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Transparent mirrors: rays, waves and localization

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Abstract. A stack of $N \gg 1$ transparent plates with randomly varying thicknesses (e.g. viewgraphs) reflects light perfectly, as a result of the accumulation of reflections from interfaces at the air gaps separating the plates. Two theories of this effect are discordant. The naive ray theory assumes that the random phases associated with the thickness variations make all the reflections incoherent, and predicts that the transmitted intensity decays as $1/N$. This theory is wrong because some distinct multiply reflected waves have identical path lengths and so superpose coherently. The true decay is exponential: exact averaging of the logarithm of the transmitted intensity over the random phases, assuming these are uniformly distributed modulo 2π, gives the transmitted intensity as $\tau^{2N} = \exp\{-2N \log(1/\tau)\}$, where τ is the intensity transmittance of a single interface. Transparent mirrors are naked-eye examples of the localization of light, for which the localization length (inverse decay exponent) can be calculated exactly. Experiments confirm the exponential decay.

Zusammenfassung. Ein Stapel von $N \gg 1$ transparenten Platten mit beliebig variierender Dicke (z.B.Overhead-Folien) reflektiert Licht perfekt. Diese perfekte Reflexion ist das Ergebnis der Akkumulation von Reflexionen an den Grenzflächen der Luftspalte zwischen den Platten. Die zwei Theorien dieses Effektes stimmen nicht überein. Die naive Strahlen-Theorie nimmt an, daß die beliebigen Phasen, die mit den Variationen der Plattendicke verknüpft sind, zu absolut inkoherenten Reflexionen führen und sagt einen $1/N$ Verlauf für die durchgelassene Intensität voraus. Diese Theorie ist falsch, da einige der mehrfach reflektierten Wellen identische Weglängen haben und daher koherent überlagern. Der wahre Verlauf ist exponential: exakte Mittelung des Logarithmus der durchgelassenen Intensität über die beliebigen Phasen, unter der Annahme sie seien gleichmässig verteilt modulo 2π, führt zur durchgelassenen Intensität $\tau^{2N} = \exp\{-2N \log(1/\tau)\}$, wo τ die durchgelassene Intensität einer einzelnen Grenzfläche ist. Durchsichtige Spiegel sind Beispiele der Lokalisation des Lichtes für das unbewaffenete Auge, für die die Lokalisationslänge (inverser Exponentialkoeffizient) exakt berechnet werden kann. Experimente bestätigen den exponentialen Verlauf.

1. Introduction

It is well known that a pile of transparent plates can act as a mirror, and the first of many theoretical treatments of the effect was apparently (Tuckerman 1947) given by Fresnel as long ago as 1821. Nowadays the phenomenon is a matter of common observation (Hecht and Zajac 1974): a stack of microscope slides, or viewgraphs (overhead-projector transparencies), or a roll of transparent microwave food wrap, gives clear bright reflections. Such a stack is a sequence of optical elements consisting of alternating *spaces* and *faces*. The spaces are transparent films or air gaps, and the faces are the boundaries (interfaces) separating the spaces. The mirror effect arises from single and multiple reflections at the faces, and the phases of the reflected waves are determined by the spaces. (Ordinary, that is, non-microwave, food wrap ('clingfilm') has films in close contact, so there are no air gaps and the mirror effect is often weak.)

In the familiar transparent mirrors (e.g. viewgraphs), the spaces of the same kind (air or film) are not identical, but have thicknesses varying erratically over several wavelengths, giving rise to random phases that can be regarded as independently and uniformly distributed modulo 2π. It is precisely this case of 'strong disorder' with which we shall be concerned here. Since all the light that is not transmitted is reflected, the development of the transparent mirror as the number N of faces increases is related to the decay of the transmitted intensity with N. Disorder is essential; without it—that is, for a stack of identical films—most frequencies would be transmitted (Born and Wolf 1959) and the reflections would be dim.

