Quantum Indistinguishability: Spin-statistics without Relativity or Field Theory?

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Abstract. We review the formulation of quantum mechanics for identical spinning particles with wavefunctions that are singlevalued when permuted configurations are identified. The identification requires the spins to be smoothly permuted along with position variables, so spin is represented in a position-dependent 'transported basis', rather than the usual fixed basis. The simplest transported basis, constructed in terms of spins represented as pairs of commuting harmonic oscillators, gives the correct connection between spin and statistics. More complicated constructions can give the wrong exchange sign. The theory is generalized to incorporate additional properties such as isospin, colour and strangeness. Some remarks about the relation between this approach and those based on relatitivity and/or field theory are given.

1. INTRODUCTION

In the nonrelativistic quantum mechanics of a fixed number of identical particles, the relation between spin and statistics (SS) sits awkwardly on top of the theory, as a separate postulate: it is simply asserted that the wavefunctions of particles with integer-plus-half spin change sign when the variables describing their spin and position are exchanged, whereas wavefunctions for particles whose spin is an integer do not change sign. Here we will investigate the possibility that SS is already contained in the theory, as a hidden consequence of imposing geometrical requirements that follow naturally from indistinguishability. We will give brief nontechnical summaries of, and comment on, the ideas in two recent papers [1] [2].

If SS is an awkward addition to nonrelativistic quantum theory, there is awkwardness too in the attempt to be described here. The explanation of any phenomenon on the basis of a well-established physical theory must begin with assumptions about how the theory is to be applied in the particular case under consideration, followed by deductions that lead unambiguously and precisely to the phenomenon. An explanation is more convincing if it is fruitful, in the sense of successfully predicting phenomena that have not yet been observed.

Here the 'phenomenon' is already known: it is SS, the principle determining how wavefunctions behave under exchange. Of course, this principle has many consequences (e.g. the Pauli exclusion principle) that are fundamental to our description of the world, but it is hard to see how an explanation of SS itself can have any new experimental consequences (though we live in hope that such pessimism will prove wrong). Thus, the degree to which any purported explanation of SS is accepted - once it has been agreed that the deductive part is technically correct - must hinge on the naturalness of the assumptions, and so involves elements of subjectivity and aesthetics. In [1, 2], 'naturalness' was interpreted as: constructing the quantum theory of identical particles using the same principles that are accepted and routinely applied in the quantum physics of nonidentical particles (or individual identical particles).

A further awkwardness is that SS involves not some experimental number whose prediction with increasing accuracy might increase confidence in the theory (like the fine-structure constant), but is simply a sign.

Nevertheless, SS cries out for understanding. There are many derivations based on relativity, beginning with [3], that have been comprehensively reviewed [4], and we will comment a little on these in section 6. And there have been many attempts at derivations that do not involve relativity, based on a variety of different assumptions [5]. We think that previous derivations have lacked a crucial geometrical ingredient, described in section 2, that follows from indistinguishability.

For simplicity, we will concentrate on the understanding of SS for two identical particles with spin S. However, the extension to N particles involved interesting technical challenges that have resulted in beautiful mathematical constructions by Atiyah [6], and we will mention one of these briefly in section 4.

2. SINGLEVALUEDNESS UNDER EXCHANGE

In the position representation, the wavefunction $|\Psi\rangle$ describing the state of two identical particles with spin S depends on the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. Only the relative position $\mathbf{r}= \mathbf{r}_2- \mathbf{r}_1$ is relevant here, so we write $|\Psi(\mathbf{r})\rangle$. Exchange of positions corresponds to $\mathbf{r}\to-\mathbf{r}$.

The central assumption in [2] was that the wavefunction for identical particles must be singlevalued under exchange. Thus

$$|\Psi(\mathbf{r})\rangle = |\Psi(-\mathbf{r})\rangle. \tag{1}$$

At first this seems absurd, but our $|\Psi\rangle$ differs from the more familiar wavefunction in a crucial respect, namely that it has, built into it, the property that exchange of positions $\mathbf{r}\to-\mathbf{r}$ is automatically accompanied by exchange of spin states, so that $\mathbf{r}\to-\mathbf{r}$ corresponds to complete exchange. Only then can indistinguishability be demanded, and the condition (1) imposed. To invoke singlevaluedness for identical particles is not a new idea [7-9]; but the characteristic feature of our approach is the systematic incorporation of spin exchange along with exchange of positions.

