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LETTER TO THE EDITOR

Orbit bifurcations and spectral statistics

M V Berry†, J P Keating‡§ and S D Prado‡

† H H Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TL, UK
‡ School of Mathematics, University of Bristol, Bristol BS8 1TW, UK
§ BRIMS, Hewlett-Packard Labs, Filton Road, Stoke Gifford, Bristol BS12 6QZ, UK

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Abstract. Systems whose phase space is mixed have been conjectured to exhibit quantum spectral correlations that are, in the semiclassical limit, a combination of Poisson and random-matrix, with relative weightings determined by the corresponding measures of regular and chaotic orbits. We here identify an additional component in long-range spectral statistics, associated with periodic orbit bifurcations, which can be semiclassically large. This is illustrated for a family of perturbed cat maps.

It has been conjectured that in the semiclassical limit the quantum spectral statistics of classically integrable systems are generically Poissonian, and that those of classically chaotic systems are generically given by the average over an appropriate random-matrix ensemble [1–4]. In between these two extremes lie systems whose phase space is mixed; that is, in which regular and irregular motion coexist. Such systems are said to exhibit *soft chaos* [5]. For these, it has been suggested that the quantum spectral statistics are a combination of Poisson and random-matrix, with relative weightings determined by the corresponding measures of the regular and chaotic orbits [6]. Our purpose here is to identify in this case an additional component in the long-range statistics that is associated with periodic orbit bifurcations and which can be semiclassically large.

A semiclassical theory for long-range spectral statistics has been developed [1, 7–9] based on Gutzwiller's trace formula [10], which relates the quantum density of states $d(E) = \sum_n \delta(E - E_n)$ to classical periodic orbits:

$$d(E) = \bar{d}(E) + \frac{1}{\pi \hbar^{\beta}} \sum_p \sum_{n=1}^{\infty} T_p A_{p,n} \cos\left(\frac{nS_p}{\hbar}\right) \tag{1}$$

where $\bar{d}$, the mean density, is $\mathcal{O}(\hbar^{-f})$ for a system with f degrees of freedom; the periodic orbits, labelled p, have action S_p (defined here to include the Maslov index), period $T_p = \mathrm{d}S_p/\mathrm{d}E$, and stability amplitude $A_{p,n} = |\det(M_p^n - 1)|^{-1/2}$, M_p being the monodromy matrix; for isolated orbits $\beta = 1$, and for orbits that lie in a d-dimensional family in phase space $\beta = (d+1)/2$. This theory explains the universality of spectral correlations on energy scales of the order of, but large compared with, the mean level separation, $\bar{d}^{-1}$, in the semiclassical limit for completely integrable systems, in which all orbits are confined to f-dimensional tori in phase space (i.e. $\beta = (f+1)/2$ in (1)), and for strongly chaotic systems, in which all orbits are isolated and unstable. It also describes the system-specific deviations from universality, on length scales that are semiclassically large compared with $\bar{d}^{-1}$, associated with the short-time dynamics.

In systems whose phase space is mixed, some periodic orbits are isolated and unstable, and others lie on tori in stable islands. It is then natural to conjecture that the spectral statistics will have both Poissonian and random-matrix components, weighted by the relative measures of regular and chaotic orbits [6, 11]. We shall call this the *phase-space volume rule*. There are two arguments that suggest why it might be correct. First, in the semiclassical limit some states are expected to condense onto stable islands, and others onto the surrounding chaotic sea, their relative densities being determined by the respective volumes in phase space [12]. The result then follows if the associated regular and irregular sub-spectra are assumed statistically independent. Second, the same result is obtained if in the semiclassical expressions for long-range spectral statistics, obtained using (1), the regular and chaotic orbits are treated as being uncorrelated, and their respective contributions are added incoherently. It has, furthermore, been demonstrated that the phase-space volume rule gives good approximations for various spectral statistics in a wide range of systems (for a recent review see [13]).

