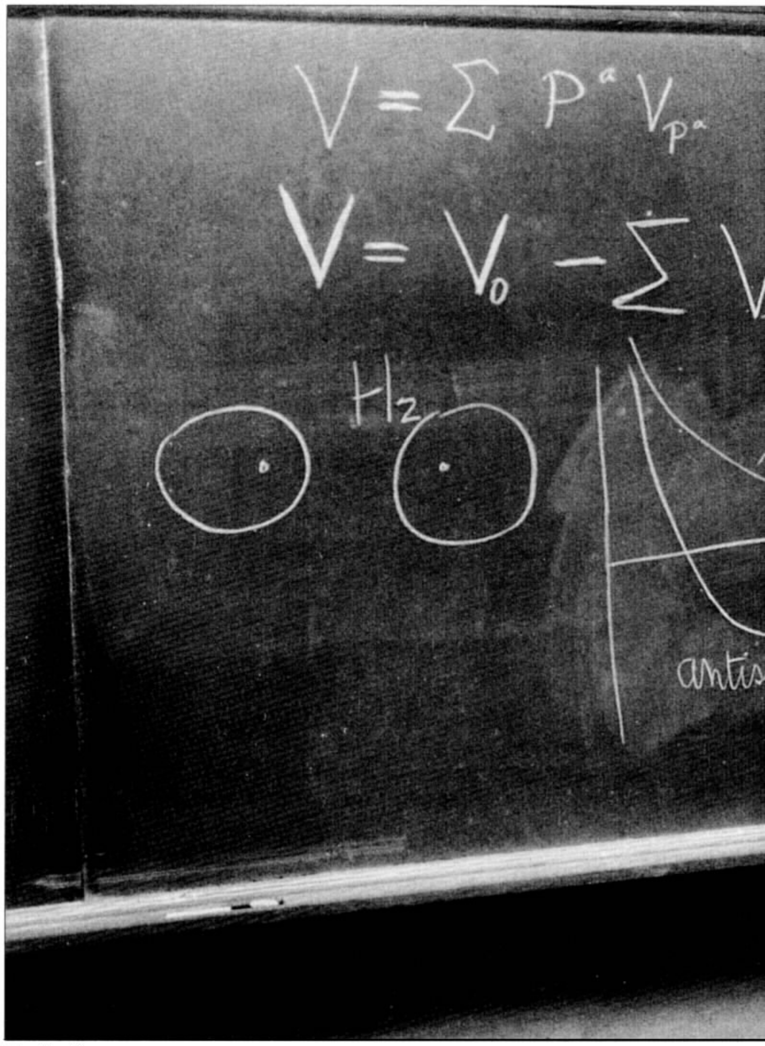
Paul Dirac published the first of his papers on "The Quantum Theory of the Electron" seventy years ago this month. The Dirac equation, derived in those papers, is one of the most important equations in physics

Paul Dirac: the purest soul in physics

Michael Berry



EACH day, I walk past the road where Paul Adrien Maurice Dirac lived as a child. It is pleasant to have even this tenuous association with one of the greatest intellects of the 20th century. Paul Dirac was born at 15 Monk Road in Bishopston, Bristol, on 8 August 1902, and educated at the nearby Bishop Road Primary School. The family later moved to Cotham Road, near the University of Bristol, and in 1914 the young Dirac joined Cotham Grammar School, formerly the Merchant Venturers.

Dirac was a student at Bristol University between 1918 and 1923, first in electrical engineering and then in applied mathematics. Much later, he said: "I owe a lot to my engineering training because it [taught] me to tolerate approximations. Previously to that I thought...one should just concentrate on exact equations all the time. Then I got the idea that in the actual world all our equations are only approximate. We must just tend to greater and greater accuracy. In spite of the equations being approximate, they can be beautiful."

Because Dirac was a quiet man – famously quiet, indeed – he is not well known outside physics, although this is slowly changing. In 1995 a plaque to Dirac was unveiled at Westminster Abbey in London and last year Institute of Physics Publishing, which is based in Bristol, named its new building Dirac House.

