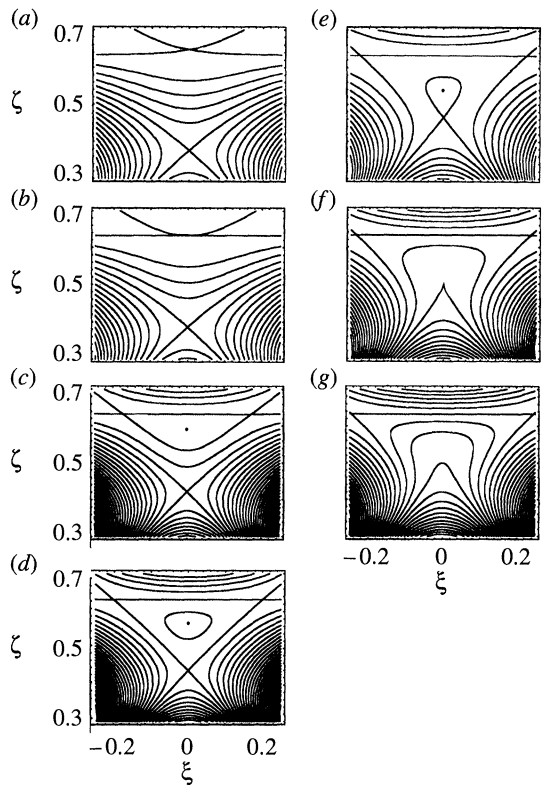

Figure 2. Contour maps of axial sections ζ, ξ (with $\eta = 0$) of the (rotationally symmetric) scaled potential energy (3.5) of the top, in the space above a base in the shape of a uniformly magnetized disk, for different scaled masses. (*a*) $M = 0.81000$; (*b*) $M = M_{-} = 0.818147$; (*c*) $M = 0.84000$; (*d*) $M = M_{\mathrm{SC}} = 0.847837$; (*e*) $M = 0.85500$; (*f*) $M = M_{+} = 0.85865$; (*g*) $M = 0.86200$.

momentum is conserved, because the potential (3.5) is rotationally invariant, and so is the total energy (kinetic plus potential). But because the horizontal and vertical variables are not separable, motion is probably nonintegrable, with chaos being more pronounced near the saddles. Although the exact motion in the potential (3.5) is worth studying further, it should be remembered that this is an approximation, valid near the axis: farther away, there will be nonquadratic corrections, which for non-circular base magnets will break the degenerate ring of saddles into a finite number of isolated critical points (which might include weak minima).

For the more realistic model of the base as a uniformly magnetized square slab (side $2a$) with an unmagnetized central hole (radius w), (3.1) gives

$$\phi_0(z) = 2\pi\rho \frac{z}{\sqrt{z^2 + w^2}} - 8\rho \arcsin\left\{ \frac{z}{\sqrt{2(z^2 + a^2)}} \right\}. \tag{3.11}$$

In the Levitron™, $w/a \approx \frac{1}{4}$. Then, stability analysis based on (2.17) (slightly more complicated than for the disk) leads to the result that the top can float stably in the rather narrow region

$$3.976 < z/w < 4.360. \tag{3.12}$$

The value of the stable height – between about 4.0 and 4.4 hole radii above the

centroid of the base – is in comfortable agreement with observation (I measured $\zeta = 4.3$, where now $b = z/w$). The mass of the top must be carefully chosen so that the solution of (2.15) lies in the stable interval. From (2.15), the change of equilibrium height $\mathrm{d}\zeta$ resulting from a change $\mathrm{d}m$ of the mass is

$$\mathrm{d}\zeta = -\frac{\mathrm{d}m}{m}\left|\frac{\phi_2}{\phi_3}\right|. \tag{3.13}$$

In the stable interval, the amplification factor $|\phi_2/\phi_3|$ decreases from infinity to a smallest value which for (3.11) with $w/a \approx \frac{1}{4}$ is 7.05; the value at the midpoint of the interval is 12.2. For the washers supplied with the Levitron$^{\mathrm{TM}}$, $\mathrm{d}m/m$ ranges from 0.003 to 0.06. Therefore the effect of adding the lightest washer is, roughly (that is, using the midpoint value of the amplification factor), to change the height by $\mathrm{d}\zeta \sim 0.04$, that is, by about one-tenth of the stable interval. For the heaviest washer, $\mathrm{d}\zeta \sim 0.7$, which is about twice the stable interval.