One of our aims in returning to this ancient problem is to point out the discordance between two theories of transparent mirrors. The first is the 'naive ray' picture (e.g. Stokes 1862, Baumeister *et al* 1972), in which it is considered that the random phases render all waves incoherent, so that the multiple reflections between the faces can be treated with intensities rather than amplitudes. The result is that the light intensity decreases linearly with N. The second is a wave picture, in which the multiple reflections are incorporated into amplitudes and the resulting intensity appropriately (that is, logarithmically) averaged over the phases. This leads to a transmittance that is simply the product of the transmittances of the individual elements, so that the effect of multiple reflections on the transmitted light cancels by destructive interference. The light intensity decays exponentially with N, rather than linearly. Unsurprisingly the wave picture is right and the ray picture is wrong. We pinpoint the error. There is an irony in the wave result, because the formula that it produces has been criticized (Tuckerman 1947) on the erroneous grounds that it appears to neglect the multiple reflections that were incorporated into the naive ray theory.

The discordance resembles that between classical mechanics and the small-$\hbar$ limit of quantum mechanics in the limit of long times (Robbins and Berry 1992, Berry 1991), with the reflectivity of an individual interface playing the role of $\hbar$ and N playing the role of time.

Our other main aim is to connect the exponential decay with localization theory (Ziman 1979, Pendry 1994, Erdös 1982). It is a well known consequence of the theory of products of random matrices (Furstenberg 1963) that for large N the exponential describes not only the average decay but also the decay for almost all individual stacks. Many studies have explored the application of this theory to the difficult calculation of the decay exponent (inverse localization length) of piles of plates, for example numerically (Kondilis and Tzanetakis 1992) or at critical incidence (Bouchaud and Le Doussal 1986), and in more dimensions, for example in a study of the reflectivity of white paint (Anderson 1985). For strong disorder and normal incidence, however, it is possible to calculate the exponent *exactly* by the random phase approximation (Anderson *et al* 1980); this point was made earlier (Baluni and Willemsen 1985) in the equivalent context of acoustic transmission in a layered medium. Here we explain how the exact result can also be obtained from a general (and usually approximate) method for calculating exponents.

We have carried out experiments to measure the transmitted intensity as a function of N, and confirmed the exponential decay with the correct exponent.

For the optical phenomena we are interested in, absorption in the material of the films is unimportant, and so for simplicity of exposition we will ignore it almost completely.

2. Stack recursion equations for waves

Consider light with wavenumber k travelling along the x-axis and interacting with an optical element, denoted 1 (figure 1(a)), consisting of an arbitrary combination of spaces with refractive index n_i and widths d_i, separated by faces. Let the transmission and reflection amplitudes

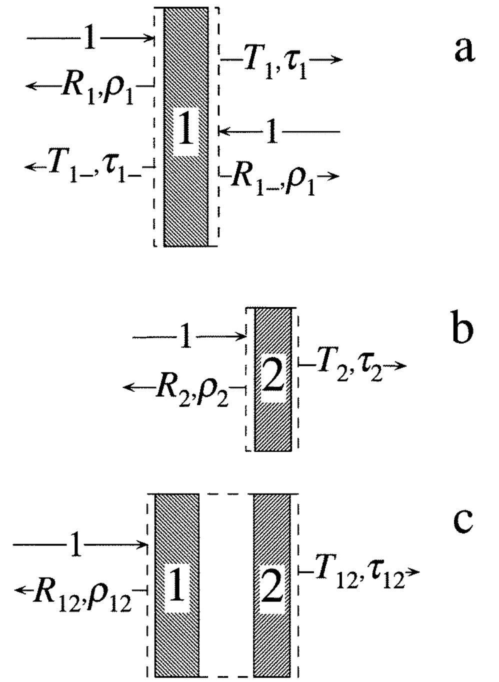

Figure 1. (a) Reflection and transmission coefficients for forward and backward ($-$) waves incident on an optical element 1 (arbitrary combination of faces and spaces); (b) reflection and transmission from an arbitrary optical element 2; (c) reflection and transmission from the combination 12 of the elements. Latin letters denote amplitude coefficients, and Greek letters denote intensity coefficients ($|\text{amplitude}|^2 \times$ refractive index).

be T_1 and R_1 for waves incident from the left ('forward' waves) and T_{1-} and R_{1-} for waves incident from the right ('backward' waves). In a medium with refractive index n_i, a wave such as T_i is represented by the function $(T_i/\sqrt{n_i})\exp\{ikn_ix\}$, and similarly for R_i. This definition ensures that the corresponding intensities, denoted by Greek letters, for example $\tau_1 = |T_1|^2$, $\rho_1 = |R_1|^2$, represent energy flows. Let T_2 and R_2 be the analogous transmission and reflection amplitudes for a second optical element (figure 1(b)), for forward waves.

