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Spectral twinkling

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1. – Introduction

It is known that the arrangement of energy levels of quantum systems with classical limits, and in particular the spectral statistics, is related to the chaology of the classical orbits [1-3]. If the trajectories are integrable, Poisson statistics typically apply for short-range correlations; if chaotic, the typical short-range statistics are those of the appropriate random-matrix ensemble ("typical" means: apart from exceptions, where, for arithmetical reasons, the statistics are different). The long-range spectral correlations are not universal [4], but depend on the details of the classical periodic orbits, which are isolated and unstable for chaotic systems, and fill resonant tori for integrable ones.

Much less is known about systems whose dynamics is mixed—that is, neither integrable nor chaotic—where all closed orbits are isolated but some are surrounded by tori (according to the KAM theorem). At short range, the statistics can usefully be approximated by those of a superposition of Poisson and random-matrix spectra [5]. But is there a more fundamental theory? That is the question I address in these lectures.

Arguments arrived at in conversations with Jonathan Keating—and certainly not yet amounting to a complete theory—suggest that there are characteristic statistics associated with the mixed regime, and that these are universal. They are associated with *bifurcations* [6], where combinations of stable and unstable orbits collide and transform into others, or annihilate, as a parameter (for example energy) varies—it is the ubiquity of bifurcations, after all, that characterizes mixed systems. The main result will be the suggestion that the spectral moments—describing the fluctuations in the distribution of energy levels as explained below (eq. (3))—are dominated by a competition among the different sorts of bifurcation.

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Bifurcations are singularities of the dynamics, and the statistics to be suggested here are a new example of the wider class of *singularity-dominated strong fluctuations.* This old [7-9] but still unfamiliar idea is that some variables exhibit wild (non-Gaussian) fluctuations, with very large values described by scaling laws and associated with particular geometric singularities (not fractals). It will be illustrated in sect. **2** by three very different examples—none of them new, but chosen to lead into the spectral application that is our main interest. First, we describe the quantities we intend to estimate.

Like all chaology-related statistics, the ones to be discussed here are semiclassically emergent, that is, emergent in the limit $\hbar \to 0$ of vanishing Planck's constant. In the present context, that needs to be made more explicit than it usually is. Consider a set of levels $E_1(\hbar), E_2(\hbar), \ldots, E_j(\hbar), \ldots$. Commonly, the $\hbar$-dependence is non-trivial, but for some systems (such as quantum billiards) it is a simple scaling, and the semiclassical limit then corresponds to the high-energy limit. The spectrum can be characterized by the counting function, or spectral staircase:

$$\mathcal{N}(E,\hbar) = \sum_{j=1}^{\infty} \Theta(E - E_j(\hbar)), \tag{1}$$

where Θ denotes the unit step.

It is convenient to follow the usual procedure, and separate $\mathcal{N}$ into its smooth and fluctuating parts:

$$\mathcal{N}(E,\hbar) = \mathcal{N}_{\mathrm{sm}}(E,\hbar) + \mathcal{N}_{\mathrm{fl}}(E,\hbar). \tag{2}$$

$\mathcal{N}_{\mathrm{sm}}$ is not interesting here (it is given by the Weyl rule [1] plus $\hbar$-corrections). Our focus is the semiclassical size of the spectral fluctuations, as embodied in the *spectral moments*

$$M_m(\hbar) = \langle [\mathcal{N}_{\mathrm{fl}}(E,\hbar)]^{2m} \rangle, \tag{3}$$

where $\langle \cdots \rangle$ denotes a local average over energy or parameters in the Hamiltonian. The main conjecture is that

$$\lim_{\hbar \to 0} \frac{\partial \log\{M_m(\hbar)\}}{\partial \log\{1/\hbar\}} = \nu_m, \tag{4}$$

that is

$$M_m(\hbar) \sim \frac{\text{constant}}{\hbar^{\nu_m}} \text{ (up to logarithms) as } \hbar \longrightarrow 0, \tag{5}$$

where ν_m are the *twinkling exponents*: universal numbers determined by the hierarchy of bifurcations. "Twinkling" will be explained later.

To see the importance of bifurcations, first recall the trace formulas: $\mathcal{N}_{\mathrm{fl}}(E)$ can be represented semiclassically as a sum over periodic orbits [10, 11]. In the generic case, where the orbits are isolated, the sum is over primitive periodic orbits p with energy E, and their repetitions r:

$$\mathcal{N}_{\mathrm{fl}}(E,\hbar) \approx \frac{1}{\pi}\sum_{p}\sum_{r=1}^{\infty}\frac{\sin\{S_{p,r}(E)/\hbar-\gamma_{p,r}\}}{r\sqrt{|\det([\mathbb{M}_p(E)]^r-1)|}}. \tag{6}$$

Here $S_{p,r}$ is the action of the orbit, $\mathbb{M}_p$ is the monodromy matrix describing the linearised return map on the Poincaré section, and $\gamma_{p,r}$ is the Maslov index (which will play no further part in this story).

In the integrable case, for two freedoms, with action Hamiltonian

$$H(\mathbf{I}),\quad (\mathbf{I}=\{I_1,I_2\}), \tag{7}$$

the sum is over resonant tori characterized by their winding numbers $\mathbf{W}=\{W_1,W_2\}$, and the trace formula [12] is

$$\mathcal{N}_{\mathrm{fl}}(E,\hbar) \approx \frac{1}{\pi\hbar^{1/2}}\sum_{\mathbf{W}}\frac{\sin\{2\pi\mathbf{W}\cdot\mathbf{I}_{\mathbf{W}}(E)/\hbar-\gamma_{\mathbf{W}}\}}{|\mathbf{W}|^{3/2}\sqrt{K(\mathbf{I}_{\mathbf{W}}(E))}}. \tag{8}$$

Here $\mathbf{I}_{\mathbf{W}}(E)$ are the actions of resonant tori, where the frequencies $\boldsymbol{\omega}$ are commensurate, that is (with orbit period T)

$$\boldsymbol{\omega}(\mathbf{I}_{\mathbf{W}}(E)) \equiv \nabla_{\mathbf{I}}H(\mathbf{I}_{\mathbf{W}}(E)) = \frac{2\pi}{T}\mathbf{W}, \tag{9}$$

and K is the curvature of the energy contour $H(\mathbf{I})=E$ in $\mathbf{I}$ space.

