Nonpropagating string excitations

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(Received 6 August 1997; accepted 3 September 1997)

It is shown that certain force distributions applied to a finite portion of an infinitely long string do not produce any excitation outside the region of the applied force. The existence of such nonpropagating excitations is demonstrated by a simple example, and two general theorems concerning their nature are proven. Some analogies between nonpropagating string excitations and fields produced by nonradiating sources are noted.

It has been known in radiation theory for some time that there are localized source distributions which do not produce a field outside the domain of the source (see, for example, Refs. 1–4). In this paper we show that analogous situations exist for waves on an infinitely long string. Specifically, we show that there are localized force distributions which produce no displacement of the string at any point outside the region of the applied force, and we derive general theorems concerning such situations. These theorems are one-dimensional analogues of theorems encountered with nonradiating sources (usually discussed in three dimensions).

We consider an infinitely long flexible string under tension

T and with mass per unit length μ, undergoing small displacements $y(x,t)$ from the equilibrium position, driven by a force density $f(x,t)$ (force per unit length) and localized in the region $a \leqslant x \leqslant b$. The displacement obeys the wave equation[5]

$$\mu \frac{\partial^2 y(x,t)}{\partial t^2} - T \frac{\partial^2 y(x,t)}{\partial x^2} = f(x,t). \tag{1}$$

Restricting ourselves to simple harmonic driving forces,

$$f(x,t) \equiv \mathrm{Re}\{f(x)e^{-i\omega t}\}, \tag{2}$$

where Re denotes the real part, the steady-state solution $y(x,t)$ of Eq. (1) will have the same time dependence,

$$y(x,t) \equiv \mathrm{Re}\{y(x)e^{-i\omega t}\}, \tag{3}$$

and Eq. (1) then reduces to the one-dimensional inhomogeneous Helmholtz equation,

$$\frac{d^2 y(x)}{dx^2} + k^2 y(x) = q(x), \tag{4}$$

where k is the wave number,

$$k = \frac{\omega}{v}, \quad v = \sqrt{T/\mu} \tag{5a}$$

and

$$q(x) = -f(x)/T. \tag{5b}$$

We will call $q(x)$ the effective force density, or simply the force density.

The outgoing solution of Eq. (4) is well known to be[6]

$$y(x) = \frac{1}{2ik} \int_a^b q(x') e^{ik|x-x'|} dx'. \tag{6}$$

For displacements to the right $(x>b)$ and left $(x<a)$ of the region of the applied force, $y(x)$ reduces to

$$y(x)|_R = \frac{e^{ikx}}{2ik} \int_a^b q(x') e^{-ikx'} dx' \tag{7}$$

and

$$y(x)|_L = \frac{e^{-ikx}}{2ik} \int_a^b q(x') e^{ikx'} dx'. \tag{8}$$

It is apparent from Eqs. (7) and (8) that the excitations will vanish everywhere outside the force region $a \leqslant x \leqslant b$ if

$$\tilde{q}(k) = 0, \quad \tilde{q}(-k) = 0, \tag{9}$$

with k given by Eq. (5a), and $\tilde{q}(k)$ is the Fourier transform of the force density, i.e.,

$$\tilde{q}(K) = \frac{1}{2\pi} \int_a^b q(x) e^{-iKx} dx. \tag{10}$$

Nontrivial force densities that satisfy Eq. (9) will generate displacements of the string only within the region of the applied force, and will not produce any displacement outside it. We will refer to such a situation as a *nonpropagating string excitation.*

As a simple example of a nonpropagating excitation, let $a=-L$, $b=L$, $L>0$, and let the force be constant throughout this domain:

$$q(x) = \begin{cases} Q_0 & \text{when } |x| \leqslant L, \\ 0 & \text{when } |x| > L. \end{cases} \tag{11}$$