Now imagine these two elements combined (figure 1(c)). We seek the transmission amplitudes T_{12} for the combination. Obviously this is the coherent sum of all the multiply reflected and transmitted waves, which form a geometrical series:

$$T_{12} = T_1T_2 + T_1R_2R_{1-}T_2 + T_1R_2R_{1-}R_2R_{1-}T_2 + \cdots = \frac{T_1T_2}{1 - R_{1-}R_2}. \tag{1}$$

Included in these coefficients are the random phases associated with the varying widths d_i of the spaces. In section 4 we shall calculate averages of the transmitted intensity $|T_{12}|^2$ over the d_i, and use these to iterate (1) and so obtain the transmission for a sequence of optical elements.

3. Naive ray theory

If all wave interactions are regarded as incoherent, the argument leading to (1) can still be employed, but with intensities replacing amplitudes. In this picture, light is considered as a stream of rays whose intensity, rather than amplitude, splits at each face. For the combinations of refracting plates we are interested in, intensities are reversible, so that $|R_{1-}|^2 = |R_1|^2$. Thus, the naive ray intensity transmitted by the combination 12 is

$$\tau_{12} = \frac{\tau_1\tau_2}{1-\rho_1\rho_2} \tag{2}$$

which incorporates all multiple reflections. For transparent plates,

$$\tau_1 + \rho_1 = \tau_2 + \rho_2 = 1. \tag{3}$$

Then (2) can be written as

$$\frac{1}{\tau_{12}} = \frac{1}{\tau_1} + \frac{1-\tau_2}{\tau_2}. \tag{4}$$

For a stack of N plates ($2N$ faces) with the same transmitted intensities τ, the formula (4) can be applied iteratively, with τ_2 representing the $2N$th face and τ_1 the remaining $2N-1$ faces. Then, since the transmission is unity when there are no plates, the transmission of the whole stack is

$$\tau_{N\text{ray}} = \frac{\tau}{\tau + 2N(1-\tau)}. \tag{5}$$

This is the result of the naive ray theory.

4. Exact wave averaging

Because of the randomness in the width of the spaces, the quantities in the wave recursion (1) will have random phases, and it is natural to average over them. For example, if the first optical element consists of the space n_1 for $0 < x < d$, and a face at $x = d$, then simple matching of the wave and its derivative at d gives the elementary space–face coefficients

$$R_1 = \frac{(n_1-n_2)}{(n_1+n_2)}\exp\{2\mathrm{i}n_1kd\}$$
$$T_1 = \frac{2\sqrt{n_1n_2}}{(n_1+n_2)}\exp\{\mathrm{i}n_1kd\}. \tag{6}$$

(We use the convenient convention that the phase of the reflected wave is measured from $x = 0$ and that of the transmitted wave from $x = d$.) Therefore the random phase is n_1kd, with d being the thickness of an air gap or transparent film. If the root-mean-square (rms) variation in d is Δd, the condition for the phases of R_1 to be uniformly distributed modulo 2π is

$$\Delta d \gg \frac{\pi}{n_1k} = \frac{\lambda}{2n_1} \tag{7}$$

where λ is the wavelength of the light and n_1 the index of air or film. This is certainly satisfied for uncontrolled films such as viewgraphs.

Now we can use (1) and write the average of any function of the transmitted intensity over the random phase of the second element as

$$\langle f(|T_{12}|^2)\rangle = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\, f\left(\frac{|T_1|^2|T_2|^2}{|1-|R_{1-}R_2|\exp\{\mathrm{i}\phi\}|^2}\right) \tag{8}$$

where ϕ is the phase of R_2 plus the (irrelevant) phase of R_{1-}. The choice $f(u) = \log u$ is uniquely appropriate (Anderson *et al* 1980), as will now be explained. From (8),

$$\langle\log(|T_{12}|^2)\rangle = \log(|T_1|^2) + \log(|T_2|^2) - \frac{\mathrm{Re}}{\pi}\int_0^{2\pi}\mathrm{d}\phi\,\log(1-|R_{1-}R_2|\exp\{\mathrm{i}\phi\}). \tag{9}$$

Because $|R_{1-}R_2| < 1$, the integral vanishes (this follows immediately on expanding the logarithm, or by contour integration). Thus

$$\langle\log(|T_{12}|^2)\rangle = \log(|T_1|^2) + \log(|T_2|^2) \tag{10}$$

indicating that the logarithmic averaging has decoupled the second element from the first (Baluni and Willemsen 1985).

