Geometric angle for rotated rotators, and the Hannay angle of the world

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Abstract. A simple formula is derived for the shift in angle variable (Hannay angle) arising from a slow (adiabatic) cycle of a parameter in a one-dimensional classical system. The formula is illustrated by numerical computations for different degrees of smoothness of the adiabatic driving. If the driving is smooth enough, the adiabatic invariant is sufficiently well conserved to enable fluctuations in the frequency to be neglected when computing the dynamical angle contribution to the final angle. If not (e.g. if the driving is uniform over a finite time), these fluctuations must be taken into account. The Hannay angle appears as a small change in period of a celestial body (Earth) rotating about another body (Sun) caused by the slow revolution of a third body (Jupiter).

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1. Introduction

Hannay’s angles [1] are geometric shifts in the angle variables of a classical integrable system resulting from an adiabatic cycle of its Hamiltonian. Here we study the special class of rotated rotators: one-dimensional systems with a potential energy whose (periodic) coordinate $q$ is replaced by $q - X(T)$, where the parameter $X$ increases from 0 to $2\pi$ over a time $T$ long compared with the natural rotation period. The paper has two purposes: first, to provide (long overdue) numerical demonstrations of Hannay’s angle, and second, to focus attention on a class of applications in astronomy.

In this section we derive a general formula for the Hannay angle for systems of this type. In sections 2 and 3 we show how it can be computed. As emphasized by Golin [2], this is not a trivial matter, and depends on the smoothness of the parameter function $X(T)$. In section 2 we consider $X(T)$ that are smooth enough for fluctuations in the adiabatic invariant to be neglected, giving two examples. In one, all derivatives are continuous; in the other, the first derivative is continuous but higher derivatives are not. Section 3 deals with the (exactly solvable) case where $X(t)$ increases uniformly over the time $T$ and its derivative jumps discontinuously at the beginning and end of the interval when the parameter rotation is switched on and off: in this case care must be taken when computing the dynamical part of the angle because fluctuations in the adiabatic invariant contribute finite amounts when integrated over the time $T$. In section 4 the theory is applied to compute the ‘Hannay angle of the world’.

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The Hamiltonian is
\begin{equation}
H(q,p;t) = \frac{1}{2} p^2 + V(q - X(t))
\end{equation}
where the potential function \( V \) has period \( 2\pi \). The equation of motion is
\begin{equation}
\ddot{q} = -V'(q - X(t)).
\end{equation}
This could describe a bead sliding round a circular loop of wire and forced by an external source that is slowly (i.e. \( |\dot{X}| \ll |\dot{q}| \)) moved in a circle parallel to the loop; another (approximate) realization of this system, in astronomy, will be given in section 4. We assume that the system is rotating before the parameter starts to change. Then, at \( t = 0 \), \( X \) is slowly increased from 0 to \( 2\pi \) over the long time \( T \).

Derivation of the Hannay angle begins with the result [3]
\begin{equation}
\theta_H(I) = -\partial_I \int dX \langle p \partial_X q \rangle.
\end{equation}
Here the brackets denote torus averaging, that is averaging over the angle variable for fixed action \( I \) and frozen parameter \( X \). To evaluate the integrand we start with the generating function for the canonical transformation from \( (q,p) \) to \( (\theta,I) \) at fixed \( X \):
\begin{equation}
S(q,I;X) = \int_0^q dq' \sqrt{2 \left[ E(I) - V(q' - X) \right]} = \int_{-X}^{q-X} d\xi \sqrt{2 \left[ E(I) - V(\xi) \right]}.
\end{equation}
\( E(I) \) is the energy of the torus labelled by \( I \).