Upon substituting from Eq. (11) into Eq. (10) and requiring that the two conditions (9) be fulfilled, we find that nontrivial solutions occur if and only if

$$kL = n\pi, \tag{12}$$

where $n = 1,2,\ldots$. This result shows that a *constant* localized force distribution within the region $-L \leqslant x \leqslant L$ produces a nonpropagating excitation only for certain special values of kL. Using this result in the general expression (6) for the displacement, one readily finds that

$$y(x) = \begin{cases} \dfrac{Q_0}{(n\pi/L)^2} \left[1 - (-1)^n \cos \dfrac{n\pi x}{L} \right] & \text{when } |x| \leqslant L \\ 0 \quad \text{when } |x| > L. \end{cases} \tag{13}$$

This displacement, along with the associated force density, is shown in Fig. 1 for the cases $n=1$ and $n=2$. We note that the displacement $y(x)$ given by Eq. (13) is continuous everywhere on the string, in particular at the boundary of the region of applied force. One can readily verify that the first derivative $dy(x)/dx$ is also continuous everywhere on the string. In the Appendix, we demonstrate that this behavior is a general property of excitations due to piecewise continuous, localized force distributions. This fact leads to the following theorem about nonpropagating string excitations.

Theorem I: A nonpropagating excitation on an infinitely long string and the piecewise continuous force distribution, assumed to be confined to a finite region $a \leqslant x \leqslant b$, which generates it are related by the inhomogeneous Helmholtz equation (4), subject to the boundary conditions

$$y(a) = y(b) = 0, \quad \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=b} = 0. \tag{14}$$

This theorem implies that the nonpropagating excitations are solutions to an *overspecified* Sturm–Liouville boundary-value problem, because only one of the two sets of boundary conditions (14) is required for a unique solution of the equation. Using Theorem I, one can construct numerous examples of nonpropagating string excitations. The theorem is a one-dimensional analogue of a theorem in radiation theory,[4] as is the following one.[3]

Theorem II: A force distribution, assumed to be confined to a region $a \leqslant x \leqslant b$, which generates a nonpropagating excitation on a infinitely long string, related by Eq. (4), is orthogonal to every solution of the homogeneous Helmholtz equation with wave number k.

To establish this theorem, let $y(x)$ be a nonpropagating excitation and $q(x)$ the force distribution which generates it, and let $u(x)$ be any solution of the homogeneous Helmholtz equation,

$$\frac{d^2 u}{dx^2} + k^2 u = 0. \tag{15}$$

We first multiply Eq. (4) by $u(x)$ and Eq. (15) by $y(x)$ and subtract the equations from each other. We then obtain the identity

$$u \frac{d^2 y}{dx^2} - y \frac{d^2 u}{dx^2} = q(x) u(x). \tag{16}$$

On integrating both sides of Eq. (15) with respect to x over the range $a \leqslant x \leqslant b$ and then integrating by parts on the left we obtain the relation