4. Geometric magnetism

The adiabatic assumption embodied in (2.7), and the first post-adiabatic correction to be discussed in this section, can be justified by general theory (see, for example, Berry & Robbins 1993*b*), but it instructive to give the reasoning for this particular case. The following elementary argument was given by Dr J. H. Hannay (personal communication) (see also Aharonov & Stern 1992).

The motion of the top is determined by the gravitational and magnetic forces on it. From (2.1), the force is

$$\boldsymbol{F} \equiv -mg\boldsymbol{e}_z + \boldsymbol{F}_{\mathrm{M}} = -mg\boldsymbol{e}_z + \nabla\boldsymbol{\mu}(t)\cdot\boldsymbol{B}(\boldsymbol{r}) \tag{4.1}$$

where $\boldsymbol{e}_z$ is the upward unit vector. Splitting $\boldsymbol{\mu}$ into components along and perpendicular to the instantaneous field $\boldsymbol{B}$, namely

$$\boldsymbol{\mu}(t) = \mu_{\mathrm{B}}\boldsymbol{b}(t) + \boldsymbol{\mu}_\perp(t) \tag{4.2}$$

enables the magnetic force to be written as

$$\boldsymbol{F}_{\mathrm{M}} \equiv \boldsymbol{F}_{\mathrm{A}} + \boldsymbol{F}_{\mathrm{G}} = \mu_{\mathrm{B}}\nabla B(\boldsymbol{r}) + \boldsymbol{\mu}_\perp(t)\cdot\nabla\boldsymbol{B}(\boldsymbol{r}) \tag{4.3}$$

where the dot product connects $\boldsymbol{\mu}_\perp$ and $\boldsymbol{B}$. $\boldsymbol{F}_{\mathrm{A}}$ and $\boldsymbol{F}_{\mathrm{G}}$ denote the 'adiabatic' and 'geometric' parts of the magnetic force (terms that will be explained later), whose time averages, over the precession – which is regarded as fast – will now be discussed.

With the fast top assumption, $\boldsymbol{\mu}$ satisfies the same equation of motion (2.4) as $\boldsymbol{S}$, so the separation (4.2) gives

$$\dot{\boldsymbol{\mu}} = \dot{\mu}_{\mathrm{B}}\boldsymbol{b} + \mu_{\mathrm{B}}\dot{\boldsymbol{b}} + \dot{\boldsymbol{\mu}}_\perp = \Omega\boldsymbol{b}\times\boldsymbol{\mu}_\perp. \tag{4.4}$$

The lowest-order adiabatic procedure is to regard the precession, about the instantaneous field direction $\boldsymbol{b}(t)$, as fast, and argue that the average value of the transverse component $\boldsymbol{\mu}_\perp$ is zero. However, this cannot be exactly true, as can be seen from the special case where $\boldsymbol{\mu}$ is initially parallel to $\boldsymbol{b}$. Then the last member in (3.7) would be zero, and if $\boldsymbol{\mu}$ were completely slaved to $\boldsymbol{b}(t)$ would remain so, contradicting the first member which would be proportional to $\dot{\boldsymbol{b}}$. Therefore we must be a little more careful, and allow for the precession to be about a direction slightly different from $\boldsymbol{b}(t)$.

To the next approximation – sufficient for the present purpose – the precession-averaged *velocity* $\dot{\boldsymbol{\mu}}_\perp$ can be set equal to zero, but not the component $\boldsymbol{\mu}_\perp$ itself. In this post-adiabatic approximation, the parallel and perpendicular components of (4.4) now give

$$\dot{\mu}_\mathrm{B} \approx 0 \quad \text{and} \quad \boldsymbol{\mu}_\perp \approx -\frac{\mu_\mathrm{B}}{\Omega}\boldsymbol{b}\times\dot{\boldsymbol{b}}. \tag{4.5}$$

The first equation is the conservation of the adiabatic invariant, so that $\boldsymbol{F}_\mathrm{A}$ in (4.3) is indeed the lowest-order adiabatic force that would be obtained from (2.7), and which leads to the conditions for static stability obtained in § 2 and § 3.