From \( S \) follows the equations of the transformation, namely
\begin{equation}
\theta(q,I;X) = \left( \frac{\partial S}{\partial I} \right)_{q,X} = \omega(I) \int_{-X}^{q-X} d\xi \sqrt{2 \left[ E(I) - V(\xi) \right]}
\end{equation}
and
\begin{equation}
p(q,I;X) = \left( \frac{\partial S}{\partial q} \right)_{I,X} = \sqrt{2 \left[ E(I) - V(q - X) \right]}
\end{equation}
where \( \omega(I) \) is the frequency of motion on the torus \( I \). Note that \( \theta \) is the angle variable conjugate to \( I \), and is to be distinguished from the original coordinate \( q \) which for rotators is also an angle; for frozen \( X \), the evolution of \( \theta \) is \( \theta = \theta_0 + \omega t \). Because \( V \) is periodic, neither the action
\begin{equation}
I(E) = \frac{1}{2\pi} \int p dq = \frac{1}{2\pi} \int_0^{2\pi} d\xi \sqrt{2 \left[ E - V(\xi) \right]}.
\end{equation}
nor the frequency
\[ \omega (I) = \left( \frac{\partial E}{\partial I} \right) = 2\pi \left\{ \int_{0}^{2\pi} \frac{d\xi}{\sqrt{2 \left[ E - V(\xi) \right]}} \right\}^{-1} \quad (8) \]
depends explicitly on \( X \). Therefore the frequency does not depend explicitly on time, and not only the action but also the energy is adiabatically invariant.

Combining the elementary relation
\[ \left( \frac{\partial \theta}{\partial q} \right)_{I,X} = \left( \frac{\partial p}{\partial I} \right)_{q,X} = \frac{\omega (I)}{p(q,I;X)} \quad (9) \]
with (5) and (8) gives
\[ \left( \frac{\partial q}{\partial X} \right)_{I,\theta} = -\left( \frac{\partial \theta}{\partial X} \right)_{q,I} = 1 - \frac{p(q,I;X)}{p(0,I;X)} . \quad (10) \]

Thus the angle average in (3) becomes
\[
\langle p \partial X q \rangle \equiv \frac{1}{2\pi} \int_{0}^{2\pi} d\theta p \partial X q = \frac{1}{2\pi} \int_{0}^{2\pi} dq \partial_{\theta} p \partial X q \\
= \frac{1}{2\pi} \omega (I) \int_{0}^{2\pi} dq \partial X q = \omega (I) \left[ 1 - \frac{I}{p(0,I;X)} \right] . \quad (11)
\]

Now further use of the periodicity of \( V \) yields the desired formula for the Hannay angle, and our main analytical result:
\[ \theta_{H}(I) = 2\pi \left( 1 - \partial_{I} \omega \right) = 2\pi \left( 1 - \omega \partial_{E} \omega \right) . \quad (12) \]

2. Smooth driving

After the parameter rotation, the angle variable \( \theta(T) \equiv \theta(q(T), I(T), 2\pi) \) and the total angle shift from \( \theta(0) \equiv \theta_{0} \) is
\[ \Delta \theta \equiv \theta(T) - \theta_{0} \quad (13) \]

Hannay’s angle is what remains in the adiabatic limit after subtraction of the dynamical angle
\[ \theta_{d} \equiv \int_{0}^{T} dt \omega (I(t), X(t)) \quad (14) \]
Thus
\[ \theta_H = \lim_{T \to \infty} (\Delta \theta - \theta_d) \]  
\( (15) \)

\( \Delta \theta \) can be computed quite accurately by solving the equation of motion (2) with a standard fourth-order Runge–Kutta algorithm. Therefore the accuracy with which the Hannay angle can be computed is limited by the accuracy in computing the dynamical angle.

Since for the rotators we are studying the frequency is independent of \( X \), the dynamical angle is
\[ \theta_d \equiv \int_0^T dt \omega (I (t)) \]  
\( (16) \)

Therefore the fluctuations in \( \omega \) arise entirely from fluctuations in \( I \). While \( X \) is changing, these are of order \( T^{-1} [4, 5] \), but if the change is sufficiently smooth these fluctuations are averaged away by the time integration in (16). ‘Sufficiently smooth’ means that the first derivative \( \dot{X} \) must be continuous. If this is true—and in the next section we shall see what happens if it is not—it is sufficient to approximate (16) by
\[ \theta_d = \omega (I) T \]  
\( (17) \)

where \( I \) is the initial (≈ final) action, and compute the Hannay angle as
\[ \theta_H = \lim_{T \to \infty} (\Delta \theta - \omega (I) T) \]  
\( (18) \)

For calculations, a convenient potential is
\[ V (q) = V_0 \cos q \]  
\( (19) \)

For this, (12) gives the Hannay angle as
\[ \theta_H = 2\pi \left\{ 1 - \frac{\omega^3}{\pi} \int_0^{\pi} d\xi \left[ \frac{\omega}{2 (E - V_0 \cos \xi)} \right]^{3/2} \right\} \]  
\( (20) \)

Here
\[ \omega (E) = \frac{\pi \sqrt{\frac{1}{2} (E - V_0)}}{K (m (E))} \]  
\( (21) \)

where \( K \) is the complete elliptic integral of the first kind [6], evaluated at the negative argument
\[ m (E) = 2V_0 (V_0 - E) \]  
\( (22) \)

For the choice \( V_0 = 1, E = 3/2 \), (20) gives \( \theta_H = -1.660139 \).