The point we wish to draw attention to here is that there can be an additional contribution to the long-range spectral correlations in mixed systems coming from the bifurcations of periodic orbits, and that under certain circumstances this can be semiclassically large. Bifurcations are critical events where orbits are created or destroyed by coalescence. This process is a characteristic phenomenon in systems exhibiting soft chaos when a parameter is changed. The generic bifurcations that occur in two-dimensional conservative systems, or, equivalently, one-dimensional area-preserving maps, have been classified by Meyer [14]. Altogether, one has to distinguish five qualitatively different cases corresponding to period-m-tuplings with $1 \leqslant m \leqslant 5$ (cases with $m > 5$ follow the same pattern as for $m = 5$). Semiclassically, the importance of these events is that the Gutzwiller amplitude $A_{p,n}$ in (1) diverges for the orbits involved. This is because in the derivation of the trace formula the periodic orbits, which appear as the stationary points in the action of a path integral, are assumed isolated. Obviously this fails when they coalesce. The remedy is to perform a uniform asymptotic expansion valid throughout the bifurcation process [15–19]. One result of this is to replace the divergence by a higher power β in (1) close to the bifurcation. Specifically, different types of bifurcation each have a characteristic amplitude exponent $\beta > 1$ and a characteristic width, which vanishes in the semiclassical limit, over which their contribution is anomalously large (far from the bifurcation (1) is valid).

In this letter we shall concentrate on single bifurcation events. Our aim is to demonstrate that these can have a significant influence on spectral statistics. In fact, this is immediately obvious from the preceding discussion: a bifurcating orbit makes a semiclassically larger contribution than expected to the density of states and so, for example, completely dominates the non-universal region in long-range statistics, such as the number variance, the spectral rigidity, and correlation functions of $d(E)$. This is particularly striking in the moments of the staircase function $N(E) = \int_0^E d(x)\,\mathrm{d}x$:

$$M_{2k} = \langle [N(E) - \langle N(E)\rangle]^{2k}\rangle \tag{2}$$

where $\langle\cdots\rangle$ denotes a local average around energy E. Take, for example, the second ($k = 1$) moment. For generic integrable systems one expects that semiclassically $M_2 \sim T_H/T_I$, where T_I is a characteristic (system dependent) short classical timescale, and $T_H = 2\pi\hbar\bar{d}$ is the Heisenberg time, which is $\mathcal{O}(\hbar^{1-f})$. Similarly, for chaotic systems, $M_2 \sim \ln(T_H/T_C)$, where T_C is the appropriate characteristic short classical timescale in this case. In a mixed system the phase-space volume rule implies that M_2 should be a given by a sum of such terms, where in each case T_H is multiplied by the fraction of the volume of the energy shell that is regular/chaotic respectively. However, it is clear from (1) that the contribution from a given bifurcating orbit is $\mathcal{O}(\hbar^{2(1-\beta)})$, and since $\beta > 1$ this is semiclassically larger than the whole chaotic component. Thus the characteristic powers of $\hbar$ associated with the different kinds of bifurcations form a set of exponents in the moments of the spectral staircase.

Before discussing further the consequences of these bifurcation contributions, we first illustrate the behaviour outlined above for the family of perturbed cat maps [20, 21]:

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \frac{\kappa}{2\pi} \cos(2\pi q) \begin{pmatrix} 1 \\ 2 \end{pmatrix} \bmod 1. \tag{3}$$

When the perturbation parameter $\kappa = 0$, this reduces to a cat map and is therefore uniformly hyperbolic. The perturbed maps are then guaranteed by Anosov's theorem to be strongly chaotic (i.e. all periodic orbits are isolated and exponentially unstable) for $\kappa \leqslant \kappa_{\max} = (\sqrt{3}-1)/\sqrt{5} \approx 0.33$. Outside this range bifurcations occur, producing stable islands and hence a mixed phase space. Our aim now is to illustrate the influence of one such bifurcation on the spectral statistics of the associated quantum map.

If q and p are viewed as representing, respectively, a position coordinate and its conjugate momentum, the mapping (3) is a canonical transformation on a phase space which has the topology of a two-torus, and may be obtained from the generating function $S(q'+m, q) - nq'$, where (m, n) are integer winding numbers and

$$S(q', q) = q'^2 - q'q + q^2 + \frac{\kappa}{4\pi^2} \sin(2\pi q) \tag{4}$$

is the action on the full (non-periodized) phase plane [22, 21]. Quantum mechanically, it follows from the fact that the phase space has unit volume that $\hbar = 1/(2\pi N)$, where N is an integer corresponding to the dimension of the appropriate Hilbert space of wavefunctions that are periodic in both their position and momentum representations. The $N \times N$ unitary matrix with elements

$$U_{Q',Q} = \sqrt{\frac{N}{\mathrm{i}}} \exp(2\pi \mathrm{i} N S(Q'/N, Q/N)) \tag{5}$$

then acts as a propagator on these wavefunctions which reduces in the usual way to the classical map (3) in the semiclassical limit $N \to \infty$ [23, 22, 20, 21].