When these formulas apply, there are no strong fluctuations, and the estimation of $\mathcal{N}_{\mathrm{fl}}$—whose distribution is approximately Gaussian—is fairly simple. For the chaotic case (6), the prefactor is of order $\hbar^0$; the sum diverges, but can be regularized by truncation at orbits with period equal to the Heisenberg time $\hbar$/(mean level spacing), which, together with the exponential proliferation of orbits with increasing period, leads to [4]

$$|\mathcal{N}_{\mathrm{fl}}| \sim \sqrt{\log(1/\hbar)}, \tag{10}$$

and moments (5) with all twinkling exponents $\nu_m=0$. For the two-dimensional integrable case (8), where the sum converges,

$$|\mathcal{N}_{\mathrm{fl}}| \sim \hbar^{-1/2}, \tag{11}$$

and (3) and (5) give $\nu_m=m$.

Here we are interested in cases when the trace formulas fail. This happens at bifurcations of periodic orbits. In (6), bifurcations of isolated orbits correspond to a unit eigenvalue of the monodromy matrix, so $\det(\mathbb{M}-1)$ vanishes, and the terms representing those orbits diverge. In (8) bifurcations of tori correspond to coalescence of parallel normals (cf. (9)) to the energy surface, so that K vanishes and the terms representing those orbits diverge. The formulas fail, but it is clear that bifurcations correspond to large values of $\mathcal{N}_{\text{fl}}$. How large? This has been studied by several authors [13-17], who have found (for the isolated-orbit case) corrected versions of the trace formula that incorporate the bifurcations correctly, with the result that $\mathcal{N}_{\text{fl}}$ does not diverge but rises to values that increase as $\hbar \to 0$. We extend these results to estimate the moments $M_m(\hbar)$ and hence the twinkling exponents ν_m, in sect. **3** for integrable systems and in sect. **4** for the mixed systems we are mainly interested in.

2. – Examples of singularity-dominated strong fluctuations

2·1. *Smells in random winds, and the sex life of moths.* – In some species of moth, females attract males by smell [18, 19]. They emit pheromones (a cocktail of molecules), and these spread by convection and diffusion to create an "odour plume". It is the form of this odour plume that is of interest here. It can be represented as the concentration $C(\mathbf{r}, t)$ of a passive scalar (the pheromone) with diffusion constant D, in a wind with velocity $\mathbf{v}(\mathbf{r}, t)$. Both diffusion and convection are important: diffusion alone is too slow, so if there is no wind the female does not emit pheromones (nor does she emit if the wind is too strong, because then the male would be unable to swim fast enough upwind to reach her). We are interested in the form of the plume for small D, and in particular the singularities that dominate the $D \to 0$ fluctuations of $C(\mathbf{r}, t)$ [20].

Concentration is governed by the diffusion equation with convective time derivative:

$$\partial_t C + \mathbf{v} \cdot \nabla C = D \nabla^2 C. \tag{12}$$

As physicists, we like to transform every problem into quantum mechanics, so we define

$$D = i\hbar, \qquad \mathbf{p} = -i\hbar\nabla = -D\nabla, \tag{13}$$

whereupon (12) becomes

$$i\hbar\partial_t C = \left(p^2 + \mathbf{p} \cdot \mathbf{v}\right) C = \left[\left(\mathbf{p} + \frac{1}{2}\mathbf{v}(\mathbf{r}, t)\right)^2 - \frac{1}{4}|\mathbf{v}(\mathbf{r}, t)|^2\right] C. \tag{14}$$

The required solution is that of a steady point source, so C is the zero-energy Green function of a charged quantum particle governed by the Schrödinger equation (14), with magnetic vector potential $-\mathbf{v}/2$ and electric scalar potential $-|\mathbf{v}|^2/4$. Moreover, the required small-D limit is just the semiclassical limit.

This looks promising. We have a great deal of intuition about semiclassical limits, and it is natural to bring it to bear on the plume problem. Conventional wisdom dictates that we should calculate the family of "rays of smell" emitted by the steady point source (that is with zero "Hamiltonian" given by the classical counterpart of the operator in (14)), and then find the caustic—that is, the envelope of these rays—where the field should be strongest. The moth plume would then be concentrated on "olfaction catastrophes". A classical model for this procedure is Kelvin's pattern of waves behind a moving ship [21-24], whose characteristic V shape (with angle $2\arcsin(1/3) = 38°56'$) is the caustic of rays of the Hamiltonian for deep-water gravity waves convected by the velocity (of water relative to the ship).

But the intuition is wrong, dead wrong, and in a way that bears strongly on our theme of singularities and strong fluctuations. To see why, consider the semiclassical approximation to the solution of (12) or (14). This is a sum of contributions from zero-energy rays of smell, of the form

$$C(\mathbf{r},t) \approx a(\mathbf{r},t)\exp\left[-\frac{S(\mathbf{r},t)}{D}\right], \tag{15}$$

involving the action

$$S(\mathbf{r},t) = \int_{t_0(\mathbf{r},t)}^{t} \mathrm{d}\tau L[\mathbf{r}(\tau),\dot{\mathbf{r}}(\tau),\tau] \tag{16}$$

(S is a solution of the Hamilton-Jacobi equation with $\partial S/\partial t_0 = 0$) with Lagrangian (from (14))

$$L[\mathbf{r},\dot{\mathbf{r}},t] = \frac{1}{4}(\dot{\mathbf{r}} - \mathbf{v}(\mathbf{r},t))^2 \tag{17}$$

which is never negative.