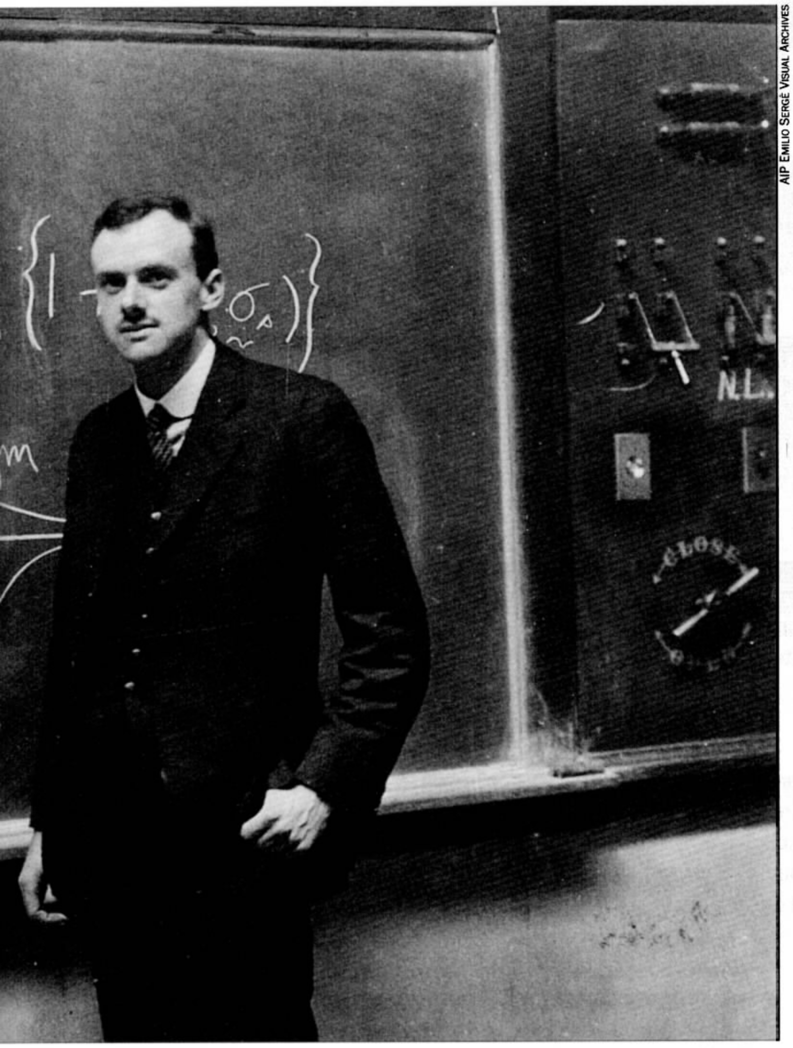
It is hard to give the flavour of Dirac's achievements in a non-technical article, because his work was so mathematical. He once said: "A great deal of my work is just playing with equations and seeing what they give."

Early days

When Dirac went to Cambridge in 1923, the physics of matter on the smallest scales – in those days this was the physics of the atom – was in ferment. It had been known for more than a decade that the old mechanics of Newton – "classical" mechanics, as it came to be called – does not apply in the microscopic world. In particular, evidence from the light coming out of atoms seemed to indicate that some quantities that in classical mechanics can take any values are actually restricted to a set of particular values: they are "quantized". One of these quantities is the energy of the electrons in an atom. This was strange and shocking. Imagine being told that when your car accelerates from 0 to 70 miles per hour it does so in a series of jumps from one speed to another (say in steps of one thousandth of a mph), with the intermediate speeds simply not existing. It did not make sense, and yet observations seemed to demand such an interpretation.

In the first attempts at a theoretical understanding, physicists tried to find the general rules for imposing these restrictions on classical mechanics – that is rules for quantization. It seemed that in order to quantize, it was necessary first to identify those quantities that do not change when their environment is slowly altered. If a pendulum is slowly shortened, for example, it swings farther and also faster, in such a way that its energy divided by its frequency stays constant. These rules worked for simple atoms and molecules but failed for complicated ones.

Dirac entered physics at the end of this baroque period. One of his first papers was an attempt at a general theory of



AIP Emilio Segrè Visual Archives

these unchanging quantities. This is a delicate problem in classical mechanics, not solved even now. It is amazing today to read that paper. In its mathematics it is quite unlike any of Dirac's later works (for example, he brings in fine differences between rational and irrational numbers), and "pre-invents" techniques developed by other people only decades later. (I say pre-invents because the paper was forgotten until recently.)

At this time the situation in atomic physics resembled that at the end of the 16th century, when the old Earth-centred astronomy had to be made ever more elaborate in the face of more accurate observations. The difficulties of the 16th and 20th centuries were resolved in the same way: by a complete shift of thought. In atomic physics this happened suddenly, in 1925, with the discovery by Heisenberg of quantum mechanics. This seemed to throw out classical mechanics completely, though it was built in as a limiting case to ensure that, on larger scales, the new mechanics agreed with more familiar experience. The quantum rules emerged automatically, but from a mathematical framework that was peculiar. For example, it involved multiplication where the result depends on the order in which the multiplication is done. It is as though 2 multiplied by 3 is different from 3 multiplied by 2. Heisenberg found this ugly and unsatisfactory. Dirac disagreed, and just a few months after Heisenberg he published the first of a series of papers in which quantum mechanics took the definitive form we still use today.