We shall be concerned with the statistical distribution of the phases θ_n, $1 \leqslant n \leqslant N$, of the eigenvalues of U. For example, if $n(\theta)$ is their counting function (i.e. the number of eigenvalues with $0 \leqslant \theta_n < \theta$), we will focus on the number variance

$$V(L; N) = \frac{1}{2\pi} \int_0^{2\pi} \left(n\left(\theta + \frac{2\pi L}{N}\right) - n(\theta) - L \right)^2 \mathrm{d}\theta \tag{6}$$

the moments

$$M_{2k}(N) = \frac{1}{2\pi} \int_0^{2\pi} \left(n(\theta) - \frac{N\theta}{2\pi} - \alpha \right)^{2k} \mathrm{d}\theta \tag{7}$$

where

$$\alpha = -\frac{1}{\pi} \Im \ln \det(I - U) \tag{8}$$

and in particular the second ($k = 1$) moment. We will later make use of the fact that both $V(L; N)$ and $M_2(N)$ can be expressed directly in terms of the traces of powers of U:

$$V(L; N) = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin^2\left(\frac{n\pi L}{N}\right) |\operatorname{Tr}(U^n)|^2 \tag{9}$$

and

$$M_2(N) = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{|\,\mathrm{Tr}(U^n)|^2}{n^2}. \tag{10}$$

When $\kappa \leqslant \kappa_{\max}$, the classical dynamics is strongly chaotic and thus the statistical properties of the eigenphases should be the same as those calculated by averaging over the circular orthogonal ensemble (COE) of random-matrix theory; that is, in the limit $N \to \infty$ $V(L; N)$ should tend to the corresponding COE number variance:

$$V^{(\mathrm{COE})}(L) = \frac{2}{\pi^2}\Bigg[\ln(2\pi L) + \gamma + 1 + \frac{1}{2}(\mathrm{Si}(\pi L))^2 - \frac{\pi}{2}\,\mathrm{Si}(\pi L) - \cos(2\pi L) - \mathrm{Ci}(2\pi L) + \pi^2 L\left(1 - \frac{2}{\pi}\,\mathrm{Si}(2\pi L)\right)\Bigg] \tag{11}$$

where γ is Euler's constant. There are, however, two important exceptional cases. First, the eigenvalue statistics of the unperturbed ($\kappa = 0$) cat map are known to be non-generic, because the quantum dynamics is periodic with a period that depends strongly and erratically on the prime factors of N [22, 24]. Second, it is a special feature of the particular map (3) that simple, κ-independent relations exist between the matrix elements (5) when N is divisible by four. These then render random-matrix theory inappropriate as a model for the eigenvalue statistics. This property is also non-generic and so in the following discussion we restrict ourselves to the range $\kappa > 0$ and values of N that are not divisible by four.

The approach to the limit $V^{(\mathrm{COE})}(L)$ as $N \to \infty$ is non-uniform in L. Semiclassically [8], when N is large and fixed, $V(L; N)$ is well approximated by $V^{(\mathrm{COE})}(L)$ for $L \ll L^*(N)$, where $L^*(N)$ is a correlation scale of the order of N that is conjugate to the discrete-time analogue of T_C, and saturates into non-universal quasiperiodic oscillations with a variance that is $\mathcal{O}(1)$ around a mean value that is approximately $(2/\pi^2)\ln N$ when $L \gg L^*(N)$. This mean saturation value is twice the second moment; that is $M_2(N) \sim (1/\pi^2)\ln N$, which coincides precisely with the leading-order asymptotics of the corresponding COE result. Numerical computations in the range $0 < \kappa \leqslant \kappa_{\max}$ confirm this behaviour. A representative sample of data is plotted in figure 1.

When κ increases beyond $\kappa_{\max}$ the dynamics becomes mixed: some orbits are stable and confined to invariant curves, whilst others remain unstable and ergodic in regions of the phase space. The phase-space volume rule implies that if the area of the stable islands is ρ, then in the limit as $N \to \infty$,

$$V(L) = V^{(\mathrm{Poisson})}(\rho L) + V^{(\mathrm{COE})}((1-\rho)L) \tag{12}$$

where

$$V^{(\mathrm{Poisson})}(L) = L \tag{13}$$

is the number variance for a Poisson distributed (i.e. uncorrelated) sequence. (This of course assumes that one can divide the phase space up into distinct islands and ergodic regions, but when N is fixed and large it is expected to provide a good approximation to $V(L; N)$ for $L \ll L^*(N)$, because ρ only has to be measured down to the scale of resolution, corresponding to an area equal to $1/N$, set by the uncertainty principle.) For the second moment, it follows from the general results discussed in the paragraph following (2) that the expression analogous to (12) is

