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Diffraction near fake caustics

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Abstract. Incoherent averaging over orientations of refracted beams of rays can generate a ‘fake caustic’, namely an intensity pattern with a geometrical singularity identical to that of a caustic produced by focusing of a family of rays (i.e. its envelope). However, the diffraction at a fake caustic is very different from the Airy fringes near a genuine caustic. We calculate this unusual type of diffraction and demonstrate its existence experimentally with a rotating prism.

Zusammenfassung. Inkoherente Mittelung über die Orientierungen von gebrochenen Strahlenbündeln kann eine ‘falsche Kaustik’ erzeugen, nämlich ein Intensitätsmuster mit einer geometrischen Singularität identisch zu der einer Kaustik, die durch Fokusierung einer Familie von Strahlen (d.h. ihre Einhuellende) hervorgebracht wurde. Die Beugung in einer falschen Kaustik aber ist sehr verschieden von den Airy Streifen nahe einer echten Kaustik. Wir berechnen diesen ungewöhnlichen Typ einer Beugung und zeigen ihre Existenz experimentell mit einem rotierenden Prisma.

1. Introduction

Caustics are focal singularities of geometrical optics. The most familiar example is the rainbow, where the focusing is directional, and occurs because the deflection function $D_{\rm Geom}(i)$, of a sun-ray striking a spherical raindrop at a place where the angle of incidence is i, possesses a minimum (Tricker 1970, Greenler 1980). Near the minimum deviation $D_{\rm min}$, $D_{\rm Geom}(i)$ has the form

$$D_{\rm Geom}(i) = D_{\rm min} + \tfrac{1}{2}D_2(i - i_{\rm min})^2 + \cdots \qquad (1)$$

and the geometrical-optics directional intensity of a single emergent ray is

$$I_{\rm Geom}(D, i) = K\delta\{D - D_{\rm Geom}(i)\} \qquad (2)$$

where K is a constant. Light from the whole drop will be concentrated near $D_{\rm min}$, close to which the intensity is

$$I_{\rm Geom}(D) = \int {\rm d}i\, I_{\rm Geom}(D, i) = \frac{\Theta(D - D_{\rm min})K\sqrt{2}}{\sqrt{D_2(D - D_{\rm min})}} \qquad (3)$$

where Θ denotes the unit step function. This inverse square-root singularity characterizes the simplest type of caustic. More complicated types of focusing are classified by catastrophe theory (Berry and Upstill 1980).

Fake caustics, with which we will be concerned here, possess the same singularity (3) as genuine caustics, but are produced by a different mechanism, namely the sum of intensities of parallel beams refracted into different directions by objects with flat faces. Ice-crystal halos (Tricker 1970, Greenler 1980) are produced in this way: each crystal in a randomly-oriented mass acts as a prism, and the halo we see arises collectively from all the crystals.

To see how the fake caustic arises, consider a prism with vertex angle A and refractive index n, illuminated by a collimated beam of monochromatic light perpendicular to its axis, and oriented so that the light hits the first face at an angle of incidence i (figure 1). The ray deflection is given by (1), where, by elementary geometry,

$$\begin{aligned} D_{\rm min} &= 2\arcsin\left\{n\sin\left(\tfrac{1}{2}A\right)\right\} - A \\ i_{\rm min} &= \arcsin\left\{n\sin\left(\tfrac{1}{2}A\right)\right\} \\ D_2 &= \frac{2(n^2-1)\tan\left(\tfrac{1}{2}A\right)}{n\cos\left(\tfrac{1}{2}A\right)\sqrt{1 - n^2\sin^2\left(\tfrac{1}{2}A\right)}}. \end{aligned} \qquad (4)$$

The directional intensity of the beam deflected by the prism is given by (2). The singular intensity (3) can now be made to arise in two different ways. First, the prism can be rotated at a uniform rate, as in the experiment to be described later, and the light collected as it emerges at a given deflection angle. Second, there can be an ensemble of prisms randomly rotated about a common axis, so that the distribution of incidence angles i is uniform, at least near minimum deviation. With ice crystals this is a model for the simplest halo phenomenon, namely the parhelion (mock sun); the more common circular halos arise when the axes are randomly oriented too, but in this case the singularity is different (Berry 1994)—a step discontinuity rather than an inverse square root.

