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LETTER TO THE EDITOR

Clusters of near-degenerate levels dominate negative moments of spectral determinants

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Abstract

The negative moments of spectral determinants $\left\langle \prod_n |1-(E+\mathrm{i}\delta)/E_n|^{-k}\right\rangle$ diverge when $\delta \to 0$ as $\delta^{-\nu(k)}$. For a spectrum with equally distributed levels, the exponent $\nu(k) = k-1$. For random-matrix ensembles, with parameters $\beta = 1$ (orthogonal), 2 (unitary), 4 (symplectic), we argue that the divergences for each k are determined by competitions between near-degenerate level clusters whose sizes depend on k, and we conjecture that

$$\nu(k) = \mathrm{int}[(k-1)/\beta+1]\big((k-1+\tfrac{1}{2}\beta) - \tfrac{1}{2}\beta\,\mathrm{int}[(k-1)/\beta+1]\big).$$

For Poisson-distributed levels, unrestricted clustering leads to the δ-divergence of the moments increasing with the number N of levels in the interval considered, and $\nu(k) = N(k-1)$.

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The energy levels E_n of a quantum system are the zeros of its spectral determinant $\zeta(E)$ (Voros (1992)), defined as

$$\zeta(E) \equiv \prod_n f(E_n)\left(1-\frac{E}{E_n}\right) \tag{1}$$

with $f(E_n)$ chosen to make the product converge. Therefore the negative moments of $\zeta(E)$—averages of $|\zeta(E)|^{-k}$ over a range of real E including many levels, with $k > 0$—are infinite if $k > 1$. The infinities can be removed by giving E a small imaginary part δ. Then the divergence of the moments as $\delta \to 0$ reflects the nature of the spectrum. In particular, degeneracies or near-degenerate clusters involving $p > 1$ levels should powerfully influence the divergence of the negative moments, because $|\zeta(E)|^{-k}$ is large in the vicinity of such clusters. The larger the value of p, the stronger the divergence, suggesting that large clusters will dominate the negative moments. On the other hand, large clusters are rare, and this reduces their influence.

Our aim here is to study the competition between these two effects, as embodied in the leading-order power-law δ-dependence of the moments, for spectra whose fine structure is given by the orthogonal, unitary and symplectic ensembles (Gaussian or circular) of random-matrix theory, and also for the Poisson ensemble of independent random numbers. Such spectra include the energy levels of classically chaotic and integrable quantum systems, and the zeros of the Riemann zeta function, regarded as eigenvalues (Berry and Keating (1999)).

For the random-matrix ensembles, we will argue that moments with different k are dominated by clusters with different sizes p. This is the outcome of a competition between different p, with victory going to the p-value that for each k generates the largest negative exponent in the power law. This phenomenon, of moment exponents being determined by a competition, is familiar (Berry (2000)) as 'singularity-dominated strong fluctuations'; in previous examples, the competing elements have been caustics (Berry (1977), Hannay (1982), Hannay (1983)) or periodic orbits (Berry *et al* (2000)). For the Poisson ensemble, the lack of level repulsion leads to exponents that increase with the number of levels in the interval considered.

We consider the spectral determinant near energy E, scaled so as to have a value of order unity and with the spectrum locally unfolded in terms of the smoothed spectral density $d_{\rm sm}(E)$ (that is, the mean scaled level density is unity). Thus we define

$$Z_E(x,\delta) \equiv \frac{\zeta(E+(x+{\rm i}\delta)/d_{\rm sm}(E))}{\langle|\zeta(E)|\rangle_x} \tag{2}$$

where the average is over a range of many levels near E, and study the scaled moments

$$M(-k,\delta) \equiv \langle|Z_E(x,\delta)|^{-k}\rangle_x. \tag{3}$$

We will be interested in the behaviour of these negative moments as $\delta \to 0$. (The positive moments $M(k,0)$—the δ-regularization is unnecessary here—were calculated by Keating and Snaith (2000).) Here and hereafter we have dropped the explicit E-dependence, and where necessary assume that E is in the high-lying part of the spectrum, that is, in the semiclassical regime.