The second equation in (4.5) gives $\boldsymbol{F}_\mathrm{G}$ as

$$\boldsymbol{F}_\mathrm{G} = -\frac{\mu_\mathrm{B}}{\Omega}(\boldsymbol{b}\times\dot{\boldsymbol{b}})\cdot\nabla\boldsymbol{B}(\boldsymbol{r}) = \frac{S_\mathrm{B}}{B}(\boldsymbol{b}\times\dot{\boldsymbol{b}})\cdot\nabla\boldsymbol{B}(\boldsymbol{r}). \tag{4.6}$$

(Use has been made of (2.5) and the fast-top assumption that $\boldsymbol{S}$ is parallel to $\boldsymbol{\mu}$, and $S_\mathrm{B} \equiv \boldsymbol{S}\cdot\boldsymbol{b}$.) The change $\dot{\boldsymbol{b}}$ is caused by the motion of the top through the inhomogeneous field $\boldsymbol{B}(\boldsymbol{r})$, so that

$$\dot{\boldsymbol{b}} = (\boldsymbol{v}\cdot\nabla)\boldsymbol{b}(\boldsymbol{r}). \tag{4.7}$$

Thus the post-adiabatic force $\boldsymbol{F}_\mathrm{G}$ depends on the velocity as well as the position of the top:

$$\boldsymbol{F}_\mathrm{G} = \frac{S_\mathrm{B}}{B^3}\left[\boldsymbol{B}\times(\boldsymbol{v}\cdot\nabla)\boldsymbol{B}\right]\cdot\nabla\boldsymbol{B}(\boldsymbol{r}). \tag{4.8}$$

Here the vector product connects $\boldsymbol{B}$ and $\boldsymbol{B}$ inside [ ], and the second scalar product connects the vectors [ ] and $\boldsymbol{B}$. This is Hannay's derivation.

Some vector algebra enables (4.8) to be written in the form

$$\boldsymbol{F}_\mathrm{G} = \boldsymbol{v}\times\boldsymbol{B}_\mathrm{G}(\boldsymbol{r}) \tag{4.9}$$

where $\boldsymbol{v} = \dot{\boldsymbol{r}}$ and the effective 'magnetic field' $\boldsymbol{B}_\mathrm{G}$ depends on the components of the actual magnetic field $\boldsymbol{B}$ according to

$$\boldsymbol{B}_\mathrm{G} = -\frac{S_\mathrm{B}}{B^3}\left(B_x\nabla B_y\times\nabla B_z + B_y\nabla B_z\times\nabla B_x + B_z\nabla B_x\times\nabla B_y\right). \tag{4.10}$$

The force $\boldsymbol{F}_\mathrm{G}$ is called geometric magnetism. It is 'magnetism' because it has the same velocity-dependence as the Lorentz force: it is as though the top carried a unit electric charge responding to the field $\boldsymbol{B}_\mathrm{G}$. It is 'geometric' because of another role played by $\boldsymbol{B}_\mathrm{G}(\boldsymbol{r})$: it is the vector whose flux through a loop in $\boldsymbol{r}$ space gives the quantum geometric phase (Shapere & Wilczek 1989) or the classical Hannay angle (Hannay 1985) acquired by a spin transported round that loop. The formula (4.10) appeared previously in a study of the geometric phase for quantum spins (Berry 1986). Geometric magnetism is thus a post-adiabatic force of reaction (Berry & Robbins 1993*a*,*b*), of the 'fast' spin $\boldsymbol{S}(t)$ of the top on its 'slow' centre-of-mass motion $\boldsymbol{r}(t)$.

There is a curious hierarchy of geometric magnetic reactions. In the argument just given, the motion of $\boldsymbol{S}$ (precession) is regarded as fast, that is, slaved to the slow variables $\boldsymbol{r}$, and $\boldsymbol{S}$ reacts magnetically on $\boldsymbol{r}$. However, Berry & Robbins (1993*a*) showed that the precession can itself be regarded as geometric, because $\boldsymbol{S}(t)$ – the motion of the *axis* of the top – is a slow variable compared with the *spin* of the top; that is, $\boldsymbol{S}$ is slaved to the spin. We showed that the precession, averaged over the nutation (wobbling) of the axis, can be considered as a geometric reaction, caused

by a monopole source of magnetism at the fixed point of the precession (here this is the centre of mass of the top, but for the more familiar gravitationally precessing top it is the point of contact with the surface on which the top is spinning).