From the form of the geometrical-optics singularity (3) alone, fake caustics cannot be distinguished from

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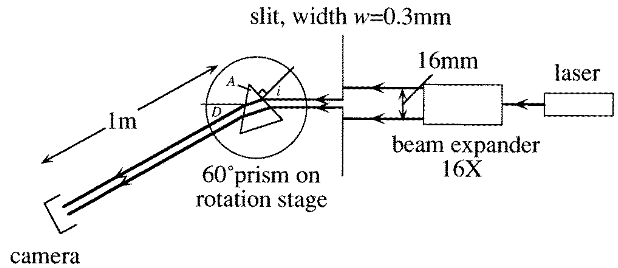

Figure 1. Experiment for observing a fake caustic.

genuine ones. However, in reality the geometrical singularities are softened and decorated on fine scales by wave effects, and these—as embodied for example in the far-field diffraction patterns—are dramatically different for genuine and fake caustics, which are therefore unmasked by diffraction. For the genuine rainbow caustic, the light near the minimum deviation (rainbow ray) is obtained (Airy 1838, Tricker 1970, Nussenzveig 1992) as the coherent sum of the amplitudes of wavelets diffracted by the smooth wavefront emerging from the droplet, and is given by the Airy function (Abramowitz and Stegun 1972):

$$I_{\text{rainbow}}(D) = K\,\text{Ai}^2(-\rho(D)) \tag{5}$$

where, for light of wavelength λ and droplet radius a,

$$\rho(D) = (D - D_{\min})\frac{(n^2-1)^{1/2}}{(4-n^2)^{1/6}}\left(\frac{4\pi a}{3\lambda}\right)^{2/3}. \tag{6}$$

The oscillations of the Airy function are visible in the sky as supernumerary rainbows.

For fake caustics, the diffraction pattern is the incoherent sum, over the distribution of angles of incidence, of the slit diffraction intensities arising because the prism transmits a beam of finite width. In section 2 we give a more detailed derivation of the formula for this diffraction pattern, obtained previously by Berry (1994). That investigation was prompted by the possibility that diffraction from ice crystals might produce supernumerary halos, analogous to supernumerary rainbows. The conclusion was that the contrast in the diffraction fringes is too small to be seen in the sky. In section 3 we show that it can, however, be demonstrated in a laboratory experiment, and our main purpose is to draw attention to this unusual kind of diffraction.

2. Diffraction pattern

The light beam emerging from a small prism will spread by diffraction as well as being deflected through the geometrical angle $D_{\text{Geom}}(i)$. The spreading can be decoupled from the diffraction by exploiting the fact that the same far-field pattern will be observed from a small slit of width w combined with a bigger prism within the Fresnel length (figure 1).

Thus the far-field intensity for a beam with incidence angle i is

$$I(D,i) = K\frac{w}{\lambda}\left[\frac{\sin\{\pi w(D - D_{\text{Geom}}(i))/\lambda\}}{\pi w(D - D_{\text{Geom}}(i))/\lambda}\right]^2. \tag{7}$$

Here we have chosen the normalization so that $I \to I_{\text{Geom}}$ in the limit $\lambda/w \to 0$ of geometrical optics. The fake caustic diffraction pattern arises from averaging over i; since we are only interested in the pattern close to the geometrical singularity $D = D_{\min}$, this can be written as the infinite integral

$$I(D) = K\frac{w}{\lambda}\int_{-\infty}^{\infty}\mathrm{d}i \times\left[\frac{\sin\left\{\pi w\left(D - D_{\min} - \frac{1}{2}D_2(i-i_{\min})^2\right)/\lambda\right\}}{\pi w\left(D - D_{\min} - \frac{1}{2}D_2(i-i_{\min})^2\right)/\lambda}\right]^2. \tag{8}$$