To compute the Hannay angle numerically, we integrated the equation of motion
\[ \ddot{q} = V_0 \sin (q - X (t)) \]  
\( (23) \)
to find $q(T)$, starting with $q_0 = 0$ (i.e. $\theta_0 = 0$) and a given value of $\dot{q}_0$ (which fixes the initial energy). To transform to the final angle variable $\theta(T)$, we used the following formula, valid at the initial and final times:

$$\theta (q) = 2\pi \text{int} \left( \frac{q}{2\pi} \right) + \omega \int_0^{q'} \frac{d\xi}{\sqrt{2[E - V_0 \cos \xi]}}.$$  \hfill (24)

Here $q' \equiv q - 2\pi \text{int}(q/2\pi)$.

We carried out calculations for two different parameter functions (figure 1):

\[ X_A (t) = \begin{cases} 
0 & (t \leq 0) \\
\pi \left\{ \tanh \left[ \tan \left( \pi \left( \frac{t - \frac{1}{2}}{T} \right) \right) \right] + 1 \right\} & (0 < t < T) \\
\frac{2\pi}{T} & (t \geq T)
\end{cases} \]  \hfill (25)

and

\[ X_B (t) = \begin{cases} 
0 & (t \leq 0) \\
\pi \left\{ 1 - \cos \left( \frac{\pi t}{T} \right) \right\} & (0 < t < T) \\
\frac{2\pi}{T} & (t \geq T)
\end{cases} \]  \hfill (26)

For $X_A$, all derivatives are continuous, whereas for $X_B$ the second and higher derivatives are discontinuous at $t = 0$ and $t = T$.

Figure 2 shows the results of these calculations, with $V_0 = 1$ and $\dot{q}_0 = 1$, i.e. $E = 3/2$. The quantity $\theta(T) - \omega T$ is plotted against $1/T$ for a range of large $T$ values. To compute $\theta_H$ we fitted the data with a quadratic function and extrapolated to the origin $T = \infty$. For $X_A(t)$, this gave a value differing from theory (equation (20)) by only 0.006%. For $X_B(t)$, the corresponding figure was 0.003%, although convergence to this value was slower than for $X_A$, as expected since $X_B$ is less smooth. Similar excellent agreement was found with other initial conditions.
Figure 2. Approach to the Hannay angle: data computed as explained in section 2, for $V_0 = 1$, $\theta_0 = 0$, $E = 3/2$, with Runge–Kutta step size 0.01. Asterisks: $X_A(t)$, giving asymptotic value $\theta_H = -1.660237$. Circles: $X_B(t)$, giving asymptotic value $\theta_H = -1.660184$. The theoretical value is $\theta_H = -1.660139$.

3. Uniform driving, with discontinuous end derivatives

Uniform driving means

$$X_C(t) = \begin{cases} 0 & (t \leq 0) \\ \Omega t & (0 < t < T) \\ 2\pi & (t \geq T) \end{cases} \quad (27)$$

where $\Omega \equiv 2\pi / T$ is the slow frequency of the system. Although the discontinuous derivative in $X_C$ makes numerical evaluation of the Hannay angle problematic, as we shall see, uniform driving is important in applications and its simple form enables the motion to be found exactly by moving to a coordinate frame where the parameter is frozen.