The main idea is that the multiplied objects – objects that represent variables we can measure in experiments – should be thought of as operations. An experiment is an operation, of course, even though its result is a number. With this interpretation, it is not surprising that the order matters: we all know that putting on our socks and then our shoes gives a result different from putting on our shoes and then our socks. Dirac found the one simple rule by which a multiplied by b differed from b multiplied by a, and from which the whole of quantum mechanics follows.

The same unification was soon found to include Schrödinger's way of doing quantum mechanics, where the state of a system is represented by a wave whose strength gives the probabilities of the different possible results of measurements on it. For a while this seemed completely different from the framework that Heisenberg had used, but it quickly emerged that in fact each represents Dirac's operators in a different way. It seemed miraculous.

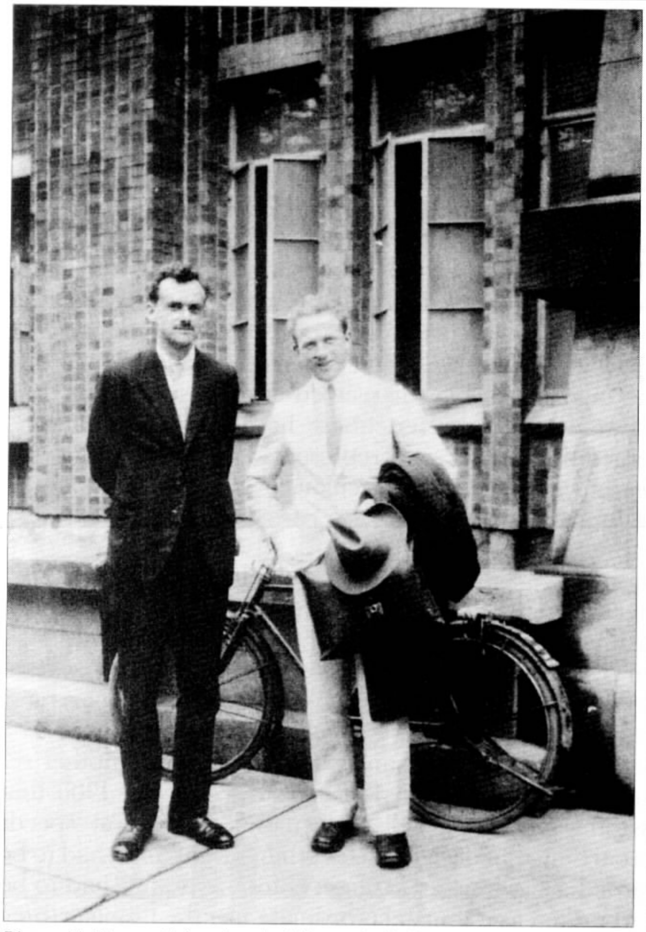
The Dirac equation

Although brilliant – in Einstein's words, "the most logically perfect presentation of quantum mechanics" – this was a reformulation of physics that had, admittedly only just, been discovered. Dirac's main contribution came several years later, when (still in his mid-twenties) he made his most spectacular discovery.

Before quantum mechanics, there had been another revolution in physics, with Einstein's discovery in 1905 that Newton's mechanics fails for matter moving at speeds approaching that of light. To get things right, time had to be regarded as no longer absolute: before-and-after had to be incorporated as a fourth co-ordinate like the familiar three spatial co-ordinates that describe side-to-side, forward-and-backward and up-and-down. Just as what is side-to-side and what is forward-and-backward change when you turn, so time gets mixed in with the other three co-ordinates when you move fast. Now, in the 1920s, came quantum mechanics, showing how Newton's mechanics failed in a different way: on microscopic scales. The question arose: what is the physics of particles that are at the same time small *and* moving fast?

This was a practical question: the electrons in atoms are small, and they move fast enough for the new quantum mechanics to be slightly inaccurate, since it had been constructed to have as its large-scale limit Newton's mechanics rather than Einstein's. From the start people tried to construct a quantum theory concordant with relativity, but failed to overcome technical obstructions: in particular, their attempts gave probabilities that were negative numbers – something that is nonsense, at least in the usual meaning of probability. The question boiled down to this: what are the right sort of quantum waves describing electrons? And what is the wave equation that governs the dynamics of these waves, while satisfying the requirements of relativity and giving sensible physical predictions?