$$M_2(N) \sim aN + \frac{1}{\pi^2}\ln(bN) \tag{14}$$

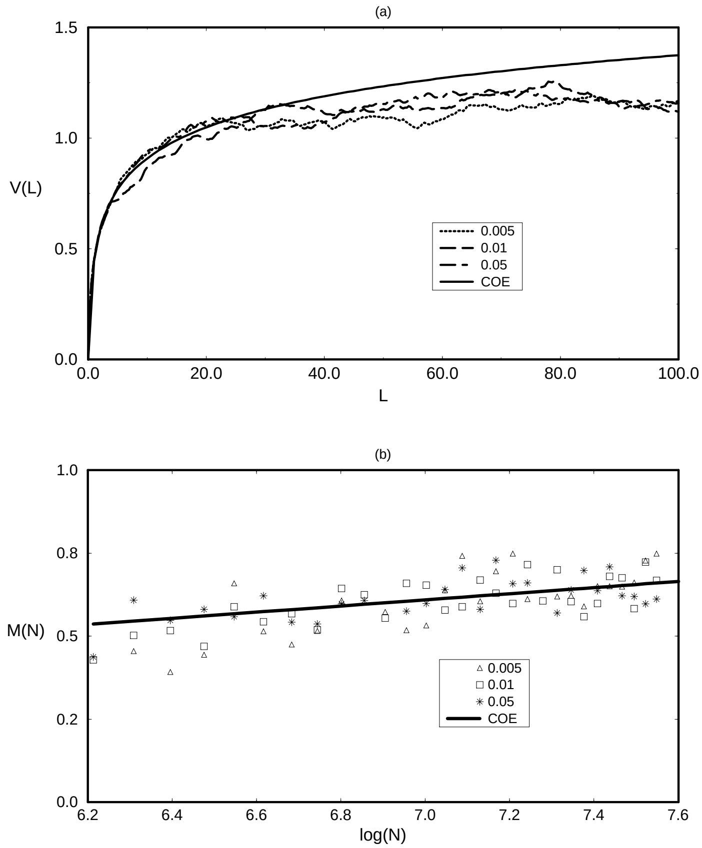

Figure 1. (*a*) $V(L)$ versus L for $N = 1999$, and (*b*) $M_2(N)$ versus $\log N$, for the values of κ indicated in the legend. In each case the continuous line represents the COE result.

where a and b are constants. Numerically, (12) is indeed a good approximation for many values of κ and N. A typical set of data is plotted in figure 2.

Whilst the number variance for the perturbed cat maps conforms to the simple picture outlined above for many values of κ, there are some for which it shows qualitatively different behaviour. As an example of this, we take the case when $\kappa = \kappa^* = 5.943\,38$. As may be seen from figure 3(*b*), the phase space is then almost entirely ergodic; that is, there are no islands on the scale that affects the spectral statistics for the values of N we shall be concerned with. $V(L; N)$ should thus be well approximated by $V^{(\mathrm{COE})}(L)$ before reaching a mean saturation value of $(2/\pi^2)\ln N$. The actual number variance, calculated numerically for $N = 1567$, is shown as triangles in figure 3(*a*). Also shown in this figure is the difference (squares) between $V(L; N)$ and $V^{(\mathrm{COE})}(L)$ (full curve). The point to note is that rather than saturating, the data actually 'lifts off', reaching a much higher value than expected ($(2/\pi^2)\ln 1567 = 1.490\ldots$). We will now demonstrate that this is caused by the bifurcation of a periodic orbit of the classical map (3) at $\kappa = \kappa^*$.

It was shown in [21] that for the map (3) $\mathrm{Tr}\, U$ can be expressed in the form

$$\mathrm{Tr}\, U = \sqrt{(N/\mathrm{i})} \sum_{j=0}^{1} \int_{-\infty}^{\infty} \exp\left(2\pi \mathrm{i} N S_j(q)\right) \mathrm{d}q \qquad (15)$$