Defining the moving coordinate by

$$\xi \equiv q - \Omega t \quad (28)$$

simplifies the equation of motion to

$$\ddot{\xi} = -V'(\xi) \quad (29)$$

with initial conditions related to those in the original frame by

$$\xi_0 = q_0, \quad \dot{\xi}_0 = \dot{q}_0 - \Omega \quad (30)$$

The moving energy

$$E \equiv \frac{1}{2} \dot{\xi}^2 + V(\xi) \quad (31)$$

is conserved. It is related to the non-conserved energy in the original frame, namely

$$E(t) = H(q(t), p(t); t) = \frac{1}{2}\dot{q}^2 + V(q(t) - \Omega t) \quad (32)$$
by
\[ \mathcal{E} = E(t) - \Omega \dot{q}(t) + \frac{1}{2} \Omega^2 \]
Solving (31) gives the exact solution \( \xi(t) \) in the moving frame:
\[ t = \int_{\xi_0}^{\xi(t)} \frac{d\xi}{\sqrt{2[E - V(\xi)]}} \quad \text{for} \quad 0 < t < T \tag{34} \]
In the original frame, the solution is
\[ q(t) = \xi(t) + \Omega, \quad \dot{q}(t) = \Omega + \sqrt{2[E - V(\xi(t))]} \tag{35} \]
For the potential (19), the moving coordinate can be expressed exactly in terms of the Jacobi elliptic function \( \text{sn} \) [6], as
\[ \xi(t) = 2 \arcsin \{ \text{sn}(u(t) | \mathcal{M}) \}, \quad u(t) = (t + \beta) \sqrt{\frac{2}{E - V_0}}, \quad \beta = \int_{0}^{\theta_0} \frac{d\xi}{\sqrt{2[E - V_0 \cos \xi]}} \tag{36} \]
where \( \mathcal{M} = m(E) \) is given by (22). The period of motion in the moving frame is
\[ \tau(E) = \frac{2\pi}{\omega(E)} = 2K(\mathcal{M}) \sqrt{\frac{2}{E - V_0}} \tag{37} \]
When using these formulae, care must be taken to choose the correct branch of the \( \arcsin \) function for each phase of the motion.

The procedure for computing the angle shift \( \Delta \theta \) is as follows. Select the total time \( T \), choose the initial angle \( \theta_0 \), and specify the initial torus by \( E \). Use (24) to obtain the initial coordinate as \( q_0 = 2 \arcsin[\text{sn}(u_0|m)] \) with \( u_0 = \theta_0 \sqrt{(E - V_0)/\omega(E)} \). Invert (32) for the initial speed \( \dot{q} \), and substitute into (33) to get the moving energy \( \mathcal{E} \). Then compute \( q(T) \) using (35) and (36). Finally, use (24) again to obtain the final angle \( \theta(T) \), and hence \( \Delta \theta \).
Figure 5. Adiabatic vanishing of the deviation $\delta \theta_H$ (equation (38)), that is convergence to the Hannay angle, for (*) $\log_{10}(|\langle \delta \theta \rangle|); (+) \log_{10}(|\langle \delta \theta^2 \rangle|)$. Averaging is over initial angle. The slope of both curves is close to $-1$, showing convergence as $1/T$.

Figure 3 shows the result of calculating the Hannay angle from $\Delta \theta$ using the naive subtraction (18), in which fluctuations of the adiabatic invariant are neglected. Clearly this procedure fails, because as $T \to \infty$ the limit in (18) converges not to the constant value of $\theta_H$ given by (20) but to a curve with strong dependence on $\theta_0$. The correct procedure is to subtract from $\Delta \theta$ the full dynamical angle (16), incorporating the action fluctuations: as figure 4 shows, this does lead to $\theta_H$ in the adiabatic limit. In the different context of multidimensional systems, the need to include the action fluctuations in the dynamical angle has been emphasized before [7] as a way to minimize pollution of the Hannay angle by temporary capture into resonances in near-integrable systems.

Figure 4. Exact numerical calculation of the Hannay angle $\Delta \theta - \theta_d$, as a function of the initial angle $\theta_0$, for uniform driving (section 3), with $V_0 = 1$, $E = 3/2$. Full curves from bottom to top: $T = 5 \times 10^2$, $10^3$, $5 \times 10^3$, $10^4$, $10^5$. The theoretical value, calculated from (20), is $\theta_H = -1.660139$. 
$\theta_d$ was determined to a sufficient approximation by computing the contribution from a single cycle in the moving frame, and then multiplying by the appropriate number of cycles given by the ratio $T/\tau(\mathcal{E})$. The deviations

$$\delta \theta_H \equiv \Delta \theta - \theta_d - \theta_H$$

(38)

for finite $T$ can be seen in figure 4. On average, these deviations vanish as $1/T$, as can be seen from figure 5.