To incoporate spin exchange, it is necessary to represent the spin part of the state in a way that is unusual but unavoidable. Following [1], we write the complete state as

$$|\Psi(\mathbf{r})\rangle = \sum_M \psi_M(\mathbf{r})|M(\mathbf{r})\rangle. \tag{2}$$

Here $M\equiv\{m_1,m_2\}$ labels the spin state of the particles, with m denoting the z component of spin; exchange of spins corresponds to $M\to\overline{M}\equiv\{m_2,m_1\}$. $|M(\mathbf{r})\rangle$ is the *transported spin basis*, that is, a basis for representing spins in a way that depends on the relative position of the particles, with the exchange requirement

$$|M(-\mathbf{r})\rangle = (-1)^K|\overline{M}(\mathbf{r})\rangle, \tag{3}$$

where K is an integer (see [1, 2]), implying that exchange generates a sign rather than some more general phase factor. The coefficients $\psi_M(\mathbf{r})$ describe the spatial dependence of the state.

The representation (2) is to be contrasted with the familiar expansion in terms of a fixed spin basis $|M\rangle$, namely

$$|\Psi(\mathbf{r})\rangle_{\text{fixed}} = \sum_M \psi_M(\mathbf{r})|M\rangle. \tag{4}$$

As was shown in [1], the coefficients $\psi_M(\mathbf{r})$ – which are, after all, the physically measurable quantities (up to a single overall phase) - are the same as those in (2) (see also section 3 later). With $|\Psi(\mathbf{r})\rangle_{\text{fixed}}$, there is no spin exchange accompanying position exchange, so indistinguishability is not incorporated and there is no justification for imposing the singlevaluedness requirement (1). With (2), however, the application of (1) implies that any sign change (3) of the transported basis is compensated by a sign change in the coefficients:

$$\psi_{\overline{M}}(-\mathbf{r}) = (-1)^K \psi_M(\mathbf{r}). \tag{5}$$

This is the usual form in which SS is assumed. Of course, to reproduce *the* SS, rather than a generic form of SS, it is necessary to show that

$$K = 2S. \tag{6}$$

This requires consideration of the transported basis $|M(\mathbf{r})\rangle$, as will be described in section 3.

It is important to emphasize that with the representation (2), SS, in the form (5), emerges as a quantization condition implied by singlevaluedness. This brings SS into the same framework as other derivations within elementary quantum mechanics. For example the quantization of a component $m\hbar$ of orbital angular momentum using wavefunctions requires singlevaluedness of $\exp(im\phi)$ under $\phi\rightarrow\phi+2\pi$, reflecting the fact that these two angles represent the same point. And in the Aharonov-Bohm effect [10, 11] a similar application of singlevaluedness is required to get a definite (and experimentally confirmed) prediction for quantum scattering by inaccessible magnetic flux. In fact, in every situation that we know in elementary quantum mechanics, wavefunctions representing the same configuration are the same (up to choice of gauge).

In effect, the incorporation of the transported basis as in (2) enables **r** and -**r** to be regarded as the same point in the configuration space of the two particles. This identification changes the topology of the configuration space, making it nonorientable and non-simply-connected. The space is the direct product of the centre of mass, the separation distance $r=|\mathbf{r}|$, and the projective plane (2-sphere with antipodal points identified) that represents directions $\mathbf{r}/r$. An intrinsic procedure would be to construct quantum mechanics by erecting two-spin bundles on this base space, and this has been systematically carried out [12]. However, we will here continue to use the more elementary approach of regarding $\mathbf{r}$ as a euclidean vector and then imposing singlevaluedness under $\mathbf{r}\to-\mathbf{r}$. (This is analogous to the common and convenient procedure of regarding azimuth angles ϕ as variables with values on the real line $-\infty<\phi<\infty$, and then insisting that functions are periodic, rather than considering functions whose domain is the circle.)

3. TRANSPORTED BASIS

The basis $|M(\mathbf{r})\rangle$ is a set of $(2S+1)^2$ spinors, because the z quantum numbers m_1 and m_2 of the two spins can range from $-S$ to $+S$. Each spinor is required to be a singlevalued and smooth function of $\mathbf{r}$. In addition, we impose the parallel-transport requirement

$$\mathbf{A}_{M,M'}\equiv \mathrm{i}\langle M'(\mathbf{r})|\nabla M(\mathbf{r})\rangle=0, \qquad (7)$$

to guarantee the vanishing of the curvature of the connection between neighbouring positions $\mathbf{r}$ ('flat exchange'). (For further discussion, see [12].)