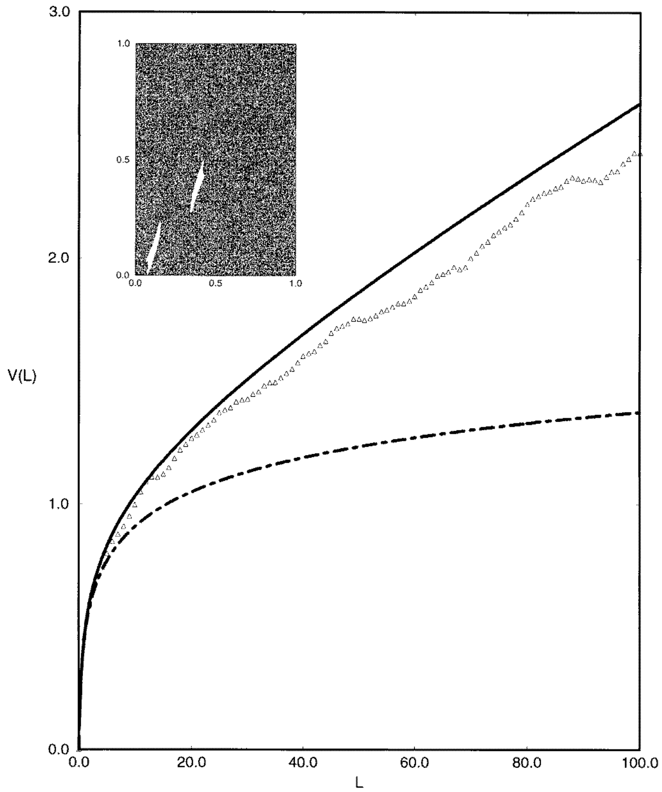

Figure 2. $V(L)$ versus L, for $\kappa = 6.5$ and $N = 1999$. The full curve represents the volume rule (12), the chain curve the COE result, and the triangles the numerical data. The inset is a phase-space plot of typical orbits. The islands have an area $\rho \approx 0.0126$.

where $S_j(q) = S(q, q) - jq$. When $\kappa \leqslant \kappa_{\max}$ the j-sum has a simple interpretation: the terms are in one-to-one correspondence with the fixed points of the underlying classical map. Thus (15) may be viewed as an exact trace formula. A generalization of this result, relating $\mathrm{Tr}\, U^n$ to period-n orbits, and holding for a general class of perturbed cat maps, was also derived in [21]. The usual trace formula may be obtained by evaluating the integral in (15) using the method of steepest descent. This involves expanding the exponent to quadratic order around the real saddle point $q_j(\kappa)$, defined to be the (unique, when $\kappa \leqslant \kappa_{\max}$) real solution of $S_j'(q) = 0$. The result is that

$$\mathrm{Tr}\, U \approx \sum_{j=0}^{1} \frac{\exp(2\pi \mathrm{i} N S_j(q_j))}{\sqrt{S_j''(q_j)}}. \tag{16}$$

Note that when $\kappa > 0$ there are also infinitely many complex solutions of the saddle-point equation. These correspond to tunnelling orbits and give rise to contributions to the trace that are exponentially small in N.

As κ increases beyond $\kappa_{\max}$ the complex solutions of the saddle-point equation approach the real-q axis in complex conjugate pairs, until at $\kappa = \kappa^*$ the first pair coalesce there. For $\kappa > \kappa^*$ this pair then separate, each moving along the real-q axis. Dynamically, this is a tangent bifurcation: the birth of two real fixed points, one stable and the other unstable, from two complex fixed points. We illustrate this by showing in figure 3(c) orbits in a small