For N plates there are $2N$ faces. If the sth face transmits intensity $\tau_s = |T_s|^2$, then iteration of (10) gives the effective (in the sense of log-averaged) transmission of the whole stack as

$$\tau_{N\text{eff}} \equiv \exp\langle\log(|T_{12\ldots 2N}|^2)\rangle = \prod_{s=1}^{2N}\tau_s = \exp\left\{\sum_{s=1}^{2N}\log\tau_s\right\}. \tag{11}$$

If the plates are made of the same material, all the τ_s have the same value, say τ, since the transmitted intensity is the same for air-to-glass as for glass-to-air (T_1 in equation (6) is symmetric under exchange of n_1 and n_2). Then

$$\tau_{N\text{eff}} = \exp\{-2N\log(1/\tau)\}. \tag{12}$$

It is hard to imagine simpler formulae than (11) and (12). The exact effective transmission for the whole stack is precisely what would be obtained by ignoring all multiple reflections and multiplying the transmissions at each interface. Of course, reflections have not been ignored: they are included but cancel exactly. One consequence of this is that the form of the expressions is independent of which combination of successive faces and spaces are regarded as the optical elements comprising the stack. For example, in a stack of viewgraphs, an element could be a single face together with the immediately preceding space (air or plastic); or, an element could be a single film together with the preceding air gap, that is two faces and two spaces.

Moreover, the formula also holds in the presence of weak absorption. This corresponds to the condition

$$\bar{d} \ll \frac{\pi}{k\,\mathrm{Im}\,n} = \frac{\lambda}{2\,\mathrm{Im}\,n}. \tag{13}$$

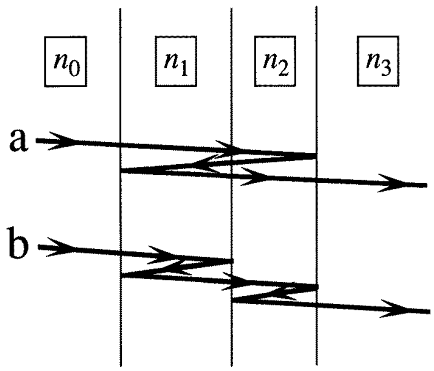

Figure 2. The simplest two different paths with the same length. Path a has no internal reflections; path b has two internal reflections, one of which contributes a phase π because the following space has larger index.

For the films we are interested in, this is comfortably satisfied (see section 7). Then the average over d is still a phase average, and (12) can be written, after using (6), with the explicit decay exponent

$$2\log\left(\frac{1}{\tau}\right) = 2k\bar{d}\,\mathrm{Im}\,n + \log\left\{\frac{(n+1)^4}{16n^2}\right\} \qquad (14)$$

where $\bar{d}$ is the average thickness of a film and now n denotes the real part of the refractive index.

5. Discordance

The ray and wave formulae (5) and (12) are very different. At first sight this is surprising because averaging over random phases would seem equivalent to regarding waves as incoherent. But the waves are not all incoherent: in the multiple-reflection expansion of the wave recursion equation (1) there can be different paths with the same length (that is, they visit each space the same number of times, albeit in different orders) and so contribute coherently to the wave amplitude. The simplest pair of such paths is shown in figure 2.

According to the exact result (12), the net effect of this coherence must be destructive, rather than constructive, interference, making the transmission decay exponentially rather than linearly. To see how this can arise, we note first that magnitude of paths with the same length can be ordered by the number of reflections at internal faces (in figure 2, two for the path b and zero for path a). Second, each such reflection by a face where the following space has a higher index contributes π to the phase associated with the path. In figure 2 this occurs for one of the additional reflections in the path b, so that this path interferes destructively with the dominant path a. It is clear that this is a general mechanism, not restricted to the paths in figure 2.