Dirac's construction of his wave equation for the electron – published in two papers in the *Proceedings of the Royal Society (London)* in February and March 1928 – contained one of those outrageous leaps of imagination shared by all great advances in thought. He showed that the simplest wave satisfying the requirements was not a simple number but had four components (see box overleaf). This seemed like a complication, especially to minds still reeling from the unfamiliarity of the "ordinary" quantum mechanics. Four components! Why should anybody take Dirac's theory seriously?



Dirac with Werner Heisenberg in Chicago in 1929.

The Dirac equation

The Dirac equation for an electron moving in an arbitrary electromagnetic field can be written in many ways. In Dirac's original papers it is written as

$$\left[p_0 + \frac{e}{c}A_0 + \alpha_1\left(p_1 + \frac{e}{c}A_1\right) + \alpha_2\left(p_2 + \frac{e}{c}A_2\right) + \alpha_3\left(p_3 + \frac{e}{c}A_3\right) + \alpha_4 mc\right]\psi = 0$$

where $p_0 = ih\partial/c\partial t$ (the energy operator), e is the charge on the electron, A_0 is the scalar potential associated with the electromagnetic field, c is the speed of light, α_i are 4×4 matrices derived from the Pauli matrices, $p_1 = -ih\partial/\partial x$ is a momentum operator ($p_2 = -ih\partial/\partial y$, $p_3 = -ih\partial/\partial z$), A_i are the three components of the electromagnetic vector potential, m is the mass of the electron and ψ is the wavefunction of the electron.

The wavefunction ψ is a 4×1 column vector (also known as a spinor) and each element is a function of space and time, representing the spin state (up or down) of the electron and the associated positron solution. As explained in the main text, the equation was able to explain the results of all of the experiments at the time, to explain the origin of electron spin and to predict the existence of antimatter.

The equation can be written in more compact form. In §67 of *The Principles of Quantum Mechanics* (4th edn, Oxford University Press) it is written as

$$\left\{p_0 + \frac{e}{c}A_0 - \rho_1\left(\sigma, \mathbf{p} + \frac{e}{c}\mathbf{A}\right) - \rho_3 mc\right\}\psi = 0$$

where ρ_1 and ρ_3 are 4×4 matrices (related to α_i and the Pauli matrices), σ is a three-component vector of 4×4 matrices, and $\mathbf{p}$ is a three-component vector of momentum operators. The version of the equation in Westminster Abbey is even more compact and reads $i\gamma\cdot\partial\psi = m\psi$ where γ is a 4×4 matrix and ∂ is a 4-vector.

First, and above all for Dirac, the logic that led to the theory was, although deeply sophisticated, in a sense beautifully simple. Much later, when someone asked him (as many must have done before) "How did you find the Dirac equation?" he is said to have replied: "I found it beautiful." Second, it agreed with precise measurements of the energies of light emitted from atoms, in particularly where these differed from ordinary (non-relativistic) quantum mechanics.

There are two more reasons why the Dirac equation was compelling as the correct description of electrons. To understand them, you should realize that any great physical theory gives back more than is put into it, in the sense that as well as solving the problem that inspired its construction, it explains more and predicts new things. Before the Dirac equation, it was known that the electron spins. The spin is tiny on the scale of everyday but is always the same and plays a central part in the explanation through quantum mechanics of the rules of chemistry and the structure of matter. This spin was a property of the electron, like its mass and its electric charge, whose existence simply had to be assumed before quantum mechanics could be applied. In Dirac's equation, spin did not have to be imported: it emerged – along with the magnetism of the electron – as an inevitable property of an electron that was both a quantum particle and a relativistic one.