Now, caustics correspond to singularities in the semiclassical amplitude in (15), softened by the correct small-D asymptotics into power law divergences. But these are totally obscured by the negative exponential "phase" factor in (15), which for small D obliterates all rays except those with zero action, that is zero Lagrangian, that is the particular rays satisfying

$$\dot{\mathbf{r}}(t) = \mathbf{v}(\mathbf{r},t). \tag{18}$$

These very special rays are the "streak lines" of fluid mechanics, corresponding to fluid particles that have passed through the female moth (for a steady wind $\mathbf{v}(\mathbf{r})$ they coincide with streamlines). They—not the caustics—are the singularities dominating the odour plume. In the limit of zero D, $C(\mathbf{r},t)$ is a delta-function concentrated on the streak line—typically a complicated coil downstream of the female. For small but finite D, (15) shows that the plume is Gaussian-broadened away from the streak line and essentially zero elsewhere. On the streak line, C is of order $1/D$, and the transverse width of the broadened plume is of order $\sqrt{D}$ (the typically high anisotropy of the broadening, associated with exponential instability of the convecting flow, will not be discussed here. though it is interesting).

The message of this example is that the classical limit is so singular that vanishing real $\hbar$ and vanishing imaginary $\hbar$ (*i.e.* vanishing D) are utterly different, to the extent that they are dominated by different singularities.

Now we use the singularity structure of the plume to estimate the moments of the concentration fluctuations, that is $\langle C(\mathbf{r},t)^m\rangle$, in a calculation that is a simple model for more sophisticated examples to follow. We write

$$
\begin{aligned}
(19)\quad & \langle C(\mathbf{r},t)^m\rangle \sim \\
& \sim (\text{size of } C)^m \times (\text{probability of encountering the plume} \sim \text{area of diffuse streak}) \sim \\
& \sim \frac{1}{D^m} \times \left(\sqrt{D}\right)^2 = \frac{1}{D^{m-1}}.
\end{aligned}
$$

These are the singularity-dominated strong fluctuations, characterized by a (here very simple) power law. The D dependence might be biologically interesting: the different chemicals in the pheromone cocktail have different D and hence different fluctuations, which might give the male a clue as to how far away the female is—a sort of smell fluctuation spectroscopy [20].

2·2. *van Hove singularities and kin.* – Consider the family of smooth functions

$$
(20)\qquad H(\mathbf{r};\mathbf{P}), \qquad \mathbf{r} = \{x,y\}, \qquad \mathbf{P} = \{P_1, P_2, \dots\},
$$

of two variables $\mathbf{r}$ and arbitrary number of parameters $\mathbf{P}$. Form the integral

$$
(21)\qquad I(E;\mathbf{P}) = \iint \mathrm{d}x\,\mathrm{d}y\,\delta\{E - H(\mathbf{r};\mathbf{P})\} = \oint_{H=E} \frac{\mathrm{d}s}{|\nabla H|},
$$

capturing the variations of H across its contours (with arc length s).

Much physics is described by this formalism. In a two-dimensional flow with velocity potential H, I is the circulation time of fluid particles with $H = E$; here x and y are spatial coordinates. In a one-dimensional dynamical system with Hamiltonian H, I is the period of orbits with energy E; here x is the position coordinate and y the conjugate momentum. In a two-dimensional crystal with band structure $H(\mathbf{r})$, I is the density of states; here x and y are both momenta. There are still more interpretations.

We are interested in the size of $I(E;\mathbf{P})$ when I is a smooth random function of E and $\mathbf{P}$, and in particular the probability distribution $P(I)$ for large I [8], which we call the *probability tail.* $P(I)$ is defined as

$$
(22)\qquad P(I) = \langle \delta(I - I(E;\mathbf{P}))\rangle.
$$

In accordance with our general philosophy, we seek to identify the singularities that dominate the probability tail. First consider the variation of E with the parameters $\mathbf{P}$ fixed. I varies smoothly, except at critical points $E = E_c$ where $\nabla_{\mathbf{r}} H = 0$. Critical points are extrema, where closed contours shrink to zero, and saddles, where contours cross. The behaviour of $I(E)$ near E_c is

$$\begin{aligned}(23) \qquad & \text{extrema}: I(E) \sim \operatorname{Re}\left\{\left[\pm (E-E_c)^{1/2}\right]\right\} \quad (+:\text{minima}; -:\text{maxima}),\\ & \text{saddles}: I(E) \sim \log\left(\frac{1}{|E-E_c|}\right).\end{aligned}$$

In crystals, these are the van Hove singularities. Obviously, large values of I correspond to saddles. The probability tail is

$$(24) \qquad P(I) \sim \int \mathrm{d}E\,\delta\left(I - \log\left(\frac{1}{|E|}\right)\right) \sim \exp[-I].$$

This is a pretty weak tail. We can produce far stronger tails by varying the parameters $\mathbf{P}$ to make critical points coalesce, so that

$$(25) \qquad \frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2} - \left(\frac{\partial^2 H}{\partial x \partial y}\right) = 0, \quad \text{as well as } \nabla_{\mathbf{r}} H = 0.$$

These coalescences of critical points in families of functions are the singularities that dominate the probability tail. Their study is *catastrophe theory* [25, 26]. One result of this branch of mathematics is a list of polynomial normal forms [27-29] for $H(\mathbf{r};\mathbf{P})$, each representing a different topology of coalescence of critical points, that is, a different catastrophe. In each such polynomial, the number of parameters is the *codimension* K, and the number of variables (in our case one or two) is the *corank*. The classification is complete for corank 1; these are the cuspoid singularities. For corank 2 the list is complete up to very high K but not for all K.