Figure 2 involves three faces. With only two, as for a single plate, all multiple reflections have different path lengths, and phase averaging does lead to incoherence. This can be seen by direct averaging the intensity over the thickness of the second element in figure 1; from (8) with $f(u) = u$,

$$\begin{aligned}\langle |T_{12}|^2\rangle &= \frac{|T_1|^2|T_2|^2}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\,\frac{1}{|1-|R_{1-}R_2|\exp\{\mathrm{i}\phi\}|^2}\\ &= \frac{|T_1|^2|T_2|^2}{1-|R_{1-}|^2|R_2|^2}.\end{aligned} \qquad (15)$$

This is identical with the naive ray theory recursion formula (2), provided two conditions are satisfied. First, the reflected intensity from the first element is reversible, that is $|R_{1-}|^2 = |R_1|^2$; this is the case for sequences of films (even with absorption, if the elements are correctly chosen). Second, the first element must not be composite. Then $|T_1|^2 = t$ and $|R_1|^2 = 1-\tau$, and (15) yields the ray formula (5) for two faces (that is, one film: $N = 1$). If the first element is composite, at least three faces are involved and the argument of the preceding paragraph applies. In (15) this means that $|R_{1-}|^2$ and $|T_1|^2$ still contain unaveraged phases within the first element, and are not equal to the corresponding classical ray intensities. Indeed, it can be shown that for three faces, or more than one film, further averaging of (15) does not yield the ray formula (5) (in contrast to what has been stated—see, for example, p 67 of Macleod (1986)), and indeed gets very complicated. Thus the 'incoherent' appearance of (15) is misleading.

The fact that the discordance originates in the ray theory's erroneous treatment of multiple reflections suggests that it will give correct results whenever such reflections can be neglected, that is, when the reflection from each individual face is weak (i.e. $\rho = 1-\tau \ll 1$) and the stack small enough for even weak reflections not to accumulate (i.e. N not too large). Indeed, (5) and (12) give

$$\tau_{N\mathrm{eff}} \approx \tau_{N\mathrm{ray}} \approx 1 - 2N\rho$$

when

$$N\rho = N(1-\tau) \ll 1. \qquad (16)$$

It is possible to regard $\rho \ll 1$ as analogous to the classical limit $\hbar \to 0$ of quantum mechanics, and $N \to \infty$ as analogous to the long-time limit. In the classical limit, the effects of wave interference in a quantum system can be neglected, but after a sufficiently long time (that gets longer as $\hbar$ gets smaller), the system reveals its quantum nature, and the classical predictions fail. One manifestation of this is the quantum suppression of classical chaos (Casati and Chirikov 1995). For transparent mirrors, (16) shows that the ray theory works when $\rho \ll 1$, but only while $N\rho \ll 1$: for sufficiently many films, multiple reflections are unavoidable however small ρ is, and the decay of transmitted intensity reveals the wave nature of light. The fact that for transparent mirrors phase averaging alone is not sufficient to reproduce the results of the ray theory—because of the coherence phenomenon illustrated in figure 2 and discussed above—also has analogies in quantum mechanics. For a quantum bound system with time-reversal symmetry, the fact that most classical periodic orbits have a time-reversed counterpart with the same action (i.e. phase) gives rise to coherence that changes the universality class of the statistics of the energy levels (Berry 1985).

6. Connection with localization

Each individual stack of N films will have different sets of $2N$ phases drawn from the uniform distributions, so that the curves of transmitted intensity against N for different stacks will fluctuate about the average (12). However, it follows from the theory of wave localization (Ziman 1979) that for $N \gg 1$ the intensity for almost all individual stacks will show the same exponential decay, whose common exponent (inverse localization length) is

$$-E \equiv \lim_{n\to\infty} \frac{\log(|T_{12\ldots 2N}|^2)}{N} = -2\log\frac{1}{\tau}. \qquad (17)$$

No averaging is necessary. The exponential decay shows that the light in the stack is localized by the disorder. The phenomenon is very general, and also occurs when the disorder is not strong, where the refractive index fluctuates from film to film, and where the index and phase fluctuations are (locally) correlated.

For electrons, the wave–ray discordance underlies the breakdown of Ohm's law for wires that are so thin that they are effectively one-dimensional (Erdös 1982). The resistance is proportional to the reciprocal of the transmission coefficient, which because of localization rises exponentially with length, instead of linearly as would be the case in the Ohmic regime where all reflections are regarded as incoherent.

Mathematically, this type of localization in one dimension is a consequence of Furstenberg's theorem (Furstenberg 1963) on products of $N \gg 1$ random matrices: under very general conditions, the elements of the matrix product, and indeed any norm of the matrix product, grow exponentially with the same exponent. This exponent is half of that giving the decay of the transmitted intensity.