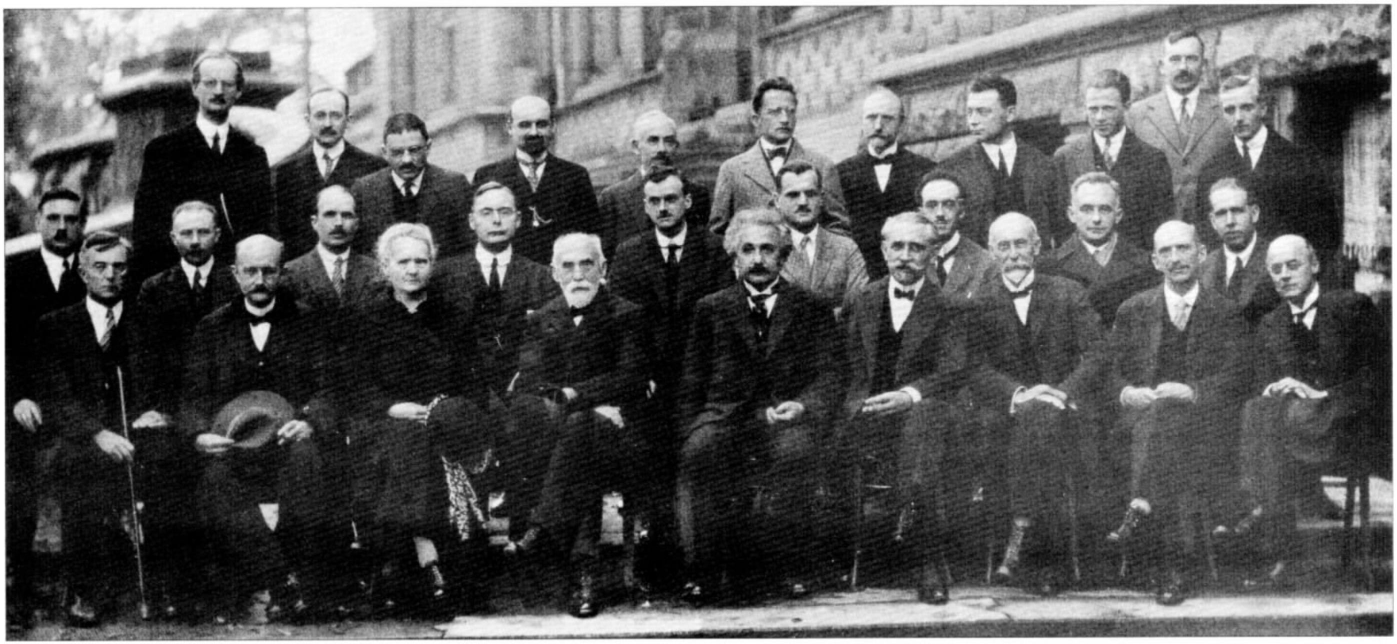
So, electron spin was the third reason for believing Dirac's mathematically inspired equation. The fourth came from a consequence of the equation that was puzzling for a few years at first. Related to its four components was the fact that any solution of the equation where the electron had a positive energy had a counterpart where the energy was negative. It gradually became clear that these counterpart solutions could be interpreted as representing a new particle, similar to the electron but with positive rather than negative charge; Dirac called it an "anti-electron", but it soon came to be known as the positron. If an electron encounters a positron, Dirac predicted, the two charges cancel and the pair annihilates, with the combined mass transforming into radiation in the most dramatic expression of Einstein's celebrated equation $E = mc^2$. Thus was antimatter predicted. When the positron was discovered by Anderson in 1932, Dirac's immortality was assured. Dirac and Schrödinger shared the Nobel Prize for Physics in 1933.

Nowadays, positrons are used every day in medicine, in PET (positron emission tomography) scanners that pinpoint interesting places in the brain (e.g. places where drugs are chemically active). These work by detecting the radiation as the positrons emitted from radioactive nuclei annihilate with ordinary electrons nearby.

Other achievements

Having explained spin, it was natural for Dirac to try to explain electric charge, and in particular the mysterious fact that it is quantized: all charges found in nature are multiples of the charge on the electron. In classical electricity, there is no basis for this: charges can have any value.

In 1931 Dirac gave a solution of this problem in an application of quantum mechanics so original that it still astounds us to read it today. He combined electricity with magnetism, in a return to the 18th-century notion of a magnet being a combination of north and south magnetic poles (magnetic



The 1927 Solvay Congress in Brussels was attended by most of the leading physicists of the time. Dirac is in the second row, on Einstein's right.The other delegates are (left to right): front row; I Langmuir, M Planck, Madame Curie, H A Lorentz, A Einstein, P Langevin, Ch E Guye, C T R Wilson, O W Richardson; second row; P Debye, M Knudsen, W L Bragg, H A Kramers, P A M Dirac, A H Compton, L V de Broglie, M Born, N Bohr; back row; A Piccard, E Henriot, P Ehrenfest, E D Herzen, T H de Donder, E Schrödinger, E Verschaffelt, W Pauli, W Heisenberg, R H Fowler, L Brillouin.

charges), in the same way that a charged body contains positive and negative electric charges. That symmetry was lost in the 19th century with the discoveries of Oersted, Ampère and Faraday, culminating in Maxwell's synthesis of all electromagnetic and – in another example of getting out more than you put in – optical phenomena. In its place came a greater simplicity: there are only electric charges, whose movement generates magnetism (and now the motive power for much of our civilisation). The absence of isolated magnetic poles – magnetic monopoles – was built into classical electromagnetism, and also the quantum mechanics that grew out of it.