The catastrophes possess remarkable homogeneity properties, to be described in more detail in the next section, leading to two indices β_j and γ_j for the catastrophe labelled j. β_j describes [30] the geometry of the $\mathbf{r}$-contours of H when they are most singular, that is when the parameters are chosen to make all the critical points coalesce, and γ_j describes the neighbourhood of the singularity in $\mathbf{P}$ space. For any given catastrophe, the probability tail $P_j(I)$ can be determined by a scaling argument [8] (not trivial, and not given here) whose result is

$$(26) \qquad P_j(I) \sim \frac{1}{I^{(1+(\gamma_j+1)/\beta_j)}} \quad (I \gg 1).$$

For a random function H sufficiently extended in $\mathbf{r}$, all singularities can occur for some parameters $\mathbf{P}$, and each will give rise to a probability tail of the form (26). The complete probability tail will be dominated by the singularity—if there is one—whose tail decays slowest, that is where $(\gamma_j + 1)/\beta_j$ is smallest. This is a competition, whose outcome must be determined by calculation using the normal forms of all the singularities that have been classified. The result [8] is that the smallest value of $(\gamma_j + 1)/\beta_j$ is 8 (corresponding to the two umbilic singularities with codimension 3) so that

$$P(I) \sim I^{-9}. \tag{27}$$

(The exponent is different if H has a special form restricting the singularities in the competition.)

2·3. *Twinkling starlight.* – Stars twinkle (scintillate) because their light reaches us after passing through the (time- and space-) varying refractive index $n(\mathbf{r}, t)$ of the Earth's turbulent air. Although $n - 1 \ll 1$, for stars near the horizon the atmosphere is thick enough for the light rays to form caustics—and geometrical optics is appropriate because the wavelengths in starlight are much smaller than the scales in turbulence. So, the strong twinkling of starlight is a rapid succession of caustics [31-33] across the eye. Each passage of a caustic is accompanied by a large spike in the intensity $I = |\psi^2|$ of the light wave ψ. Caustics are the singularities that dominate the statistics of I.

For wave number k $(= 2\pi/\text{wavelength})$ the quantitative effect of caustics can be seen dramatically in the intensity moments [7]

$$M_m(k) = \langle I^m \rangle. \tag{28}$$

In geometrical optics $(k = \infty)$, I varies with distance or time x across a caustic as $\mathrm{Re}\left[1/\sqrt{(\pm x)}\right]$, so that the first moment I_1 exists (as it must by energy conservation), but all higher moments are infinite. Diffraction softening of the geometrical singularities makes the moments finite, with the following large-k behaviour:

$$M_m(k) \sim k^{\nu_m}, \quad \text{as } k \longrightarrow \infty, \tag{29}$$

where ν_m are the twinkling exponents. The analogy with (5) (with k replacing $1/\hbar$) is obvious.

To see the origin of (29), and the way the ν_m are calculated, we note that for large k the local asymptotics near a caustic are dominated by its local structure, described in terms of one of the catastrophes, labelled j, by the canonical diffraction integral ("diffraction catastrophe")—for example. over a wavefront in the atmosphere—

$$\Psi_j(\mathbf{y}; k) = k \iint \mathrm{d}\mathbf{r} \exp\left[ik\Phi_j(\mathbf{r}; \mathbf{y})\right]. \tag{30}$$

Here Φ_j is the polynomial normal form of the singularity. the collective notation $\mathbf{y} = \{y_1, y_2\} = \{y_m\}$ denotes all quantities the wave depends on (position, time, parameters describing the turbulent atmosphere ...). and the prefactor ensures that the saddle-point approximation gives the amplitude of the geometrical Ψ_j independent of k.

The average in (28) can be evaluated as an integral over all variables $\mathbf{y}$, and is dominated by the caustic, where the critical points of ψ (rays) coalesce, and the saddle-point approximation fails. To estimate this average, we will use the following *wavelength scaling law* for diffraction catastrophes [7]:

$$\Psi_j(\mathbf{y};k) = k^{\beta_j}\Psi_j(\{y_m k^{\sigma_{m,j}}\},1). \tag{31}$$

This is derived by first scaling $\mathbf{x}$ to eliminate k from the $\mathbf{y}$-independent terms (the "germ") in the exponent in (30), giving the index β_j, and scaling $\mathbf{y}$ to eliminate the k-dependence of the remaining terms (the "unfolding") and thereby determine the indices $\sigma_{m,j}$. A simple exercise illustrating (31) is the cusp catastrophe, where the polynomial and indices are

$$\begin{gathered}\Phi_{\text{cusp}}(x_1,x_2;y_1,y_2) = x_1^4 + x_1^2 y_2 + x_1 y_1 + y_2^2,\\ \beta_{\text{cusp}} = \frac{1}{4}, \qquad \sigma_{1,\text{cusp}} = \frac{3}{4}, \qquad \sigma_{2,\text{cusp}} = \frac{1}{2}.\end{gathered} \tag{32}$$

The index β_j [30] describes the short-wave divergence of the wave amplitude at the most singular part of the caustic (amplitude $\sim$ (wavelength)$^{-\beta_j}$), and the index $\sigma_{m,j}$ describes the shrinking of the associated diffraction fringes in the different directions y_m (fringe size $\sim$ (wavelength)$^{\sigma_{m,j}}$). The additional index

$$\gamma_j = \sum_m \sigma_{m,j} \tag{33}$$

is important; it describes the asymptotic wavelength scaling of the hypervolume of the main diffraction maximum in $\mathbf{y}$ space (hypervolume $\sim$ (wavelength)$^{\gamma_j}$). For the cusp (23), $\gamma = 5/4$.