Localization and Furstenberg's theorem are connected by *transfer matrices*, relating the forward- and backward-moving waves on the right side of each optical element to those on the left. It is not hard to show (e.g. with the help of figure 1(a)) that the matrix for the sth element is

$$\mathbf{m}_s = \frac{1}{T_{s-}}\begin{pmatrix} T_sT_{s-} - R_sR_{s-} & R_{s-} \\ -R_s & 1 \end{pmatrix}. \qquad (18)$$

For stacks of dielectric films (even absorbing ones), det $\mathbf{m} = 1$, so that the forward and backward transmission coefficients are the same. For transparent films, unitarity of the $\mathbf{S}$ matrix (relating ingoing and outgoing waves) gives

$$|T_s|^2 + |R_s|^2 = |T_{s-}|^2 + |R_{s-}|^2 = 1, \quad R_{s-}T_s^* + R_s^*T_{s-} = 0 \qquad (19)$$

so that $\mathbf{m}_s$ can be written

$$\mathbf{m}_s = \begin{pmatrix} \dfrac{1}{T_s^*} & -\dfrac{R_s^*}{T_s^*} \\ -\dfrac{R_s}{T_s} & \dfrac{1}{T_s} \end{pmatrix}. \qquad (20)$$

The matrix for the whole stack, regarded as $2N$ space–face combinations, is

$$\mathbf{M}_{2N} = \begin{pmatrix} \dfrac{1}{T_{12\ldots 2N}^*} & -\dfrac{R_{12\ldots 2N}^*}{T_{12\ldots 2N}^*} \\ -\dfrac{R_{12\ldots 2N}}{T_{12\ldots 2N}} & \dfrac{1}{T_{12\ldots 2N}} \end{pmatrix} = \mathbf{m}_1\mathbf{m}_2\ldots\mathbf{m}_{2N}. \qquad (21)$$

Thus the transmitted intensity is

$$|T_{12\ldots 2N}|^2 = \frac{1}{|[\mathbf{M}_{2N}]_{22}|^2} \qquad (22)$$

so that the growth of the elements of the matrix product does indeed give the decay of the transmission.

Usually, it is difficult to calculate decay exponents, because the explicit formula (Ziman 1979) provided by Furstenberg's theorem requires knowledge of the distribution of vectors that is invariant under the action of $\mathbf{m}$ when averaged. For strong disorder, however, this programme has been carried out (Baluni and Willemsen 1985) and gives the exact result (17). We now give an alternative derivation that is based on Furstenberg's theorem but does not require knowledge of the invariant distribution.

Using the fact that the logarithm of the transmitted intensity is self-averaging, and the freedom to choose any norm for the matrix product, we can write the decay exponent, defined by the first equality in (17), in terms of the trace, as

$$E = 2\lim_{N\to\infty}\frac{\mathrm{Re}\langle \log \mathrm{Tr}\mathbf{M}_{2N}\rangle}{N} \qquad (23)$$

where the average is over all the random phases. The justification of this formula is that the growth of $\mathbf{M}_{2N}$ can be determined by using as norm the modulus of its larger eigenvalue, which gets closer to the trace as N increases (because det $\mathbf{M}_{2N} = 1$). We can therefore envisage a renormalization method of calculating E, where the limit is approximated by $N = 1, 2, \ldots$. This is an example of resurgence (see, e.g., Berry and Howls 1991), where an asymptotic property of a long string of matrices (the exponent E) is approximated by averages over short strings. In general we expect this method to give very accurate results, by analogy with a related and very successful technique based on zeta functions (Mainieri 1992). For strong disorder the method is exact for all N, as we now show.

From (6) and (20), the matrix for a single element (space plus face) is

$$\mathbf{m}_s = \frac{1}{\sqrt{\tau}}\begin{pmatrix} \exp(\mathrm{i}\phi_s) & \pm\sqrt{\rho}\exp(-\mathrm{i}\phi_s) \\ \pm\sqrt{\rho}\exp(\mathrm{i}\phi_s) & \exp(-\mathrm{i}\phi_s) \end{pmatrix} \qquad (24)$$

where ϕ_s is the random phase associated with the space (air or film) and $\pm$ depends on whether the larger refractive index is to the left or right of the face. Because of the real part in (23), we can pull out phase factors from all the matrices, and, defining the complex variables $z_s = \exp(2i\phi_s)$, write E as a $2N$-fold contour integral around unit circles:

$$E = 2\log\left(\frac{1}{\tau}\right) + 2\lim_{N\to\infty}\frac{1}{N(2\pi i)^{2N}}\oint\frac{dz_1}{z_1}\cdots\oint\frac{dz_{2N}}{z_{2N}} \times \log\mathrm{Tr}\left[\begin{pmatrix} z_1 & \pm\sqrt{\rho} \\ \pm z_1\sqrt{\rho} & 1\end{pmatrix}\cdots \cdots\begin{pmatrix} z_{2N} & \pm\sqrt{\rho} \\ \pm z_{2N}\sqrt{\rho} & 1\end{pmatrix}\right]. \qquad (25)$$