5. Geometric extension of stability

The equation of motion of the top under gravity and the two magnetic forces (equations (4.1), (4.3) and (4.9)) is

$$m\ddot{\boldsymbol{r}} = -mg\boldsymbol{e}_z + \mu_{\mathrm{B}}\nabla B(\boldsymbol{r}) + \boldsymbol{v}\times\boldsymbol{B}_{\mathrm{G}}(\boldsymbol{r}). \tag{5.1}$$

For the static (adiabatic) magnetic force in (4.3), (2.14) and (2.10) give

$$\nabla B = \mathrm{sgn}\phi_1\left[\frac{1}{4}\left(\frac{\phi_2^2}{\phi_1} - 2\phi_3\right)\boldsymbol{R} + \phi_2\boldsymbol{e}_z\right]. \tag{5.2}$$

For the geometric magnetic field (4.10) on the axis, (2.9) and (2.13) give

$$\boldsymbol{B}_{\mathrm{G}} = S_{\mathrm{B}}\,\mathrm{sgn}\phi_1\frac{\phi_2^2}{4\phi_1^2}\boldsymbol{e}_z. \tag{5.3}$$

Thus the vertical motion is unaffected by geometric magnetism, and depends only on the static forces of gravity and $\boldsymbol{F}_{\mathrm{A}}$, as discussed in § 2 and § 3 and embodied in the conditions (2.17*a*,*b*).

Geometric magnetism does however affect the horizontal motion, whose acceleration is given by the following linear equation, whose coefficients depend on height z:

$$\ddot{\boldsymbol{R}} = \frac{g}{4}\left(\frac{\phi_2}{\phi_1} - 2\frac{\phi_3}{\phi_2}\right)\boldsymbol{R} + S_{\mathrm{B}}\,\mathrm{sgn}\phi_1\frac{\phi_2^2}{4m\phi_1^2}\,\dot{\boldsymbol{R}}\times\boldsymbol{e}_z \tag{5.4}$$

(μ_{B} has been eliminated using equation (2.15)). The definition

$$u(t) \equiv x(t) + \mathrm{i}y(t) \tag{5.5}$$

enables (5.4) to be written in the scalar form

$$\ddot{u} = \alpha(z)u + \mathrm{i}\beta(z)\dot{u}. \tag{5.6}$$

The general solution is

$$u(t) = u_+\exp\{\mathrm{i}\omega_+(z)t\} + u_-\exp\{\mathrm{i}\omega_-(z)t\} \tag{5.7}$$

where

$$\omega_\pm(z) = \tfrac{1}{2}(\beta \pm \sqrt{\beta^2 - 4\alpha}). \tag{5.8}$$

Horizontal stability requires real $\omega_\pm$, that is $\beta^2 > 4\alpha$, for which (5.4) gives

$$\frac{S_{\mathrm{B}}^2}{m^2 g} > G(z) \equiv 32\left|\frac{\phi_1}{\phi_2}\right|^3\left(\frac{\phi_1\phi_3}{\phi_2^2} - \frac{1}{2}\right) \tag{5.9}$$

(in writing the modulus sign, equation (2.17*a*) has been invoked). This is automatically satisfied if the condition (2.17*c*) for horizontal stability in the lowest-order adiabatic approximation is satisfied, since then $G(z)$ is negative. If $G(z)$ is positive, (2.17*c*) is violated – this occurs, for example, when $\zeta > \sqrt{\frac{2}{5}}$ for the circular base. Nevertheless, (5.9) shows that geometric magnetism can provide post-adiabatic stabilization if the top is spun fast enough.

To estimate the size of this effect, let the top have a vertical axis, radius of gyration d, and spin frequency ν, and evaluate the last member of (5.9) with the potential (3.2) of the disk with radius a. Then the horizontal motion will be stable if $\zeta = z/a$ satisfies

$$\frac{S_B^2}{m^2 g a^3} = \frac{4\pi^2\nu^2 d^4}{g a^3} > \frac{G(z)}{a^3} = \frac{16(5\zeta^2-2)(\zeta^2+1)^3}{\zeta^5}, \quad \text{i.e.} \quad \zeta - \sqrt{\tfrac{2}{5}} < \frac{81\pi^2\nu^2 d^4}{686 g a^3}. \tag{5.10}$$

For the Levitron™, $a \approx 5$ cm (approximating the base by a disk), $d \approx 1.13$ cm (=radius/$\sqrt{2}$, approximating the top by a disk), and for hand spinning $\nu \sim 20$ Hz. Then stability requires $\zeta - \sqrt{\frac{2}{5}} < 0.0062$. In this case, geometric magnetism contributes only a modest increase of about 5% in the statically stable interval $\frac{1}{2} < \zeta < \sqrt{\frac{2}{5}}$. However, if the top could be made to spin faster (perhaps by encircling it with a hoop – like a toy gyroscope – and pulling a string wound round its axis) the geometric effect could be greatly enhanced. For example, if $\nu = 40$ Hz geometric magnetism would increase the interval of stability by 20%; 100 Hz would double it (but would destabilize the top for other reasons – see the end of § 6).