Now we can estimate the contribution $M_{j,m}$ of the catastrophe j to the moment M_m. From (28) and (31)

$$\begin{aligned}M_{j,m} &\sim \int \frac{\mathrm{d}\mathbf{y}}{k^{\gamma_j}}\left|k^{\beta_j}\int \mathrm{d}\mathbf{x}\exp[i\Phi_j(\mathbf{x},\mathbf{y})]\right|^{2m} \sim\\ &\sim (\text{maximum wave amplitude})^{2m} \times (\text{diffraction hypervolume}) \sim\\ &\sim k^{2m\beta_j-\gamma_j}.\end{aligned} \tag{34}$$

The different catastrophes compete to dominate M_m by having the largest value of $2m\beta_j - \gamma_j$. So in (29) the twinkling exponents are

$$\nu_m = \max_j\left(2m\beta_j - \gamma_j\right). \tag{35}$$

To implement this programme, it is necessary first to specify which catastrophes are allowed to enter the competition. This is a question of physics, in which dimensionality

TABLE I. – *The first few singularities with corank* 1 *and* 2.

Singularity j	Codimension	β_j	γ_j
A_j ($j \geq 2$) (cuspoids)	$j-1$	$\frac{j-1}{2(j+1)}$	$\frac{(j+2)(j-1)}{2(j+1)}$
D_j ($j \geq 2$) (umbilics)	$j-1$	$\frac{j-2}{2(j-1)}$	$\frac{j^2-2j+2}{2(j-1)}$
E_6 (symbolic umbilic)	5	5/12	5/2
E_7	6	4/9	26/9
E_8	7	7/15	49/15
$X_{j+5}(j \geq 4)$	$j+3$	1/2	7/2
$J_{j+4}(j \geq 6)$	$j+2$	1/2	4

is paramount. For waves in three space dimensions, where wavefronts have dimension 2, singularities with corank 1 and 2 occur; for waves in two space dimensions, only corank 1 singularities (the cuspoids) are permitted. Then the list of relevant normal forms must be supplied. To illustrate the calculation, I show in table I the scaling exponents for some singularities with corank 1 and 2 (more extensive tables can be found elsewhere [7]), and in table II the twinkling exponents deduced from (35). All are rational numbers, unlike the somewhat analogous critical exponents associated with phase transitions.

From table II, it is clear that ν_m, and also the codimension of the dominating singularity, increase with m. This is not surprising: high moments describe large rare fluctuations, and so do high-codimension singularities. The same observation explains the mathematical fact, mysterious on first encounter, that the competition has a winner at all: for each m, the exponents $2m\beta_j - \gamma_j$ do indeed increase and then decrease with increasing codimension.

This theory gives the colour (k) dependence of the intensity fluctuations in strong twinkling. Observation of the effect is difficult because high moments (especially of strongly fluctuating quantities) require long runs before they converge. Nevertheless, a preliminary experiment [34], comparing the fluctuations of red and blue laser light reflected from randomly rippling water, did support the results of the singularity-competition theory.

TABLE II. – *The first few twinkling exponents from singularities with corank* 1 *and* 2.

m	2	3	4	5	6	7	8	9
Twinkling exponent ν_m	0	1/3	1	5/3	5/2	7/2	9/2	11/2
Dominant singularity	A	A_2, D_4	D_4	D_4, E_6	E_6, X	X	X	X

3. – Spectral twinkling for integrable systems: superpoisson fluctuations

For the Hamiltonian (7), where the contours $H = E$ in $\mathbf{I}$ space (E-contours) are closed, so that the spectrum is discrete, semiclassical (and in some cases exact) energy levels are obtained by quantizing the torus actions:

$$E_{\mathbf{N}} = H(\mathbf{I}_{\mathbf{N}}), \qquad \mathbf{I}_{\mathbf{N}} = \mathbf{N}\hbar + \gamma \qquad (\mathbf{N} = \{n_1, n_2\}, \gamma = \{\alpha, \beta\}). \tag{36}$$

Poisson summation [12] of (1) leads (exactly) to (2) with

$$\begin{gathered} \mathcal{N}_{\mathrm{sm}}(E,\hbar) = \frac{\mathcal{A}(E)}{\hbar^2}, \\ \mathcal{N}_{\mathrm{fl}}(E,\hbar) = 2\mathrm{Re}\sum_{\mathbf{W}\neq 0} \exp[2\pi i \mathbf{W}\cdot\gamma] J_{\mathbf{W}}(E,\hbar), \\ \mathbf{W} = \{w_1, w_2\} \qquad (-\infty < w_1 < \infty, 0 \leq w_2 < \infty), \end{gathered} \tag{37}$$

where $\mathcal{A}$ is the area of the E-contour, and

$$J_{\mathbf{W}}(E,\hbar) = \frac{1}{2\pi i \hbar W^2} \oint_{H=E} \mathrm{d}s \frac{\mathbf{W}\cdot\omega(s)}{\omega(s)} \exp\left[\frac{2\pi i \mathbf{I}(s)\cdot\mathbf{W}}{\hbar}\right], \tag{38}$$

where ω is the frequency (9) and s is arc length along the $\mathbf{I}$ contour.

In the s-integral, small-$\hbar$ asymptotics selects the resonant tori (9) on the E-contour, with winding numbers $\mathbf{W}$, for which the phase $\mathbf{W}\cdot\mathbf{I}(s)$ is stationary, and leads to the approximation (8). But this fails for bifurcating resonant tori, where two or more rational normals to the E-contour coincide and $\mathbf{W}\cdot\mathbf{I}(s)$ is stationary to higher order. Then $J_{\mathbf{W}}$ is locally approximated by one of the diffraction catastrophes (30), with k replaced by $1/\hbar$ and the integral over the single variable s rather than $\mathbf{r}$—which restricts possible singularities to the corank-1 cuspoids. The singularities, as E or parameters in H vary, make $J_{\mathbf{W}}$, and hence $\mathcal{N}_{\mathrm{fl}}$, rise to larger values than (11), in fact to values of order $\hbar^{-(\beta_j+1/2)}$, where β_j are the cuspoid exponents in table I and the extra 1/2 is a consequence of the fact that there is one integration variable in (38), rather than two as in (30). Soon I will give an explicit example.