Parallel transport implies that $|M(\mathbf{r})\rangle$ inhabits an ambient space, within which it is smoothly transported, that is larger than the $(2S+1)^2$-dimensional space of the fixed basis $|M\rangle$. Without enlargement, (7) would imply that $|M(\mathbf{r})\rangle$ is independent of $\mathbf{r}$ and so unable to satisfy the fundamental exchange requirement (3). We find it necessary to emphasize that this enlargement is in no way undesirable or unphysical. Nor is it unfamiliar: in [1] we give the analogy of light in a coiled optical fibre, where transversality implies that in a frame whose z axis is along the the local propagation direction the electric field vector can be described with only two components (x and y) whereas a fixed basis requires all three components. An even simpler analogy is that in a space $\mathbf{r}=\{x, y, z\}$, each vector in a field $\mathbf{v}(\mathbf{r})$ possesses only one component when described in a local frame $\{x_1, x_2, x_3\}$ whose x_3 axis is directed along $\mathbf{v}$, but requires three components in the ambient space $\{x, y, z\}$.

The transformation between the transported and fixed bases is described by a unitary operator $\mathbf{U}(\mathbf{r})$, such that

$$|\Psi(\mathbf{r})\rangle = \mathbf{U}(\mathbf{r})|\Psi(\mathbf{r})\rangle_{\text{fixed}}, \quad |M(\mathbf{r})\rangle = \mathbf{U}(\mathbf{r})|M\rangle. \tag{8}$$

An immediate consequence is the equality

$$\psi_M(\mathbf{r}) = \langle M(\mathbf{r})|\Psi(\mathbf{r})\rangle = \langle M|\Psi(\mathbf{r})\rangle_{\text{fixed}} \tag{9}$$

asserted after (4). As described in [1, 2], $\mathbf{U}(\mathbf{r})$ also generates dynamical variables (e.g. momentum and spin) in the transported basis from their more familiar fixed counterparts, thereby guaranteeing that all local physics (for example Schrödinger equations derived from hamiltonians) is the same as in the fixed basis.

The main technical content of [1] was a construction of $\mathbf{U}(\mathbf{r})$. In this, each of the two spins was represented by the formalism of Schwinger [13], by two harmonic oscillators: a_1 and b_1 for one particle, and a_2 and b_2 for the other. Each of these four oscillators had its own creation and annihilation operator, with operators corresponding to different oscillators commuting. The dimension of the ambient space, on which $\mathbf{U}$ acts and within which the $(2S+1)^2$ states $|M(\mathbf{r})\rangle$ are smoothly transported, is the number of ways that $4S$ quanta can be distributed among four oscillators, namely $(4S+1)(4S+2)(4S+3)/6$; for $S=1/2$, there are 10 such states, in contrast to the four fixed-basis and transported-basis states.

Exchange was incorporated using the following insight. For a single spin, interchanging the number of quanta in its a and b oscillators corresponds to replacement of m by $-m$, equivalent to a rotation of the axis of quantization from z to $-z$. Therefore, interchanging the quanta in the 1 and 2 oscillators corresponds to exchanging the spin states of the two particles and can be used to define an 'exchange angular momentum', analogous to spin, which generates 'exchange rotations' $\mathbf{U}(\mathbf{r})$ from z to $\mathbf{r}$, whose effect is precisely to generate a transported basis with the desired exchange property (3). With this construction, in which each spin is decomposed into its 'atomic spin bosons', it was possible to make an explicit calculation of the exchange sign in (3), and the result was the correct SS sign (6). The calculation was extended in I from two to N identical particles.

In [1] we suggested that any construction of the transported basis that was smooth, singlevalued and parallel-transported would lead to the correct exchange sign. This was wrong. In [2] we exhibited two 'perverse' constructions that satisfy these requirements but which, unlike those based on the Schwinger formalism, lead to the wrong sign.