Dirac wondered if there was any way that magnetic monopoles could be brought into quantum physics without spoiling everything that had grown out of assuming that they did not exist. He found that this could be done, but only if the strength of the monopole (the "magnetic charge") was linked to that of the electric charge, and if both were quantized. This solved the original problem: for consistency with quantum mechanics, the existence of even one monopole anywhere in the universe would suffice to ensure that electric charge must be quantized. The implication is compelling: to account for the quantization of electricity, magnetic poles must exist. After this, Pauli referred to Dirac as "Monopoleon".

Alas, no magnetic monopole has ever been found. Perhaps they do not exist, or perhaps (and there are hints of this in the theory) positive and negative monopoles are so tightly bound together that they have not been separated. Much later, Dirac referred to this theory as "just a disappointment". However, the mathematics he invented to study the monopole – combining geometry with analysis – now forms the basis of the modern theories of fundamental particles.

There were two other seminal contributions to physics in those early years. I have space only to mention them. Dirac applied quantum mechanics to the way light and matter interact. This made him realize that it was necessary to quantize not only particles but the electromagnetic field itself, and led him to the first consistent theory of photons (which had been discovered several decades previously in the beginnings of quantum mechanics). This led to the elaborate and thriving quantum field theories of today.

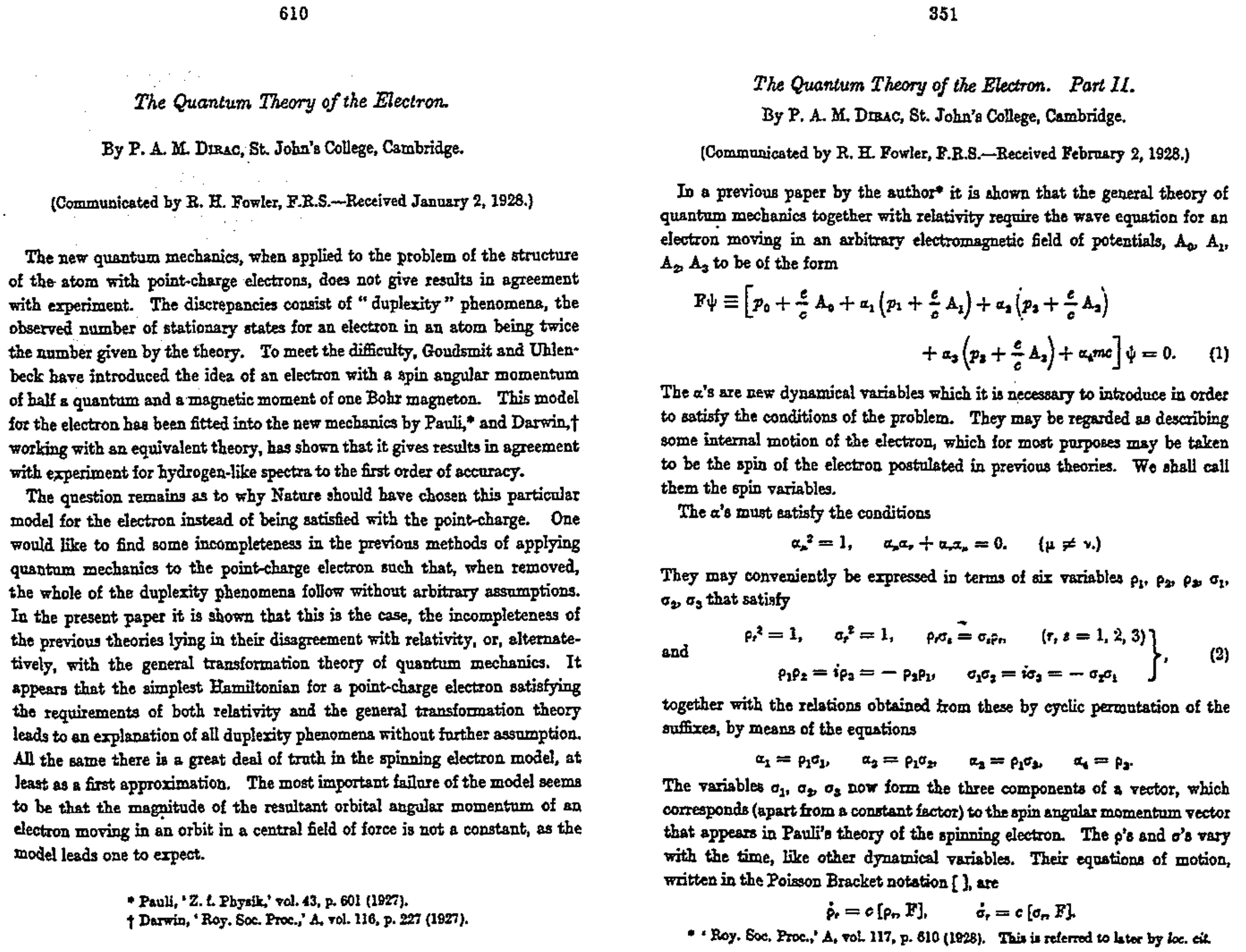
Dirac also showed how quantum waves for many electrons had to be constructed, incorporating the philosophically intriguing fact that any two of these particles are absolutely identical and so cannot be distinguished in any way. This produced the definitive understanding of earlier rules about how quantum mechanics explains the periodic table of the elements, and provided the basis for the theory of metals and the interior of stars.