The intensity can be expressed in dimensionless form as

$$I(D) = K\left(\frac{2\pi w}{\lambda D_2}\right)^{1/2} f(\eta) \tag{9}$$

where

$$\eta \equiv \frac{\pi w(D - D_{\min})}{\lambda} \tag{10}$$

is a scaled coordinate crossing the fake caustic, and $f(\eta)$ is the *fake caustic diffraction function*

$$f(\eta) \equiv \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}u\left[\frac{\sin(\eta - u^2)}{\eta - u^2}\right]^2. \tag{11}$$

In the appendix we show how to evaluate this integral in terms of the Fresnel functions $\mathrm{C}(x), \mathrm{S}(x)$ defined by

$$\mathrm{C}(x) + \mathrm{i}\mathrm{S}(x) \equiv \int_0^{\infty}\mathrm{d}t\,\exp\left\{\mathrm{i}\frac{\pi}{2}t^2\right\}. \tag{12}$$

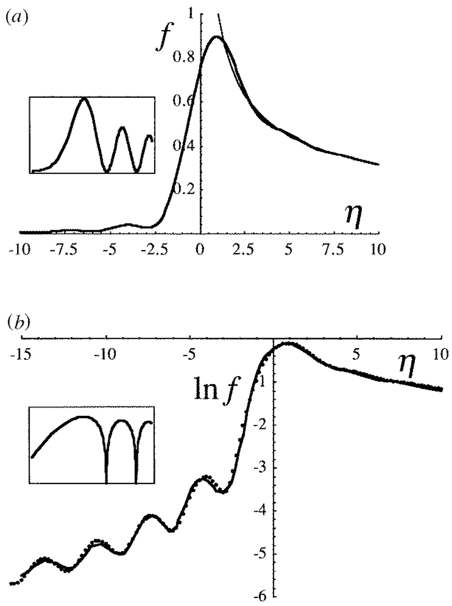

Figure 2. (*a*) Thick curve, fake caustic diffraction function (13); thin curve, geometrical-optics approximation (15); inset, genuine caustic (rainbow) diffraction function $\mathrm{Ai}^2(-\rho)$. (*b*) Full curve, measured intensity from a scanned photograph of a fake caustic; dots, $\ln f(\eta)$, calculated from (13); inset, $\ln\{\mathrm{Ai}^2(-\rho)\}$.

The result is

$$f(\eta) = \frac{\cos\left(2\eta + \frac{1}{4}\pi\right)}{\eta\sqrt{2\pi}} + \frac{1}{2\sqrt{|\eta|}}\left[\left(2 - \frac{1}{2\eta}\right)\mathrm{C}\left\{2\sqrt{\frac{|\eta|}{\pi}}\right\} + \mathrm{sgn}(\eta)\left(2 + \frac{1}{2\eta}\right)\mathrm{S}\left\{2\sqrt{\frac{|\eta|}{\pi}}\right\}\right]. \tag{13}$$

(This formula corrects a typographical error in Berry (1994).)

From the known asymptotic behaviour of C and S (Abramowitz and Stegun 1972) it is easy to establish the following limiting forms for $f(\eta)$:

$$f(\eta) \approx \begin{cases} \dfrac{1}{\sqrt{\eta}} - \dfrac{\sin\left(2\eta + \frac{1}{4}\pi\right)}{4\eta^2\sqrt{2\pi}} & (\eta \gg 1) \\ \dfrac{4}{3\sqrt{\pi}} & (\eta = 0) \\ \dfrac{1}{4|\eta|^{3/2}} - \dfrac{\sin\left(2\eta + \frac{1}{4}\pi\right)}{4\eta^2\sqrt{2\pi}} & (\eta \ll 1). \end{cases} \tag{14}$$

In the geometrical-optics limit ($\lambda \to 0$ for fixed D, corresponding to $|\eta| \gg 1$), the surviving term gives

$$f(\eta) \to f_{\mathrm{G}}(\eta) = \frac{1}{\sqrt{\eta}}\Theta(\eta) \tag{15}$$

which is the scaled version of the (fake or genuine) caustic singularity (3).

Figure 2 shows the fake caustic diffraction function and its logarithm, and for comparison, the Airy intensities across a genuine caustic. The fake caustic fringes differ from the Airy fringes in several ways. First, they have very low contrast. Second, they are strongest on the dark side of the fake caustic. Third, from the scalings (6) and (10) it is clear that the fake caustic fringes are much finer than their genuine counterparts: the separation between adjacent fake fringes is $\Delta\eta = \pi$, corresponding to an angle

$$\Delta D = \frac{\lambda}{w} \tag{16}$$

whereas the spacing of the Airy fringes is proportional to $(\lambda/a)^{2/3}$.