The first perverse construction applies to spin S=0, and reflects a question frequently asked by people sceptical of the arguments in [1]: can the exchange of two spinless particles be accompanied by a fermionic sign? In this perverse construction, they can. The single transported state is represented as the unit vector

$$|M(\mathbf{r})\rangle = |\{0,0\}(\mathbf{r})\rangle = \mathbf{r}/|\mathbf{r}|. \qquad (10)$$

This is singlevalued, smooth, and parallel-transported, and involves the extended spin space spanned by the three basis states $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$, of which only one (e.g. $\mathbf{e}_z$) corresponds to the fixed-spin state $|\{0,0\}\rangle$. The operator $\mathbf{U}(\mathbf{r})$ is then rotation from $\mathbf{e}_z$ to $\mathbf{r}$. Under $\mathbf{r}\rightarrow-\mathbf{r}$, $|M(\mathbf{r})\rangle$ changes sign fermionically, rather than being bosonically invariant.

The second perverse construction in [2] applies to spin S=1/2, and consists in replacing all commutators in the Schwinger formalism by anticommutators. For this 'anti-Schwinger' construction, the exchange sign is +1, rather than the fermionic -1.

We have not found a general principle to exclude these perverse constructions, and others that generate the wrong exchange sign. However, all the perverse constructions we have found are defective in one or more ways, described in [2]. For example, the spin-zero fermion construction fails the test of simplicity, because (10) is decomposable into a constant (unity) that satisfies the requirements - and is what Schwinger gives - and a superfluous factor with no intrinsic connection to spin. And the anti-Schwinger construction is special in that it applies only to S=1/2 and so fails to describe the statistics of composite objects that can have any spin (there are generalizations of anti-Schwinger for higher spins, but they are cumbersome).

By contrast, the Schwinger construction applies for all spins (and also for all N - see section 4), generates the transported basis without superfluous factors, and also is intrinsically related to spin. In the reformulation of our approach by constructing N-spin bundles on the identified configuration space [12], the Schwinger construction emerges as the simplest implementation of the geometrical requirements.

4. ATIYAH'S CONSTRUCTION for *N* PARTICLES

In [1] we extended the Schwinger construction to the general case of N particles (with permutations instead of exchanges). The N spins are built from $2N$ oscillators, from whose creation and annihilation operators it is possible to construct 'permutation angular momenta', generating 'permutation rotations' and thence the transported basis states. We showed that any such construction must yield the correct SS sign. But we were unable to exhibit an explicit construction, analogous to the exchange rotation from z to $\mathbf{r}$ for two particles. We reduced the problem to that of finding a unitary NxN matrix $U_{ij}(\mathbf{R})$, smoothly dependent on the positions $\mathbf{R}=\{\mathbf{r}_1,...\mathbf{r}_N\}$, with the property that any permutation of the $\mathbf{r}_i$ results in the corresponding permutation of its columns, up to an overall phase.

Recently, Atiyah [6] has produced several such constructions. Here we will describe the simplest. The matrix $U_{ij}(\mathbf{R})$ is obtained from the polar decomposition of an NxN matrix $V_{ij}(\mathbf{R})$, each of whose columns $\mathbf{v}_j(\mathbf{R})$ is associated with the jth particle, so that the required permutation property is assured. It suffices to explain $\mathbf{v}_1(\mathbf{R})$.

Let the directions $(\mathbf{r}_2\text{-}\mathbf{r}_1)/|\,\mathbf{r}_2\text{-}\mathbf{r}_1|,...(\mathbf{r}_N\text{-}\mathbf{r}_1)/|\,\mathbf{r}_N\text{-}\mathbf{r}_1|$ of the other particles, as seen from 1, be described by their complex stereographic coordinates $\zeta_2,...\ \zeta_N$ (that is, the real and imaginary parts of ζ_j are the cartesian coordinates of the intersection with the equatorial plane of the line joining the south pole of the unit sphere centred on $\mathbf{r}_1$ to the point where the vector connecting $\mathbf{r}_1$ to $\mathbf{r}_j$ intersects the sphere). Then the components $v_{m,1}$ of $\mathbf{v}_1$ are the coefficients in the expansion

$$P_1(z)=\prod_{n=2}^{N}\left(z-\zeta_n\right)=\sum_{m=1}^{N}\frac{z^{m-1}}{\sqrt{(m-1)!(N-m)!}}v_{m,1}. \qquad (11)$$

The orthogonalization leading to $U_{ij}(\mathbf{R})$ requires the columns $\mathbf{v}_j(\mathbf{R})$ to be independent, that is $\det V_{ij}(\mathbf{R})\neq 0$ for any configuration $\mathbf{R}$ where no two particles coincide. At present this is a plausible conjecture, proved for N=3 and some special cases (e.g. N particles in a line), but not generally; this problem remains open. Independence of the columns has however been shown for a more elaborate version of this construction [6].