6. Adiabatic conditions

Since all principal moments of inertia of the top are roughly the same size, the condition for the top to be fast (spin parallel to angular velocity and along the symmetry axis) is

$$\text{spin angular velocity } 2\pi\nu \gg \text{precession angular velocity } |\Omega|. \tag{6.1}$$

Using (2.5) and (2.15), and introducing the radius of gyration d of the top and the radius a of the base, this can be expressed in terms of the magnetic potential as

$$\nu \gg \nu_{\min} = \frac{1}{2\pi d}\sqrt{g\left|\frac{\phi_1}{\phi_2}\right|}. \tag{6.2}$$

In terms of $\nu_{\min}$, the precession frequency is

$$\frac{\Omega}{2\pi} = \frac{\nu_{\min}^2}{\nu}. \tag{6.3}$$

The adiabatic condition is that the precession is much faster than the rate (4.7) at which the driving field $\boldsymbol{b}$ is changing, that is

$$|\Omega| \gg |(\boldsymbol{v}\cdot\nabla)\boldsymbol{b}|. \tag{6.4}$$

From (2.9) and (2.13) follows

$$\boldsymbol{b} = \left\{x\frac{\phi_2}{2|\phi_1|}, y\frac{\phi_2}{2|\phi_1|}, -\mathrm{sgn}\phi_1\right\} + \ldots \tag{6.5}$$

so (6.3) depends only on the transverse speed $v_\perp$ of the top. After a little algebra, the adiabatic condition becomes

$$v_\perp \gg \frac{2\sqrt{g}}{d}\left(\left|\frac{\phi_1}{\phi_2}\right|\right)^{3/2}\frac{\nu_{\min}}{\nu}. \tag{6.6}$$

A related quantity is the frequency of the top in its small oscillations about equilibrium. Here I consider only the vertical (bobbing) motion, for which (5.1) gives the frequency

$$\nu_z = \frac{1}{2\pi}\sqrt{g\left|\frac{\phi_3}{\phi_2}\right|} = \frac{1}{2\pi}\sqrt{\frac{g}{a}\frac{(4\zeta^2-1)}{\zeta\ (\zeta^2+1)}} \tag{6.7}$$

where the last equality refers to the circular disk base.

In the following numerical estimates, we use the values $a = 5$ cm, $d = 1.13$ cm, $n = 20$ Hz. Over the stability interval $\frac{1}{2} < \zeta < \sqrt{\frac{2}{5}}$, the potential (3.2) implies that the quantity $|\phi_1/\phi_2|$ varies from $0.73a$ to $0.83a$. Then

$$\nu_{\min} \approx \frac{0.88}{2\pi}\sqrt{\frac{ga}{d^2}} \tag{6.8}$$

This gives $\nu_{\min} \sim 8.7$ Hz, which is well below hand-spinning speed for the top From (6.3), the precession frequency is $\Omega/2\pi = 3.8$ Hz. Therefore the Levitron$^{\text{TM}}$ is a fast top.

From (6.6) follows $v_\perp \gg 387\ \text{cm s}^{-1} \times (8.7/\nu\ (\text{Hz}))$. For hand spinning, the horizontal speed of the top is much slower than this upper limit, so the adiabatic condition is comfortably satisfied too. This can also be seen from (6.7), which for the mass M_{SC} corresponding to the most stable motion ($\zeta_{\text{SC}} = 0.565373$) gives

$$\nu_z = \frac{0.61106}{2\pi}\sqrt{\frac{g}{a}} \approx 0.69\ \nu_{\min}\frac{d}{a} \tag{6.9}$$

This gives $\nu_z = 1.4$ Hz, which is roughly the observed bobbing frequency, and several times less than the precession frequency.