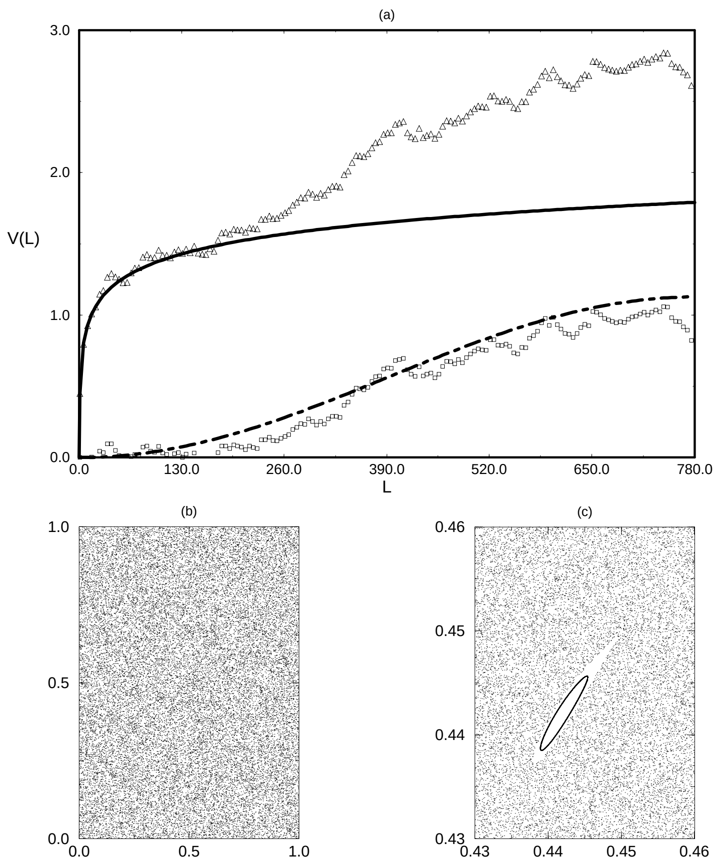

Figure 3. (*a*) $V(L)$ versus L for $\kappa = \kappa^*$, with the full curve representing the COE prediction, triangles the numerical data, squares the difference between these, and the chain curve the contribution from the $n = 1$ term in (9) calculated using (18). A phase-space orbit plot is shown in (*b*) for $\kappa = \kappa^*$, and a part of the phase space is plotted in (*c*) for $\kappa = 5.945$, just after the bifurcation has occured. The island is centred on the newly created stable orbit.

part of the phase space that includes the bifurcation point when $\kappa = 5.945$ (i.e. just after the bifurcation has occurred); the island of invariant curves surrounds the newly created stable orbit.

The leading-order asymptotic approximation (16) holds for all κ if the j-sum is extended to include all saddle points on the real-q axis, except at bifurcation points, where by definition $S_j''(q_j) = 0$ and so the corresponding term diverges. To remedy this divergence, and to derive a uniformly accurate approximation, one must expand to cubic order around the bifurcation point. The result of doing this is as follows. If the two solutions of the saddle-point equation corresponding to the orbits involved in the bifurcation occur at $q_j^{(+)}(\kappa)$ and $q_j^{(-)}(\kappa)$ (which form a complex-conjugate pair when $\kappa < \kappa^*$, and are real when $\kappa \geqslant \kappa^*$), and if $S_j^{(+)} = S_j(q_j^{(+)}(\kappa))$ and $S_j^{(-)} = S_j(q_j^{(-)}(\kappa))$, then

$$\mathrm{Tr}\, U \approx \sum_{j=0}^{1} \frac{\exp\left(2\pi \mathrm{i} N S_j(q_j)\right)}{\sqrt{S_j''(q_j)}} + \left(\frac{4\pi^2}{\beta_j}\right)^{1/3} N^{1/6} \exp\left(2\pi \mathrm{i} N \bar{S}_j - \mathrm{i}\frac{\pi}{4}\right) \mathrm{Ai}(z_j) \qquad (17)$$