The traces never vanish when $|z_s| < 1$, because then the matrices in the product correspond to absorbing optical elements, whereas a zero-trace the matrix would be elliptic and so represent a transparent medium. A transfer matrix represents absorption if the eigenvalues of $\mathbf{S}^\dagger\mathbf{S}$, where $\mathbf{S}$ is the scattering matrix relating incoming and outgoing waves, are less than unity, because then the energy flowing out of the element is always less than that flowing in. It follows that the absorption condition is

$$|T|^2 + |R|^2 < 1, \qquad |T|^2 + |R_-|^2 < 1,$$
$$(1 - |T|^2 - |R|^2)(1 - |T|^2 - |R_-|^2) > |T^*R_- + TR^*|^2. \qquad (26)$$

A short calculation now shows that the logarithm in (25) is never singular, and the integrals are given by their residues at $z_s = 0$, which vanish, giving the claimed result (17).

The naive ray theory can also be formulated in terms of transfer matrices, and we can ask why the general exponential argument fails, giving instead the peculiar linear decay (10). The obvious adaptation of (18) gives the ray matrix for a single element as

$$\mathbf{m}_{\text{ray}} = \frac{1}{\tau}\begin{pmatrix}\tau^2-\rho^2 & \rho \\ -\rho & 1\end{pmatrix}. \qquad (27)$$

For transparent films, $\tau + \rho = 1$, so that $\mathbf{m}_{\text{ray}}$ becomes

$$\mathbf{m}_{\text{ray}} = \frac{1}{\tau}\begin{pmatrix}\tau-\rho & \rho \\ -\rho & \tau+\rho\end{pmatrix} = \mathbf{1} + \frac{\rho}{\tau}\begin{pmatrix}-1 & 1 \\ -1 & 1\end{pmatrix}. \qquad (28)$$

This a unimodular matrix with the peculiar feature that it has degenerate eigenvalues, so that when raised to the $2N$th power (for N films) it grows linearly rather than exponentially: since

$$\begin{pmatrix}-1 & 1 \\ -1 & 1\end{pmatrix}^2 = 0 \qquad (29)$$

the $2N$th power is

$$\mathbf{m}_{\text{ray}}^{2N} = \mathbf{1} + 2N\frac{\rho}{\tau}\begin{pmatrix}-1 & 1 \\ -1 & 1\end{pmatrix},$$

i.e.

$$(\mathbf{m}_{\text{ray}}^{2N})_{22} = 1 + 2N\frac{\rho}{\tau} \qquad (30)$$

whose reciprocal reproduces the transmission (10). This result is an example of physics (in this case wrong physics) associated with degeneracies of non-Hermitian matrices; other examples are given in Berry (1994). (When there is absorption, the ray transfer matrix is no longer degenerate, and the transmitted intensity decays exponentially, but ray theory gives the wrong exponent.)

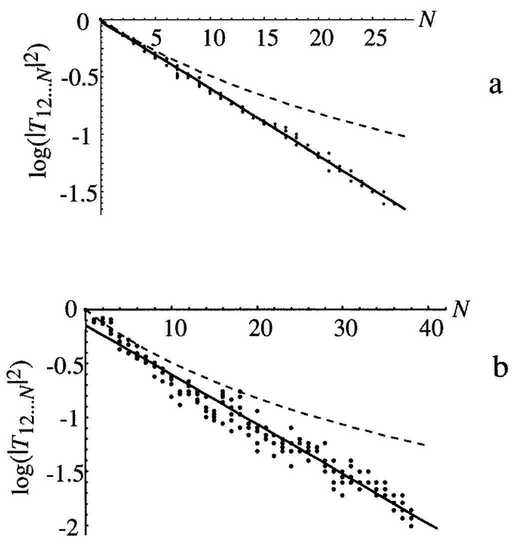

Figure 3. Dots: measurements of logarithmic transmitted intensity for stacks of N plastic films (five runs): full curve, best fit to the data; dashed curve, predictions of naive ray theory. (a) mean thickness $\bar{d}$ = 0.25 mm (best-fit slope, −0.059); (b) $\bar{d}$ = 0.1 mm (best-fit slope, −0.046).