However, a much faster spin would destabilize the top, because the adiabatic condition (6.6) would be violated: the precession frequency (6.3) would be smaller, and the axis of precession could no longer follow $\boldsymbol{b}$. Perhaps this explains the observation by W. Hones and E. Hones (personal communication) that the top indeed becomes unstable when ν is increased with air jets.

7. Connection with traps for microscopic particles

Closely analogous to the Levitron$^{\text{TM}}$ are traps for neutral particles (e.g. neutrons) with spin and magnetic moment (Paul 1990). Just as in the theory of § 2 and § 3, these confine particles by a magnetic field, and Earnshaw's theorem can be circumvented because the magnitude B, unlike the components of $\boldsymbol{B}$, can possess a minimum – indeed, the formula (2.7) for the adiabatic potential energy appeared in a paper by Vladimirskii (1960) in which these traps were proposed. There are however several differences from the top. First, the spin is not an independent parameter but a fixed quantity coupled to the magnetic moment; one consequence of this is that the neutral particle trap works only for one sense of spin, whereas the top can levitate for either. Second, the spin component along the field is quantized: it must be a multiple of $\hbar/2$. Third, in traps for neutral particles the magnetic field is small in the confinement region, so that (cf. equation (2.5)) the precession frequency is small and care must be taken to prevent non-adiabatic transitions between spin states.

It is interesting to consider the circumstances in which geometric magnetism (§ 4 and § 5) can make a significant contribution to the stability of the neutral-particle

Table 1. *Comparison of Levitron™ and particle traps*

Levitron™ and neutral particle trap	Penning trap
spinning magnet (classical or quantum)	charge
magnetic field magnitude B	electric potential
geometric magnetic field $\boldsymbol{B}_{\mathrm{G}}$	magnetic field

traps. Consider a neutron, for which $S_{\mathrm{B}} = \hbar/2$. Then the geometric magnetism stability condition (5.10), for the case where $G(z) > 0$ – that is, where the neutron would be unstable statically – can be written

$$a < L_{\mathrm{neutron}} \frac{\zeta^{5/3}}{4\,(\zeta^2+1)\,(5\zeta^2-2)^{1/3}} \tag{7.1}$$

where

$$L_{\mathrm{neutron}} \equiv \left(\frac{\hbar^2}{m^2 g}\right)^{1/3} = 7.40\ \mu\mathrm{m}. \tag{7.2}$$

The disk radius a can be regarded as a measure of the linear size of the trap. A conclusion from (7.1) is that a neutron trap dominated by geometric magnetism would have to be very small unless it operated close to the stability border $\zeta = \sqrt{\frac{2}{5}}$. (The gravitational length L_{neutron} has appeared – for essentially dimensional reasons – in a quite different context (Berry 1982): the size of the interference fringes decorating a caustic formed by neutrons sprayed upwards from a small source.)

Another interesting analogue is the Penning trap (Dehmelt 1990), for charged particles (e.g. electrons). These can be held in stable vertical equilibrium against gravity by a quadrupole electric field. The potential of this electric field is analogous (table 1) to the magnetic field strength that provides the potential (2.7) for the dipole in the spinning top. But the electric potential, unlike the magnitude of a magnetic field, cannot possess a minimum, so there is no counterpart of the small regions of static stability found for the top in § 2 and § 3. This means that Earnshaw's theorem would apply, and horizontal motion in the electric field would always be unstable. To provide horizontal stability, the Penning trap includes a strong magnetic field, giving a Lorentz force. This is analogous to the geometric magnetic force on the top (§ 4), which can provide horizontal stability even outside the interval of static stability (§ 5).

The parameter ranges in which the top and Penning trap operate are completely different. The top operates near the border of stability, where geometric magnetism is a small effect, so the discriminant $\beta^2 - 4\alpha$ in (5.8), which must be positive in order that the frequency ω be real, is small. A consequence of this is that the three modes of small oscillation about stable equilibrium have comparable frequencies ($\omega \sim \sqrt{\alpha}$), and the small oscillations are roughly in oblique ellipses, in accordance with the observed bobbing and weaving. In the Penning trap, however, the magnetic field (analogous to geometric magnetism) is large, so $\beta^2 \gg 4\alpha$ and the three frequencies are very different: Larmor winding frequency $\beta \gg$ vertical oscillation frequency $\sqrt{a} \gg$ magnetron frequency α/β.

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