Like all scientists at the highest level, Dirac was not afraid to descend from the pinnacle and discuss more down-to-earth matters. Here are two examples. Much of our knowledge comes from light scattered by matter; in particular, that is how we see. In a clever stroke of lateral thinking, Dirac realized that the quantum symmetry between waves of light and waves of matter implied that it is also possible for material particles to be scattered by light, a ghostly possibility that could be observed, as he showed in 1933 in a paper with Peter Kapitza. This was observed for the first time about ten years ago and the manipulation of atoms by laser beams is now a thriving area of applied quantum mechanics – a fact recognized with a Nobel prize last year (*Physics World* November 1997 p51).

The second example is his Second World War work. In the Manhattan Project to develop the first nuclear bombs, it was necessary to separate isotopes of uranium. One class of methods involved the centrifugal effects of fluid streams that were made to bend. Dirac put the theory of these techniques on a firm basis, and indeed his work in this field has been described as seminal.

Dirac stories

It is not my intention to write about what sort of person Dirac was. But I must mention the genre of "Dirac stories". He was so unusual in the logic and precision of his interaction with



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The Quantum Theory of the Electron.

By P. A. M. Dirac, St. John's College, Cambridge.

(Communicated by R. H. Fowler, F.R.S.—Received January 2, 1928.)

The new quantum mechanics, when applied to the problem of the structure of the atom with point-charge electrons, does not give results in agreement with experiment. The discrepancies consist of "duplexity" phenomena, the observed number of stationary states for an electron in an atom being twice the number given by the theory. To meet the difficulty, Goudsmit and Uhlenbeck have introduced the idea of an electron with a spin angular momentum of half a quantum and a magnetic moment of one Bohr magneton. This model for the electron has been fitted into the new mechanics by Pauli,* and Darwin,† working with an equivalent theory, has shown that it gives results in agreement with experiment for hydrogen-like spectra to the first order of accuracy.

The question remains as to why Nature should have chosen this particular model for the electron instead of being satisfied with the point-charge. One would like to find some incompleteness in the previous methods of applying quantum mechanics to the point-charge electron such that, when removed, the whole of the duplexity phenomena follow without arbitrary assumptions. In the present paper it is shown that this is the case, the incompleteness of the previous theories lying in their disagreement with relativity, or, alternatetively, with the general transformation theory of quantum mechanics. It appears that the simplest Hamiltonian for a point-charge electron satisfying the requirements of both relativity and the general transformation theory leads to an explanation of all duplexity phenomena without further assumption. All the same there is a great deal of truth in the spinning electron model, at least as a first approximation. The most important failure of the model seems to be that the magnitude of the resultant orbital angular momentum of an electron moving in an orbit in a central field of force is not a constant, as the model leads one to expect.

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* Pauli, 'Z. f. Physik,' vol. 43, p. 601 (1927).

† Darwin, 'Roy. Soc. Proc.,' A, vol. 116, p. 227 (1927).

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The Quantum Theory of the Electron. Part II.

By P. A. M. Dirac, St. John's College, Cambridge.

(Communicated by R. H. Fowler, F.R.S.—Received February 2, 1928.)