Now we explain the origin of the error made by neglecting the action fluctuations, by calculating $\theta_H$ as given by (18) directly from the exact solution (34). After a complete parameter cycle, we have, again using the periodicity of $V$,

$$T = \int_{q_0}^{q(T)} \frac{dq}{\sqrt{2[E - V(q)]}} = \int_{q_0}^{q(T)} \frac{dq}{\sqrt{2[E - V(q)]}} - \tau(\mathcal{E})$$

(39)

Rearrangement gives the exact relation

$$\omega(\mathcal{E}) \left[ \int_{q_0}^{q(T)} \frac{dq}{\sqrt{2[E - V(q)]}} - T \right] = 2\pi$$

(40)

This is similar to the expression for $\Delta \theta - \omega T$ obtained using (5), but involves $\mathcal{E}$ rather than $E$. To relate the two formulae, we expand the LHS of (40) to lowest order in $\Omega$ using (33) at $t = 0$. Thus

$$\Delta \theta - \omega T = \omega(E) \left[ \int_{q_0}^{q(T)} \frac{dq}{\sqrt{2[E - V(q)]}} - T \right]$$

$\omega(\mathcal{E}) \left[ \int_{q_0}^{q(T)} \frac{dq}{\sqrt{2[E - V(q)]}} - T \right] = 2\pi - \dot{q}_0 \omega \int_{q_0}^{q(T)} \frac{dq}{(2[E - V(q)])^{3/2}}$

$$\lim_{T \to \infty} 2\pi - \dot{q}_0 \omega^2 \Omega T \frac{2\pi}{2} \int_0^{2\pi} \frac{dq}{(2[E - V(q)])^{3/2}}$$

$$= 2\pi (1 - \dot{q}_0 \partial_q \omega)$$

(41)

where $E = E_0$ and $\omega = \omega(E)$. The difference between this formula and the Hannay angle (12) expresses in a concise way the error committed by neglecting action fluctuations when subtracting the dynamical angle. The dependence on initial angle, through $\dot{q}_0 = \sqrt{2(E - V(q_0))}$, is clear, and confirmed by figure 3. In the perturbative case $V << E$, the result (41) was obtained by Golin [2].

Another way to remove the fluctuations in the Hannay angle as naively computed from (18) is by averaging over $\theta_0$. This follows from (41) and

$$\langle \dot{q}_0 \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} d\theta_0 \dot{q}_0(\theta_0) = \frac{1}{2\pi} \int_0^{2\pi} dq_0 \frac{\partial \theta_0}{\partial q_0} q_0 = \omega$$

(42)

where in the last step we have used (9). (This result can also be derived by using ergodicity to replace the torus average, over $\theta_0$, by a time average.) This was also pointed out by Golin [2]. Thus, in a numerical calculation of the Hannay angle we may either compute the dynamical angle correctly as in (16) and thereby detect $\theta_H$ in a single trajectory, or pretend the action is adiabatically invariant and use the subtraction (18) followed by averaging over initial angles (phases) of the motion.
4. Celestial Hannay angles

In the absence of other planets, the Earth’s Newtonian orbit around the Sun would be a Kepler ellipse, that is, a degenerate integrable motion in the plane, with two equal frequencies. The main perturbation, from Jupiter, breaks this degeneracy, changes the orbital period, and causes the orbit to precess slowly. These effects would exist even if Jupiter were stationary. Here we discuss a small additional effect resulting from Jupiter’s revolution in its own orbit. This revolution (period 11.87 years) is slow compared with the Earth’s but fast compared with the precession (period about $10^5$ years). Therefore it seems reasonable to make the physical approximation that the orbital precession can be neglected in calculating the Hannay angle associated with the Earth’s annual motion as driven adiabatically by Jupiter.

The Hannay angle implies a shift in the position of the Earth in its orbit after one Jupiter period, that is (after dividing by 11.87) a change in the length of the year as computed from the frequency that would be appropriate for a stationary Jupiter. With the above approximation—which amounts to replacing two dimensions by one and effectively constraining the Earth to move on a fixed curve, which because of the small eccentricity of the Earth’s orbit can be further approximated by a circle—the motion is described by the Hamiltonian (1) that we have been considering.