5. EXTENDED SPIN-STATISTICS RELATIONS FOR PARTICLES WITH ADDITIONAL PROPERTIES

Returning now to two particles, we incorporate into the theory the fact that particles can be characterised not only by position and spin but by one or more further quantum properties, that we denote by P. Examples of P are isospin, strangeness and colour. We denote the values of P by p (assumed discrete), and the pair of values for two particles - and the associated exchanged pair - by

$$P \equiv \{p_1, p_2\}, \quad \overline{P} \equiv \{p_2, p_1\}. \tag{12}$$

If we regard P as describing different states of identical particles, the argument we employed in [1] to derive the spin-statistics relation can be extended by requiring the state to be singlevalued under full exchange, including $P \rightarrow \overline{P}$ as well as $\mathbf{r} \rightarrow -\mathbf{r}$.

To implement this idea, we write the state of the two particles as

$$\left|\Psi_P(\mathbf{r})\right\rangle = \sum_M \psi_{M,P}(\mathbf{r})\left|M(\mathbf{r})\right\rangle, \tag{13}$$

in which $\left|M(\mathbf{r})\right\rangle$ is the same transported spin basis as before, with the exchange sign (3) and (6). Singlevaluedness, that is

$$\left|\Psi_P(\mathbf{r})\right\rangle = \left|\Psi_{\overline{P}}(-\mathbf{r})\right\rangle, \tag{14}$$

leads to the extended spin-statistics relation

$$\psi_{\overline{M},\overline{P}}(-\mathbf{r}) = (-1)^{2S} \psi_{M,P}(\mathbf{r}). \tag{15}$$

This is consistent with the requirement that the original spin-statistics relation must hold when the P state of both particles is the same, that is $P = \overline{P}$.

In the above argument, P has been treated differently from spin, notwithstanding the fact that the operators representing P (e.g. isospin) can have the same mathematical structure as angular momenta. The reason is that such mathematical resemblance conceals a physical difference: it is spin, and not any other property P, that is uniquely related to spatial rotations, because of its connection (section 6) with galilean or Lorentz invariance.

An argument similar to that leading to (15) has been given [14] in the context of Kaluza-Klein theory.

The decision to regard the particles as identical, embodied in (14), needs further discussion. An alternative possibility would be to regard the different values p_1, p_2 as distinguishing the particles. It seems absurd to consider macroscopic objects such as apples and pears as identical particles in different states of quantum fruitiness (P). Nevertheless, it is possible to choose to do this - but the choice is inconsequential, because as is well known it leaves unconstrained the symmetry of the space-spin part of the state - the symmetry of the P part of the state can always be adjusted to satisfy (15). The extended spin-statistics relation has consequences only when superpositions of states with different p are meaningful, and the interactions are such that transitions can occur between them (so that the P physics is coherently entangled with the space-spin physics).

6. SPIN AND RELATIVITY

There is a complicated history of derivations of SS [4] using arguments that rely on relativity, involving successively more refined postulates (causality, absence of negative-energy states, hermitean fields...). This raises the question of the relation between relativistic approaches and our nonrelativistic formulation. On this subject we can make only scattered remarks.

First we should point out that our derivation was nonrelativistic in the sense that it made no use of relativity, and not in the sense of being a low-velocity approximation. Since time never entered our considerations, the exchanges we considered (involving the variables **r** and *M*) can be regarded as taking place at fixed time. But fixed time is not relativistically invariant. Regarded relativistically, our exchanges were spacelike. This makes our arguments appear complementary to the some of the quantum field theoretic ones [15, 16], which involve the creation of pairs of antiparticles, and therefore are based on timelike exchanges.

Second, although we considered only the relation between spin and statistics, and not the origin of spin itself, the widespread belief that spin is unavoidably relativistic has led to doubts about our arguments involving exchange. But the existence of spin is equally a consequence of galilean relativity as of einsteinian relativity. This point has been well made before [17, 18], and it is not necessary to repeat the general arguments. However, we think it worth outlining the galilean spin-1/2 case in the simplest and least technical way, in an argument attributed to Feynman [19]. This is done in the Appendix.