In the (admittedly rather artificial) much more general situation where H contains parameters whose variation enables all possible cuspoids to occur, the moments (3) are given by the same theory as for twinkling starlight, where the hypervolume indices γ_j must participate but where the competition for the winning exponent is restricted to

Table III. – *Superpoisson spectral fluctuation exponents for integrable systems.*

m	2	3	4	5	6	7	8	9
Superpoisson exponents $\nu_m - m$	0	1/3	3/4	5/4	9/5	12/5	3	11/3
Dominating codimension	1	1	2	2	3	3	3 and 4	4

cuspoids. The result is (4), with the exponents

$$\nu_m^{\text{superpoisson}} = m + \max_j \left(\frac{(j-1)(2m-j-2)}{2(j+1)} \right), \tag{39}$$

whose first few values are shown in table III.

This general theory represents the extreme of possible spectral fluctuations in integrable systems whose I-dependence is generic. To test it on any spectrum, natural or computed, would require energy or parameter ranges including many torus bifurcations, with $\hbar$ so small that many levels are involved in each bifurcation. We are far from such a test now. Instead, here is an example showing the effect of a single bifurcation on the asymptotics of $\mathcal{N}_{\text{fl}}$ as E varies.

Consider

$$H(\mathbf{I}) = I_1^2 + I_1^4 + I_2^2 + I_2^4 - 2I_1^2 I_2 \tag{40}$$

whose E-contours are shown in fig. 1. For $E > 5/16$ and $I_2 > 0$, there are three vertical normals to each contour; for $E < 5/16$, there is one. Thus at $E = 5/16$ (and $I_2 = 1/2$) there is a trident (pitchfork) bifurcation of the tori $\mathbf{W} = \{0, \pm 1\}$, for which the phase is $\mathbf{W} \cdot \mathbf{I}(s) = \pm I_2(s)$. The local form of the E-contours means that (38) can be mapped

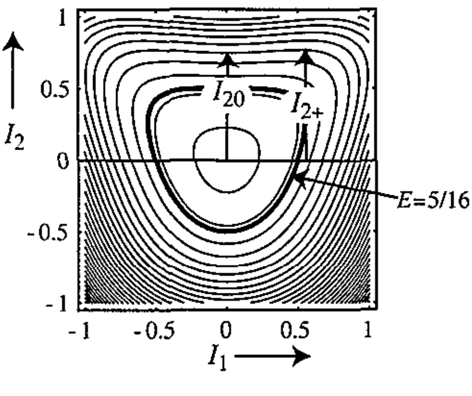

Fig. 1. – Contours for the Hamiltonian (40) and geometry associated with the resonant tori with winding numbers $\mathbf{W} = \{0, \pm 1\}$ (vertical normals).

onto the canonical integral

$$G(A) = \int_{-\infty}^{\infty} \mathrm{d}S \exp\left[i(AS^2 - S^4)\right] = \frac{\pi}{2}\sqrt{\frac{|A|}{2}} \exp\left[\frac{1}{8}iA^2\right] \times \times \left[\exp\left[-\frac{1}{8}i\pi\right] J_{-1/4}\left(\frac{1}{8}A^2\right) + \mathrm{sgn}(A) \exp\left[\frac{1}{8}i\pi\right] J_{1/4}\left(\frac{1}{8}A^2\right)\right], \tag{41}$$

which is a section through the cusp diffraction catastrophe [32, 35, 36] and where A increases through zero as E increases through 5/16.

We perform the mapping using the technique of uniform approximation [31, 37, 38]. For this, we express each E-contour in the form $I_2(I_1)$, denote the central resonant torus (with $I_1 = 0$) by the suffix 0, and the outer two tori (which are complex when $E < 5/16$) by the suffix +, and curvatures $\partial^2 I_2/\partial I_1^2$ by K. Then

$$I_{20}(E) = \sqrt{\frac{\sqrt{1+2E}-1}{2}}, \qquad I_{2+}(E) = \text{positive root of } x^4 + x - \frac{1}{4} = E, \tag{42}$$

$$A(E) = 2\sqrt{\frac{2\pi}{\hbar}|I_{2+}(E) - I_{20}(E)|}\,\mathrm{sgn}\left(E - \frac{5}{16}\right),$$

$$K_0(E) = -\frac{(2I_{20}(E)-1)}{I_{20}(E)(1+2I_{20}^2(E))}, \qquad K_+(E) = -\frac{(2I_{2+}(E)-1)}{(I_{2+}^3(E)+\frac{1}{4})}.$$

With these ingredients, the semiclassical contribution to $\mathcal{N}_{\mathrm{fl}}$ from $\mathbf{W} = \{0, 1\}$ and $\mathbf{W} = \{0, -1\}$ can be shown to be

$$\mathcal{N}_{\mathrm{fl}}(E) \approx \frac{1}{\sqrt{\pi^3 \hbar}} \mathrm{Im}\, A(E)^{1/2} \exp\left[2\pi i\left(\beta + \frac{I_{20}}{\hbar}\right)\right] \times \times \left[K_0(E)^{-1/2} - \frac{2i}{A(E)}((-2K(E)_+)^{-1/2} - K_0(E)^{-1/2})\partial_A\right] G\{A(E)\}. \tag{43}$$