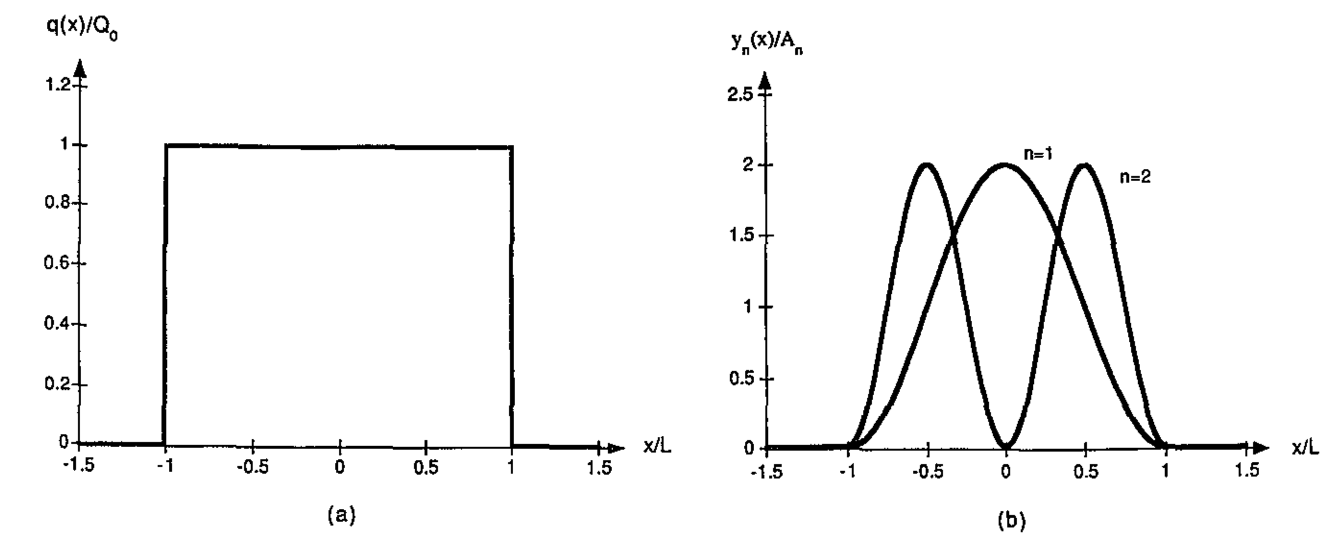

Fig. 1. Force density distribution $q(x)$, shown in (a), which produces nonpropagating excitations (b):

$$y_n(x)=A_n\left[1-(-1)^n\cos\frac{n\pi x}{L}\right],\quad \text{when } |x|\leqslant L$$
$$=0,\quad \text{when } |x|>L,$$

with $A_n=Q_0/(n\pi/L)^2$, $k=n\pi/L$, for $n=1$ and $n=2$.

$$\left[u\,\frac{dy}{dx}-y\,\frac{du}{dx}\right]_a^b=\int_a^b q(x)u(x)dx. \tag{17}$$

According to Theorem I, the left-hand side of Eq. (16) vanishes and consequently

$$\int_a^b q(x)u(x)dx=0, \tag{18}$$

as asserted by Theorem II.

We conclude by noting that in the example that we have presented, the force density which produced the nonpropagating excitation is quite simple [see Eqs. (11) and (12)]. This force density is, of course, not the only one with these properties. Any localized force density which is a piecewise continuous function of x and which obeys the conditions (9) will produce nonpropagating excitations on the string.

ACKNOWLEDGMENTS

This research was supported by the National Science Foundation and by the New York State Foundation for Advanced Technology.

APPENDIX: CONTINUITY OF $y(x)$ AND $dy(x)/dx$

Let $q(x)$ be a piecewise continuous function on a finite interval $a\leqslant x\leqslant b$, with its only discontinuities at the points $x_0=a$, $x_1,x_2,\ldots,x_n=b$. It is clear from the integral representation (6) that $y(x)$ is continuous at every point on the line, and one can readily deduce from that equation that its first derivative is continuous at every point with the possible exception of the points x_j, $j=0,1,\ldots,n$, where $q(x)$ is discontinuous. Now consider

$$\int_{x_j-\epsilon_1}^{x_j+\epsilon_2} y''(x)dx+k^2\int_{x_j-\epsilon_1}^{x_j+\epsilon_2} y(x)dx=\int_{x_j-\epsilon_1}^{x_j+\epsilon_2} q(x)dx \quad (j=0,1,\ldots,n), \tag{A1}$$

with $\epsilon_1\geqslant 0$, $\epsilon_2\geqslant 0$, which follows from the differential equation (4). From this equation it is clear that

$$y'(x)|_{x_j+\epsilon_2}-y'(x)|_{x_j-\epsilon_1}=-k^2\int_{x_j-\epsilon_1}^{x_j+\epsilon_2} y(x)dx+\int_{x_j-\epsilon_1}^{x_j+\epsilon_2} q(x)dx \quad (j=0,1,\ldots,n). \tag{A2}$$

Because $y(x)$ is continuous and $q(x)$ piecewise continuous, it is evident on proceeding to the limits $\epsilon_1\rightarrow 0$ and $\epsilon_2\rightarrow 0$ that $y'(x)$ is also continuous at the points of discontinuity of $q(x)$.

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[5]See, for example, P. M. Morse and K. Uno Ingard, *Theoretical Acoustics* (Princeton U.P., Princeton, NJ, 1986), Chap. 4.

[6]See, for example, G. Arfken, *Mathematical Methods for Physicists* (Academic, San Diego, CA, 1985), 3rd ed., Secs. 16.5, 16.6.