7. Experiment

We measured the transmittance in He–Ne laser light of stacks of two different PPC (polyester carbonate) plastic films, from a thick folder cover ($\bar{d} = 0.25$ mm) and thin viewgraph transparencies ($\bar{d} = 0.1$ mm). In each case, we cut the films into rectangles and formed these into a staircase whose treads were 2.5 mm wide, which was illuminated normally by the laser beam (1 mm wide). The transmitted intensity was measured by allowing the beam to enter the aperture (diameter 10 mm) of a photodetector after passing through the stack. By moving the staircase through the beam, we measured the transmittance of $N = 0, 1, 2\ldots$ films.

Figure 3 shows the results. It is clear that the transmittance decays exponentially as predicted by the wave theory, and that the naive ray theory is false. This is an observation of localization of light caused by macroscopic coherence (that is, coherence between waves whose path differences are large in comparison with the wavelength). Similar results were obtained with white light.

The measured decay exponents were

$$-\log(\text{transmitted intensity})/N = 0.059 \quad (\text{thick films})$$
$$= 0.046 \quad (\text{thin films}). \qquad (31)$$

We also measured the average transmitted intensity τ of the individual films over 20 runs; the result was $\tau = 0.94 \pm 0.01$ for the thick films, and $\tau = 0.94 \pm 0.05$

for the thin films. Since $0.059 = -\log(0.943)$ and $0.046 = -\log(0.955)$, the measurements of τ are, for both films, quantitatively consistent with the theoretical formula (12) with τ replaced by τ^2 because here each element is a film not a face.

A short calculation shows that the individual transmittance predicted by the theoretical formulae (8) and (11) is

$$\tau = \exp\left\langle \log\left\{\frac{16n^2}{|(n+1)^2-(n-1)^2\exp[2\mathrm{i}nkd]|^2}\right\}\right\rangle_d = \frac{16n^2}{(n+1)^4}. \tag{32}$$

We measured the refractive index by the longitudinal shift of an image viewed through the whole stack (compressed and with the films wetted with oil to reduce reflections from the faces). The result was $n = 1.58 \pm 0.01$, which was identical to the tabulated value for polycarbonate (Kroschwitz 1987). According to (32) this would give $\tau = 0.90$, substantially lower than the measured $\tau = 0.94$. However, it is notoriously difficult to get measured transmittances to agree with theory (Macleod 1986, p 3), because of surface contamination, and any layer with index less than n (irrespective of thickness) increases the transparency of a film.

Although we have employed a theory of waves travelling in one dimension, the physical situations we are interested in are not quite one-dimensional. The thickness variations from space to space (films or air gaps) through the stack also appear as variations across the individual spaces. In our experiments these variations, over the width of the laser beam, caused small lateral shifts of the beam, as well as some scattering, visible as weak multiple images and a slight halo round the emergent beam. We were careful to ensure that all this structure (whose angular diameter was several degrees) was accepted by the aperture of the photodetector. We think that the spatial thickness variations cause a degree of ensemble-averaging within each measurement, perhaps explaining why the fluctuations visible in figure 3 are smaller than those inevitable in the purely one-dimensional theory (Pendry 1994).

Finally, we emphasize that exponential decay resulting from coherent interference (that is, localization) should not be confused with the trivial exponential associated with absorption. In our experiments the effect of absorption is negligible. From published measurements (Lytle *et al* 1979, Kroschwitz 1987) of the transmittance of thick slabs of polycarbonate plastic, we infer that the absorptive intensity decay exponent in (20) is about 27 m^{-1}. For the thick films, these values correspond to 0.0068 per film, and, for the thin films, 0.0027 per film. For the thin films, we have checked the published value by measuring the reflected, incident and transmitted intensity for several stacks; for example, for 18 films, we found $\rho = 0.95 \pm 0.02$, in agreement with the expected $(1 - 0.0027)^{18} = 0.95$. The absorptive decays are negligible in comparison with the measured exponents (31), which we can therefore assert with confidence are the result of interference.

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