3. Experiment

We observed fake caustics by collecting the light emerging from a prism (figure 1) as it was smoothly rotated (manually) over a range of 10° through its minimum deviation orientation. The light, from a He–Ne laser ($\lambda = 632.8$ nm), was broadened with a beam expander and then passed through a slit of width $w = 0.3$ mm before striking the prism 60 mm away (and well within the Fresnel distance $w^2/\lambda = 142$ mm). The light left the rotating prism and was captured in the film plane of a lensless camera 1 m away.

After processing, the film was scanned (with a UMAX Gemini D-16) into a (Power Macintosh 8500-120) computer, and the graph of intensity I versus deflection D was obtained using a densitometer program (Scan Analysis). Because the response of the film was logarithmic over a limited range, it was not possible to capture the bright peak of the fake caustic and the faint fringes on the dark side in a single exposure. Therefore we combined the fringes from a long exposure with the peak from a short exposure, and joined the scanned intensity curves (at the first minimum, near $\eta = -3$).

In this way we obtained the full curve in figure 2(*b*), in perfect agreement with theory (dots). In this comparison the only manipulation of the data was a linear stretching of the (arbitrary) intensity scale. No adjustment of the angular scale was required, since this was fixed by the scaling (10): equation (16) predicts a fringe separation $\Delta D = 2.11 \times 10^{-3}$ rad, and we measured $2.1 \pm 0.2 \times 10^{-3}$ rad.

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Appendix

An obvious change of variables brings (11) to the form

$$f(\eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}x \frac{\Theta(x+\eta)}{\sqrt{x+\eta}}\left(\frac{\sin x}{x}\right)^2. \quad \text{(A1)}$$

This expresses f as the convolution of the geometric singularity with the slit diffraction function. Since

$$\frac{\Theta(x)}{\sqrt{x}} = \int_{-\infty}^{\infty} \mathrm{d}s\ \exp(\mathrm{i}xs)a(s) \quad \text{(A2)}$$

where

$$a(s) = \frac{1}{2}\sqrt{\frac{1}{\pi|s|}}\exp\left(\mp\tfrac{1}{4}\pi\mathrm{i}\,\mathrm{sgn}\,s\right)$$

and

$$\left(\frac{\sin x}{x}\right)^2 = \int_{-\infty}^{\infty} \mathrm{d}s\ \exp(-\mathrm{i}xs)b(s) \quad \text{(A3)}$$

where

$$b(s) = \tfrac{1}{4}(2-|s|)\Theta(2-|s|)$$

we find

$$\begin{aligned} f(\eta) &= 2\int_{-\infty}^{\infty} \mathrm{d}s\ \exp(\mathrm{i}s\eta)a(s)b(s) \\ &= \frac{1}{4\sqrt{\pi}}\int_{-2}^{2}\mathrm{d}s\frac{(2-|s|)}{\sqrt{|s|}}\exp\left\{\mathrm{i}\left(\eta s \mp \tfrac{1}{4}\pi\,\mathrm{sgn}\,s\right)\right\} \\ &= \frac{1}{\sqrt{\pi}}\mathrm{Re}\int_0^{\sqrt{2}}\mathrm{d}t\ (2-t^2)\exp\left\{\mathrm{i}\left(\eta t^2 - \tfrac{1}{4}\pi\right)\right\} \\ &= \mathrm{Re}\left(2+\mathrm{i}\frac{\partial}{\partial\eta}\right)\sqrt{\frac{1}{2|\eta|}}\int_0^{2\sqrt{|\eta|/\pi}}\mathrm{d}w \\ &\quad \times \exp\left\{\mathrm{i}\left(\tfrac{1}{2}\pi w^2\,\mathrm{sgn}\,\eta - \tfrac{1}{4}\pi\right)\right\} \end{aligned} \quad \text{(A4)}$$

whence (13) follows from (12) after some elementary manipulations.

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