It is not hard to think of similar examples: Mars is also adiabatically driven by Jupiter, and a geosynchronous Earth satellite is adiabatically driven by the Moon. In an extensive analysis, Golin et al [7] explore what appears to be a related effect, that is the Hannay angle of a satellite driven by slowly-rotating nonsphericities on the central body; however, we are unable to make the expected connections between their formulae and ours.

![Figure 6. Geometry for 'The Hannay angle of the world'. E is the Earth, S the Sun and J Jupiter.](image)

The geometry is shown in figure 6. In our previous notation, $T$ is the time for a single orbit of Jupiter round the Sun, $\Omega = 2\pi / T$ is the adiabatic driving frequency, and $\omega = 2\pi$ radians per year is the (fast) frequency of the Earth’s orbital motion. In the appendix we show that the approximate equation of motion for the Earth is (29) with potential (19), where the strength $V_0$ of the potential, which has the dimensions of frequency squared, is

$$V_0 = -\Omega^2 \frac{M_J R_I}{M_S R_E}$$

(43)

where $M$ and $R$ denote masses and orbital radii and E, J, S denote Earth, Jupiter and Sun. This amplitude is very small compared with the kinetic energy $\dot{q}^2 / 2$ of the Earth’s orbital
angular motion. In years\(^{-2}\) the latter is \(2\pi^2 \approx 19.7\), whereas \(V_0 \approx 1.39 \times 10^{-3}\); the ratio of potential to kinetic energies is about 1/14,000—this situation is perturbative as well as adiabatic, and motion lies far above the separatrix.

To compute the Hannay angle from (12), it is necessary first to determine how the energy depends on the action from (7). Because \(V_0\) is so small, it suffices to expand the square root in (7) to lowest non-trivial order:

\[
I = \sqrt{2E \left(1 - \frac{V_0^2}{16E^2}\right)}, \quad \text{i.e.} \quad E = \frac{I^2}{2} + \frac{V_0^2}{4I^2}
\]  

(44)

The frequency is

\[
\omega = \frac{\partial E}{\partial I} = I - \frac{V_0^2}{2I^3}
\]

(45)

Equation (12) now gives the Hannay angle as

\[
\theta_H = -\frac{3\pi V_0^2}{I^4}
\]

(46)

From (45), the action is very close to the Earth’s orbital frequency \(\omega \approx 2\pi\) per year. Combining this with (43) and using Kepler’s third law, we obtain the final result

\[
\theta_H = -3\pi \left(\frac{\Omega}{\omega}\right)^4 \left(\frac{M_J R_J}{M_S R_E}\right)^2 = -3\pi \left(\frac{R_E}{R_J}\right)^4 \left(\frac{M_J}{M_S}\right)^2
\]

(47)

for the Hannay angle per period of Jupiter.

The Hannay angle per orbit (period \(\tau = 1\) year) is

\[
\Theta_{H, \text{ orbit}} = \frac{\theta_H \tau}{T} = -3\pi \left(\frac{R_E}{R_J}\right)^{11/2} \left(\frac{M_J}{M_S}\right)^2 = -3\pi \left(\frac{\tau}{T}\right)^{11/3} \left(\frac{M_J}{M_S}\right)^2
\]

\[
= -2.04 \times 10^{-4}\text{arc sec.}
\]

(48)

We have confirmed this value by numerical integration as in section 3. \(\Theta_{H, \text{ orbit}}\) is the Hannay angle of the world. It can be interpreted in several ways. For motion so far above the separatrix, \(q \approx \theta\). Therefore the shift (48) corresponds to a shift, in one year, of the Earth’s angular position. This is equivalent to a shortening of the year, compared to what would be calculated from the dimensions of the orbit ignoring Hannay’s angle, of

\[
\frac{\Delta \tau}{\tau} = \frac{\Theta_{H, \text{ orbit}}}{2\pi} = -1.6 \times 10^{-10} = -5 \text{ms yr}^{-1}
\]

(49)

This corresponds an annual displacement of the Earth in its orbit by

\[
d_H = R_E \Theta_{H, \text{ orbit}} \approx -150 \text{ m}
\]

(50)

The numerical data for this and the analogous shifts in the orbits of Mars and a geosynchronous Earth satellite are given in table 1.