Third, there is the intriguing possibility that the field-theoretic arguments could be made to operate in reverse, in the following sense. Suppose that the nonrelativistic programme outlined here is eventually completed, so that it would become clear that SS is embedded in quantum mechanics in a fundamental way, more primitive than field theory. Then instead of deriving SS by demanding that field theory satisfy certain requirements, such as causality and energy positivity, the knowledge that SS must be true might be invoked to show that field theory already possesses these desirable properties. This would be much more satisfactory, after all, than having to impose them.

Fourth, we are not aware that there exists any relativistic field theory, for particles with spin, that involves the configuration space we use here, in which indistinguishability is incorporated geometrically by the identification of permuted configurations.

Fifth, we note that Anandan [14] has presented a relativistic generalization of our construction, in what may be a first step in establishing a bridge to the field-theoretic arguments.

APPENDIX. GALILEAN TRANSFORMATIONS AND SPIN 1/2

For a free particle without spin, with hamiltonian $H=p^2/2m$, the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t)=-\frac{\hbar^2}{2m}\nabla_{\mathbf{r}}^2\psi(\mathbf{r},t) \tag{A1}$$

is Galilean-invariant in the following sense. Under the transformation to

$$\begin{gathered} t\to t_1\equiv t-T, \quad \mathbf{r}\to\mathbf{r}_1\equiv\mathbf{R}\mathbf{r}-\mathbf{v}t-\mathbf{a}, \\ \psi(\mathbf{r},t)\to\psi_1(\mathbf{r}_1,t_1)\equiv\psi(\mathbf{r},t)\exp\left\{-i\frac{m}{\hbar}\left(\mathbf{v}\cdot\mathbf{r}_1+\tfrac{1}{2}v^2t_1\right)\right\}, \end{gathered} \tag{A2}$$

where T is a constant scalar, $\mathbf{a}$ and $\mathbf{v}$ are constant vectors, and $\mathbf{R}$ is a constant rotation matrix, the equation preserves its form:

$$i\hbar \frac{\partial}{\partial t_1} \psi_1(\mathbf{r}_1, t_1) = -\frac{\hbar^2}{2m} \nabla^2_{\mathbf{r}_1} \psi_1(\mathbf{r}_1, t_1). \tag{A3}$$

For a particle with spin 1/2, this invariance is obviously shared by the two-spinor Schrödinger equation generated by the free 2x2 matrix Hamiltonian

$$\mathbf{H} = \frac{1}{2m}(\mathbf{S}\cdot\mathbf{p})^2 = \frac{1}{2m} p^2 \mathbf{1}, \tag{A4}$$

where $\mathbf{S}$ is the vector of Pauli matrices.

In both cases, external fields with potentials $\mathbf{A}(\mathbf{r},t)$, $V(\mathbf{r},t)$ can then be introduced by minimal coupling to the particle's charge q through

$$\mathbf{p} \to \mathbf{p} - q\mathbf{A}(\mathbf{r},t). \tag{A5}$$

and addition of $qV(\mathbf{r},t)$ to H. In the spin 1/2 case, coupling to the first equation in (A4) leads to

$$\begin{aligned} \mathbf{H} &= \frac{1}{2m}(\mathbf{S}\cdot(\mathbf{p} - q\mathbf{A}))^2 + qV(\mathbf{r},t) \\ &= \frac{1}{2m}[\mathbf{p} - q\mathbf{A}(\mathbf{r},t)]^2 \mathbf{1} - \frac{q\hbar}{2m} \mathbf{S}\cdot\mathbf{B}(\mathbf{r},t) + qV(\mathbf{r},t). \end{aligned} \tag{A6}$$

where $\mathbf{B} = \nabla \times \mathbf{A}$. This is the Pauli equation, with the spin operator $\mathbf{s} = \hbar\mathbf{S}/2$ coupled to the magnetic field with a magnetic moment $\mathbf{m}$, that is

$$\frac{q\hbar}{2m} \mathbf{S} \cdot \mathbf{B} = \mathbf{m} \cdot \mathbf{B}, \quad \text{where} \quad \mathbf{m} = \frac{q\hbar}{2m} \mathbf{S} = \frac{q\mathbf{s}}{m}. \tag{A7}$$

This $\mathbf{m}$, originating in a free equation that is invariant under galilean transformations, is the same – that is, it has the same gyromagnetic ratio - as that in the corresponding Dirac equation, which is Lorentz-invariant.

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