In a previous paper by the author* it is shown that the general theory of quantum mechanics together with relativity require the wave equation for an electron moving in an arbitrary electromagnetic field of potentials, A_0, A_1, A_2, A_3 to be of the form

$$F\psi \equiv \Big[p_0 + \frac{e}{c}A_0 + \alpha_1\Big(p_1 + \frac{e}{c}A_1\Big) + \alpha_2\Big(p_2 + \frac{e}{c}A_2\Big) + \alpha_3\Big(p_3 + \frac{e}{c}A_3\Big) + \alpha_4 mc\Big]\psi = 0. \qquad (1)$$

The α's are new dynamical variables which it is necessary to introduce in order to satisfy the conditions of the problem. They may be regarded as describing some internal motion of the electron, which for most purposes may be taken to be the spin of the electron postulated in previous theories. We shall call them the spin variables.

The α's must satisfy the conditions

$$\alpha_\mu^2 = 1, \qquad \alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu = 0. \qquad (\mu \neq \nu.)$$

They may conveniently be expressed in terms of six variables ρ_1, ρ_2, ρ_3, σ_1, σ_2, σ_3 that satisfy

$$\text{and} \quad \left.\begin{array}{l} \rho_r^2 = 1, \quad \sigma_r^2 = 1, \quad \rho_r\sigma_s = \sigma_s\rho_r \quad (r, s = 1, 2, 3) \\ \rho_1\rho_2 = i\rho_3 = -\rho_2\rho_1, \quad \sigma_1\sigma_2 = i\sigma_3 = -\sigma_2\sigma_1 \end{array}\right\}, \qquad (2)$$

together with the relations obtained from these by cyclic permutation of the suffixes, by means of the equations

$$\alpha_1 = \rho_1\sigma_1, \quad \alpha_2 = \rho_1\sigma_2, \quad \alpha_3 = \rho_1\sigma_3, \quad \alpha_4 = \rho_3.$$

The variables σ_1, σ_2, σ_3 now form the three components of a vector, which corresponds (apart from a constant factor) to the spin angular momentum vector that appears in Pauli's theory of the spinning electron. The ρ's and σ's vary with the time, like other dynamical variables. Their equations of motion, written in the Poisson Bracket notation [ ], are

$$\dot{\rho}_r = c\,[\rho_r, F], \qquad \dot{\sigma}_r = c\,[\sigma_r, F].$$

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* 'Roy. Soc. Proc.,' A, vol. 117, p. 610 (1928). This is referred to later by *loc. cit.*

Dirac's papers on the quantum theory of the electron were published in the *Proceedings of the Royal Society (London) A* in 1928 (see further reading).

the world, both in and out of physics, that tales have become attached to him and have acquired a life of their own. I suppose it matters to a historian whether they are true or apocryphal (or as Norman Mailer says, "factoids"), but to us they have a deeper resonance that transcends fact. Resisting temptation, I retell just two less well known ones.

Like many scientists, Dirac was known to sleep during (other people's) lectures, and then wake and suddenly make a penetrating remark. Once, a speaker stopped, scratched his head and declared: "Here is a minus where there should be a plus. I seem to have made an error of sign." Dirac opened one eye and said: "Or an odd number of them." Another time, Dirac was at a meeting in a castle, when another guest remarked that a certain room was haunted: at midnight, a ghost appeared. In his only reported utterance on matters paranormal, Dirac asked: "Is that midnight Greenwich time, or daylight saving time?"

Dirac's writing was famous for its clarity and simplicity. Every physicist knows his *Principles of Quantum Mechanics* – such a perfect and complete summary of his views that in later years his lectures consisted of readings from it. There is the story that he was once present when Niels Bohr was writing a scientific paper – with many hesitations and redraftings, as was his custom. Bohr stopped: "I do not know how to finish this sentence." Dirac replied: "I was taught at school that you should never start a sentence without knowing the end of it."

Many physicists have spoken of Dirac with awe. John Wheeler, referring to the sharp light of his intelligence, said "Dirac casts no penumbra." Niels Bohr said: "Of all physicists, Dirac has the purest soul." He is also reported as saying (I cannot now find this quotation): "Dirac did not have a trivial bone in his body."

The mathematician Mark Kac divided geniuses into two classes. There are the ordinary geniuses, whose achievements one imagines other people might emulate, with enormous hard work and a bit of luck. Then there are the magicians, whose inventions are so astounding, so counter to all the intuitions of their colleagues, that it is hard to see how any human could have imagined them. Dirac was a magician.

Further reading

P A M Dirac 1928 The quantum theory of the electron *Proc. R. Soc. (London)* **117** 610-612

P A M Dirac 1928 The quantum theory of the electron. Part II *Proc. R. Soc. (London)* **118** 351-361

R H Dalitz (ed) 1995 *The Collected Works of P A M Dirac 1924-1948* (Cambridge University Press)

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