The Hannay effect is very small, and could be difficult to observe in the presence of several larger shifts. We mention two of these. First, even the attraction of a stationary
Table 1. Orbital data [9] and Hannay angles for three celestial examples of a test body (Te) orbiting an attractor (A) and perturbed by a slowly rotating third body (P). E: Earth; S: Sun; J: Jupiter; Ma: Mars; Sa: geosynchronous satellite; Mo: Moon; τ: orbital period of Te; T: orbital period of perturber; MA: mass of attractor; MP: mass of perturber; RT: orbital radius of Te; RP: orbital radius of P; θ_{H, orb}: Hannay angle per orbit of Te (equation (48)); Δτ: resulting change in orbital period of Te (equation (49)); d\_H: annual shift of Te along its orbit (equation (50))

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<th>Te, A, P</th>
<th>E, S, J</th>
<th>Ma, S, J</th>
<th>Sa, E, Mo</th>
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<td>1.000 day</td>
</tr>
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<td>11.867 yr</td>
<td>27.32 day</td>
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<td>RT</td>
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<td>1.5237 au</td>
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<td>RP</td>
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<td>5.2028 au</td>
<td>384 400 km</td>
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<tr>
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<td>−1.6 \times 10^{-3}</td>
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<td>Δτ (ms)</td>
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<tr>
<td>d_H (m)</td>
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Jupiter causes a perturbative change in the year. An estimate of the size of this effect, based on (45), is

\[
\frac{Δτ}{τ} = \frac{Δω}{ω} = - \frac{V_J^2}{2κ^2} = \frac{1}{2} \left( \frac{R_E}{R_J} \right)^4 \left( \frac{M_J}{M_S} \right)^2
\]

\[= -6.2 \times 10^{-10} = -20 \text{ ms yr}^{-1}. \quad (51)\]

This is about four times larger than the Hannay effect, and corresponds to an annual orbital shortfall of about 600 m.

The second effect is much larger, and arises from the fact that the orbit of the Earth is of course not a circle but an ellipse with eccentricity \(e \approx 0.0167\), which precesses because of the attractions of the other planets. The amount of this precession, measured in an inertial frame fixed relative to the solar system, is about \(α = 5.8 \times 10^{-5} \text{ rad yr}^{-1}\) [8]. Over a year (that is, between times when the line from the Sun to the Earth makes the same direction relative to the stars) this causes a radial displacement whose magnitude depends on the time of year; the maximum, occurring halfway between perihelion and aphelion, is, by a simple calculation,

\[|ΔR_E| = α e R_E \approx 140, 000 \text{ m} \quad (52)\]

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Appendix

Referring to figure 6, we let \( \mathbf{R} \) be the position vector of Jupiter relative to the Earth, and \( \sigma \) the angle made by \( \mathbf{R} \) with the position vector \( \mathbf{R}_E \) of the Earth relative to the Sun. Then

\[ R = \sqrt{R_E^2 + R_J^2 - 2R_E R_J \cos ξ}, \quad \sin σ = \frac{R_I}{R} \sin ξ. \quad (A1) \]
The tangential force and acceleration are related by Newton’s law:

\[ M_E R_E \ddot{q} = -F \sin \sigma, \quad \text{where} \quad F = G \frac{M_E M_J}{R^2}. \tag{A2} \]

Thus the equation of motion for the angular coordinate of the Earth forced by Jupiter is

\[ \ddot{q} = -G \frac{M_J R_J}{R_E} \frac{\sin \xi}{\left( R_E^2 + R_J^2 - 2 R_E R_J \cos \xi \right)^{3/2}} = -\frac{\partial V(\xi)}{\partial \xi} \tag{A3} \]

where the potential function is

\[ V(\xi) = -\frac{G M_J}{R_E^2 \sqrt{R_E^2 + R_J^2 - 2 R_E R_J \cos \xi}} + \text{constant}. \tag{A4} \]

Expanding in powers of \( R_E/R_J \), and retaining only the largest non-constant term, gives precisely the potential (19) with the moving coordinate \( \xi \), with the strength \( V_0 \) given, after applying Kepler’s third law, by (43).

The above argument gives the Hannay angle, change in orbital period, and orbital displacement for any test body \( T_e \) orbiting an attractor \( A \) and perturbed by a distant body \( P \); it is necessary only to replace \( (E, S, J) \) by \( (T_e, A, P) \).

References

[8] Murray C Private communication