A numerical test of this formula was made by constructing a spectrum from (40) and (36) with $\alpha = 0.1$, $\beta = 1/\sqrt{2}$ (to avoid symmetry degeneracies), calculating 62000 levels E_N with $\hbar = 0.005$, and ordering these into a list E_j $(j = 1, 2, \ldots)$. $\mathcal{N}_{\mathrm{fl}}$ was then calculated by straight-line interpolation between the points

$$\mathcal{N}_{\mathrm{fl}}(E_j; \hbar) = j - \frac{1}{2} - \frac{\mathcal{A}(E_j)}{\hbar^2}. \tag{44}$$

Figure 2 shows the comparison. Agreement is very good, particularly after realizing that the "sawtooth" oscillations arise from repetitions of the resonant orbits [11, 39] (that is

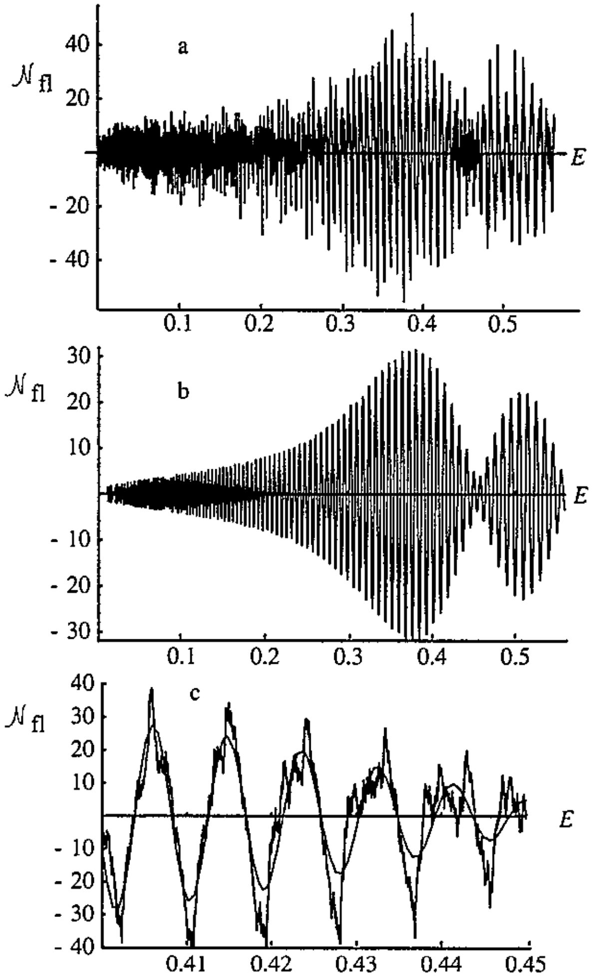

Fig. 2. – Spectral fluctuations $\mathcal{N}_{\text{fl}}$ for the Hamiltonian (40) and $\hbar = 0.005$ (62000 levels), showing effects of the bifurcation at $E = 5/16$ (inflection point of the envelope); (a) computed from (36) and (44), (b) the approximation (43), (c) magnification and comparison. Near the bifurcation, the mean spacing between levels is $\hbar^2/(\partial\mathcal{A}/\partial E) \approx 1.000 \times 10^{-5}$.

resonant tori with $\mathbf{W} = \text{integer} \times (0,1)$), and the very fast (noisy) fluctuations in $\mathcal{N}_{\text{fl}}(E)$ come from the accumulation of longer nonresonant orbits. Note in particular the large value $\mathcal{N}_{\text{fl}}(E) \sim \hbar^{-3/4}$ near the cusp-dominated region near $E \sim 5/16$, compared with $\mathcal{N}_{\text{fl}}(E) \sim \hbar^{-1/2}$ elsewhere.

4. – Spectral twinkling for mixed systems

For systems with two freedoms, periodic orbits are fixed points of the map determined by successive intersections of the Poincaré section. In terms of the generating function

ϕ, the map can be specified as

$$\begin{pmatrix} q \\ p \end{pmatrix} \longrightarrow \begin{pmatrix} q' \\ p' \end{pmatrix}, \quad \text{with } q = \partial_p \phi(q', p),\, p' = \partial_{q'} \phi(q', p). \tag{45}$$

It follows that the periodic orbits are critical points of the reduced generating function

$$\begin{gathered} \Phi(q', p) = \phi(q', p) - q'p, \\ \{\partial_q \Phi(q,p) = \partial_p \Phi(q,p) = 0\} \leftrightarrow \{q' = q, p' = p\}. \end{gathered} \tag{46}$$

Semiclassical spectral fluctuations are determined by a "diffraction integral" that can be written, with $\Phi^{(r)}$ denoting the reduced generating function for the r-times-iterated Poincaré map, and up to irrelevant factors, as [13, 16]

$$\mathcal{N}_{\text{fl}}(E) \approx \operatorname{Im} \frac{1}{\hbar} \sum_{r=1}^{\infty} \iint \mathrm{d}q\, \mathrm{d}p \exp\left[\frac{i}{\hbar} \Phi^{(r)}(q, p; E)\right]. \tag{47}$$

Periodic orbits are stationary points of the phase in this integral. If they are isolated, the stationary-phase approximation reproduces the standard result (6). This fails at bifurcations, where orbits coalesce. In principle the remedy for estimating the moments $M_m(\hbar)$ (eq. (3)) is simple, and analogous to that used for the van Hove integrals in subsect. (2·2), the twinkling integrals in subsect. **2·3**, or the torus bifurcations of sect. **3**: substitute for $\Phi^{(r)}$ the normal form corresponding to each bifurcation, use its scaling properties to determine the power of $\hbar$ that it contributes to $M_m(\hbar)$, and then allow all bifurcations to compete to obtain the twinkling indices ν_m in (5).