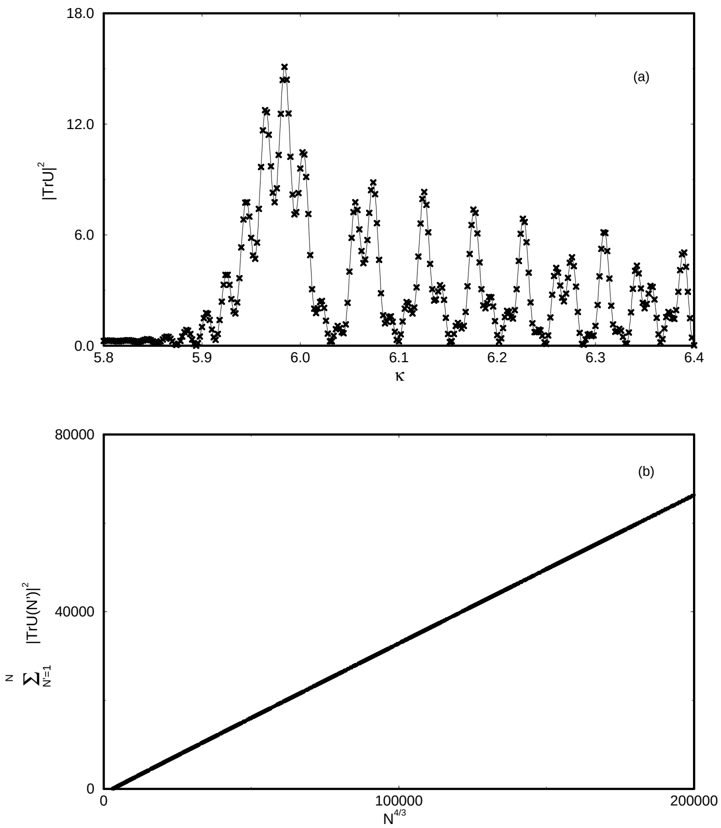

Figure 4. (*a*) $|\,\mathrm{Tr}\,U|^2$ vs κ for $N = 1567$, the line representing the approximation (17) and the crosses a direct numerical evaluation of the trace using (5), and (*b*) $\sum_{N'=1}^{N} |\,\mathrm{Tr}\,U(N')|^2$ versus $N^{4/3}$ for $\kappa = \kappa^*$, calculated in the same way.

where $\bar{S}_j = (S_j^{(+)} + S_j^{(-)})/2$, $\beta_j = \frac{3}{2}|\frac{\Delta S_j}{\epsilon_j^3}|$, and $z_j = -\mathrm{sign}(\kappa - \kappa^*)(3\pi N|\Delta S_j|)^{2/3} = -(2\pi\beta_j N)^{2/3}\epsilon_j^2$, with $\epsilon_j = (q_j^{(+)} - q_j^{(-)})/2$ and $\Delta S_j = (S_j^{(+)} - S_j^{(-)})/2$. A numerical verification of this approximation is shown in figure 4. At the bifurcation point itself, that is when $\kappa = \kappa^*$ and hence $q_j^{(+)} = q_j^{(-)}$, (17) reduces to

$$\mathrm{Tr}\,U \approx \sum_{j=0}^{1} \frac{\exp\left(2\pi \mathrm{i} N S_j(q_j)\right)}{\sqrt{S_j''(q_j)}} + \frac{N^{1/6}}{\Gamma(2/3)} \left(\frac{4\pi}{9\kappa^*}\right)^{1/3} \frac{\exp\left(2\pi \mathrm{i} N S_j^{(+)} - \mathrm{i}\frac{\pi}{4}\right)}{|\cos(2\pi q_j^{(+)})|^{1/3}} \qquad (18)$$

showing how the bifurcating orbits dominate the contributions from the pre-existing real fixed points as $N \to \infty$.

The factor $N^{1/6}$ in the contribution from the bifurcating orbits is the analogue of $\hbar^{-\beta}$ in (1); that is, $\frac{1}{6}$ is the characteristic exponent associated with a tangent bifurcation for maps. It is clear that $|\,\mathrm{Tr}\,U|^2 \sim N^{1/3}$ as $N \to \infty$, rather than being $\mathcal{O}(1)$, which would be the case far from the bifurcation. This can be verified by direct numerical calculation of the trace when $\kappa = \kappa^*$; specifically, in figure 4(*b*) we plot $\sum_{N'=1}^{N} |\,\mathrm{Tr}\,U(N')|^2$ as a function of $N^{4/3}$, which is expected to be a straight line on the basis of (18).

The consequences of the fact that $|\,\mathrm{Tr}\,U|^2 \sim N^{1/3}$ for $V(L; N)$ and $M_2(N)$ are now clear, given (9) and (10). First, for $V(L; N)$ the contribution from the short-time (small

n in (9)) dynamics increases as $N^{1/3}$. This means that both the mean and the variance of the quasiperiodic oscillations in the saturation region increase in the same way, rather than, respectively, increasing as $\ln N$ and being constant, as is the case far from a bifurcation. It is thus the explanation of the lift-off in the variance exhibited in figure 3(a), as is verified in the same figure by the fact that the difference between the COE curve and the data is well approximated by the $n = 1$ contribution to (9) calculated using the bifurcation formula (18).

The effect on the second moment is even more striking. It follows from (10) that when κ is close to κ^* the bifurcation contributes a term proportional to $N^{1/3}$ that must be added to the phase-space volume rule result (14); that is

$$M_2(N) \sim aN + \frac{1}{\pi^2}\ln(bN) + cN^{1/3} \tag{19}$$

where c is a constant. This new term is semiclassically large, and indeed dominates the contributions from all of the isolated and unstable periodic orbits. The size $\Delta\kappa$ of the range of values of κ in which it must be included may be deduced from (17): the argument of the Airy function is proportional to $|\kappa - \kappa^*|N^{2/3}$, and so $\Delta\kappa = \mathcal{O}(N^{-2/3})$. Consequently, the tangent bifurcation makes a semiclassically small contribution to averages over the second moment with respect to κ. However, for higher moments this is no longer the case; in the $2k$th moment defined by (7) it obviously gives rise to a term that is $\mathcal{O}(N^{k/3})$, and so for $k > 2$ its net contribution to an average over a κ-range that includes κ^* is semiclassically large.

The analysis and results described above extend immediately to tangent bifurcations in general maps and flows: in addition to the phase-space volume rule terms, the contribution to the $2k$th moment of the spectral staircase is $\mathcal{O}(\hbar^{-k/3})$ in a parameter range with a size that is $\mathcal{O}(\hbar^{2/3})$; and for the number variance and other long-range statistics there is a 'lift-off' to anomalously large fluctuations in the non-universal regime. They also generalize to other generic (and non-generic—see, for example [25]) bifurcations, each of which have their own characteristic $\hbar$-exponent in the trace formula and 'width' in parameter space, and to the remarkable bifurcation cascades that have been found in typical systems [26] (and which we have observed in the perturbed cat maps). This hints at a semiclassical picture for the spectral statistics in mixed systems where there is, in addition to the phase-space volume rule, a rich structure associated with the competition between the various kinds of periodic orbit bifurcations, which would be a new example of the 'singularity dominated strong fluctuations' already familiar in optics and elsewhere [27–29]. The possibility remains that a new universality will emerge when one averages over these contributions (e.g. by averaging over a system parameter in regimes where bifurcations are dense), and this, we believe, warrants further investigation.

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References

[1] Berry M V and Tabor M 1977 *Proc. R. Soc.* A **356** 375–94
[2] Bohigas O, Giannoni M-J and Schmit C 1984 *Phys. Rev. Lett.* **52** 1–4
[3] Berry M V 1987 *Proc. R. Soc.* A **413** 183–98
[4] Mehta M L 1991 *Random Matrices* (San Diego, CA: Academic)
[5] Gutzwiller M C 1990 *Chaos in Classical and Quantum Mechanics* (New York: Springer)

[6] Berry M V and Robnik M 1984 *J. Phys. A: Math. Gen.* **17** 2413–21
[7] Hannay J H and Ozorio de Almeida A M 1984 *J. Phys. A: Math. Gen.* **17** 3429–40
[8] Berry M V 1985 *Proc. R. Soc.* A **400** 229–51
[9] Bogomolny E B and Keating J P 1996 *Phys. Rev. Lett.* **77** 1472–5
[10] Gutzwiller M C 1971 *J. Math. Phys.* **12** 343–58
[11] Seligman T H and Verbaarschot J J M 1985 *J. Phys. A: Math. Gen.* **18** 2227–34
[12] Percival I C 1973 *J. Phys. B: At. Mol. Phys.* **6** L229–32
[13] Robnik M 1996 *Proc. Conf. on Nonlinear Phenomena in Complex Systems (Minsk)* in press
[14] Meyer K R 1970 *Trans. Am. Math. Soc.* **149** 95–107
[15] Ozorio de Almeida A M and Hannay J H 1987 *J. Phys. A: Math. Gen.* **20** 5873–83
[16] Sieber M 1996 *J. Phys. A: Math. Gen.* **29** 4715–32
[17] Schomerus H and Sieber M 1997 *J. Phys. A: Math. Gen.* **30** 4537–62
[18] Schomerus H 1997 *Europhys. Lett.* **38** 423–8
[19] Schomerus H and Haake F 1997 *Phys. Rev. Lett.* **79** 1022–5
[20] Basilio de Matos M and Ozorio de Almeida A M 1995 *Ann. Phys.* **237** 46–65
[21] Boasman P A and Keating J P 1995 *Proc. R. Soc.* A **449** 629–53
[22] Keating J P 1991 *Nonlinearity* **4** 309–41
[23] Hannay J H and Berry M V 1980 *Physica* **1D** 267–90
[24] Keating J P 1991 *Nonlinearity* **4** 277–307
[25] de Aguiar M A M, Malta C P, Baranger M and Davies K T R 1987 *Ann. Phys.* **180** 167–205
[26] Sadovskiíi D A, Shaw J A and Delos J B 1995 *Phys. Rev. Lett.* **75** 2120–3
[27] Berry M V 1977 *J. Phys. A: Math. Gen.* **10** 2061–81
[28] Berry M V 1982 *J. Phys. A: Math. Gen.* **15** 2735–49
[29] Berry M V 1986 *Wave Propagation and Scattering* ed B J Uscinsci (Oxford: Clarendon) pp 11–35