At the moment, however, this programme cannot be implemented properly, because the normal forms associated with Hamiltonian period-r bifurcations have not been classified as extensively as their catastrophe-theory counterparts, namely the singularities of gradient maps. Naive inspection of (46) might lead to the expectation that the normal forms are the same, but this is wrong because the period-r generating function $\phi^{(r)}$ must have the special property of possessing an r-th root, namely the reduced generating function ϕ for the primitive map.

Information is available for codimension-one and codimension-two bifurcations of period-r orbits [6, 17, 40], and is summarized in table IV. The two unfolding parameters are A and B (it is helpful to think of A as energy). To simplify writing, I have included only the relevant leading terms, and replaced all unimportant coefficients by unity. Derivation of these normal forms involves many subtleties [16, 17], and I have ignored some aspects of the normal forms that are necessary to describe the full geometry of the bifurcating orbits. The codimension-one bifurcations can be obtained by setting $B = 1$.

For each r, the $\hbar$-scaling of the contribution $\mathcal{N}_{\text{fl}}^{(r)}$ to the staircase fluctuation is, in an

TABLE IV. – *Normal forms for bifurcations of period-r orbits (after Schomerus* [17]*).*

r	$\Phi^{(r)}$
1	$p^2+q^4+Aq^2+Bq^3$
2	$p^2+q^6+Aq^2+Bq^4$
3	$(p^2+q^2)^2+A(p^2+q^2)+B\,\mathrm{Re}(p+iq)^3$
4	$p^2q^2+A(p^2+q^2)+B(p^2-q^2)^2$
5	$\mathrm{Re}(p+iq)^5+A(p^2+q^2)+B(p^2+q^2)^2$
6	$(p^2+q^2)^3+A(p^2+q^2)+B(p^2+q^2)^2$

obvious notation, for codimension two,

$$\mathcal{N}_{\mathrm{fl}}^{(r)}(A,B,\hbar)=\frac{1}{\hbar^{\beta_r}}\mathcal{N}_{\mathrm{fl}}^{(r)}\left(\frac{A}{\hbar^{\sigma_{Ar}}},\frac{B}{\hbar^{\sigma_{Br}}},1\right) \tag{48}$$

(cf. (31)). The contribution to the m-th moment is (cf. (34))

$$M_{r,m}\sim\frac{1}{\hbar^{2m\beta_r-\sigma_{Ar}-\sigma_{Br}}}=\frac{1}{\hbar^{\nu_{m,r}}}. \tag{49}$$

For codimension-one, the contribution from σ_{Br} is absent.

Tables V and VI show some indices β and σ, and twinkling exponents ν, for the codimension-one and codimension-two bifurcations. Unfortunately, there is only one case where there seems to be a clear winner of the competition amongst bifurcations, namely the first moment (mean square of $\mathcal{N}_{\mathrm{fl}}$), where period $r=4$ gives the largest exponent $\nu_1=1/2$ (for codimension one and two). Obtaining plausible twinkling exponents for higher moments requires the normal forms for much higher codimension, and these are not available at present. Therefore I regard the arguments presented here, and the results in tables V and VI, as a preliminary, not a definitive, investigation of the twinkling exponents.

It is not clear *a priori* whether these twinkling exponents describe $\mathcal{N}_{\mathrm{fl}}$ itself, or $\mathcal{N}_{\mathrm{fl}}$ after subtracting the fluctuations associated with the integrable background of undestroyed

TABLE V. – *Spectral indices and exponents for codimension-one bifurcations.*

r	β_r	σ_{Ar}	$\nu_{1,r}$	$\nu_{2,r}$	$\nu_{3,r}$
1	1/6	1/3	0	1/3	2/3
2	1/4	1/2	0	1/2	1
3	1/3	1/3	1/3	1	5/3
≥ 4	1/2	1/2	1/2	3/2	5/2

TABLE VI. – *Spectral indices and exponents for codimension-two bifurcations (negative values of ν have been set equal to zero).*

r	β_r	σ_{Ar}	σ_{Br}	$\nu_{1,r}$	$\nu_{2,r}$	$\nu_{3,r}$	$\nu_{4,r}$
1	1/4	1/2	1/4	0	1/4	3/4	5/4
2	1/3	2/3	1/3	0	1/3	1	5/3
3	1/2	1/2	1/4	1/4	5/4	9/4	13/4
4	1/2	1/2	0	1/2	3/2	5/2	7/2
5	3/5	3/5	1/5	2/5	8/5	14/5	4
6	2/3	2/3	1/3	1/3	5/3	3	13/3

(and nonbifurcating) KAM tori. However, the question might be moot, because the trend of table VI suggests that for large m the ν_m associated with bifurcations might exceed the integrable background value $\nu_m = m$ (see the remark after (11)). This would mean that as $\hbar \to 0$ the spectral fluctuation moments of mixed systems are indeed bifurcation-dominated, so they do describe the spectral statistics of mixed systems in a way that distinguishes them from both integrable and chaotic systems, by concentrating on the characteristic singularities that dominate them.

Numerical experiments to test this scenario seem very difficult. Although it is expected that generic smooth Hamiltonians (*e.g.* anharmonic oscillators) can exhibit many bifurcations of stable and unstable orbits, of low and high codimension, when energy and other parameters vary, detection of the universal twinkling exponents conjectured here would require extraordinarily small $\hbar$, in order to resolve the many bifurcations required to make the moments converge to their semiclassical values (5), especially for large m. However, regardless of our speculation here about the collective influence of bifurcations, it has been established [41], by analysis and numerical experiment, that individual bifurcations do affect the spectral statistics.

* * *

The idea described in the introduction and in sect. 4, that bifurcations affect spectral statistics in a way that is characteristic of mixed systems, has been arrived at in collaboration with J. KEATING.

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