The regularization involving $\delta > 0$ corresponds to replacing the levels by resonances of width δ. With the scaling embodied in (2) and (3), $\delta \to 0$ means that the widths tend to zero on the scale of the mean level spacing. We emphasize that this is the regime we are considering here. Without this scaling, that is without the $d_{\rm sm}$ in (2), small δ would correspond to a different regime, where the smoothing could be small in comparison with the value of E, or with the energy range associated with periodic orbits, but still large compared to, or comparable with, the mean level spacing. Then the details of degeneracies that concern us here are smoothed away, and the small-δ behaviour of the negative moments is simpler (Hughes *et al* (2001), Fyodorov (2001)).

Before proceeding to the main calculations, involving degeneracies, we study the simplest case of a spectrum without them. This is the one-dimensional harmonic oscillator, where the levels are equally spaced, and the spectral determinant is

$$Z(x,\delta) = \sin(\pi(x+{\rm i}\delta)). \tag{4}$$

The moments are

$$M(-k,\delta) \equiv \int_{-1/2}^{1/2} \frac{{\rm d}x}{|\sin(\pi(x+{\rm i}\delta))|^k}. \tag{5}$$

For small δ, the dominant contribution to the integral comes from the eigenvalue at $x = 0$. This can be approximated by

$$M(-k,\delta) = \frac{1}{\pi^k}\int_{-\infty}^{\infty} \frac{{\rm d}x}{(x^2+\delta^2)^{k/2}}\,(1+{\rm o}(1)) = \frac{A_k}{\pi^k\delta^{k-1}}(1+{\rm o}(1)) \tag{6}$$

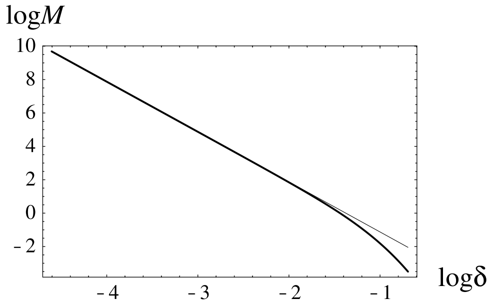

Figure 1. Plot of $\log M(-4, \delta)$ versus $\log \delta$ for a spectrum of equally spaced levels, computed exactly from the integral (5) (thick curve), and from the approximation (6) with the coefficient (7) (thin curve).

where

$$A_k = \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{(y^2+1)^{k/2}} = \begin{cases} \dfrac{\pi(k-3)!}{2^{k-3}(k/2-1)!(k/2-2)!} & (k \text{ even}) \\ \dfrac{2^{k-2}[((k-3)/2)!]^2}{(k-2)!} & (k \text{ odd}). \end{cases} \quad (7)$$

In this simple case the small-δ asymptote of a log–log plot of M versus δ should be a straight line with slope $-(k-1)$, and figure 1 confirms the accuracy of this prediction (and also of the prefactor (7)).

Now we turn to the case of principal interest, in which energies are distributed according to one of the ensembles of random-matrix theory. For the small-δ limit we are considering (equations (2) and (3)), a rigorous approach is unavailable, so the following arguments are heuristic.

Consider a cluster of p near-degenerate levels at $x_1, x_2, \ldots, x_p$, with all spacings $|x_m - x_n| \ll 1$ for $1 \leqslant (m, n) \leqslant p$. The local behaviour of the scaled spectral determinant (2) on the real energy axis is

$$Z_E(x, 0) \approx C(x-x_1)(x-x_1)\cdots(x-x_p). \quad (8)$$

The constant C is of order unity, ensuring that Z_E is of order unity away from the cluster, that is when $|x - x_m|$ is of order unity. (Hereafter C will always denote a numerical constant of order unity.)

The contribution $M_p(-k, \delta)$ of p-clusters to the negative moment $M(-k, \delta)$ is

$$M_p(-k, \delta) = C \int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_p \, \frac{P_p(x_1, \ldots, x_p)}{[(x_1^2+\delta^2)\cdots(x_p^2+\delta^2)]^{k/2}} \quad (9)$$

where P_p is the probability density for p near-degenerate levels. In writing (9), we have incorporated the x-independence of the moments. For the random-matrix ensemble with parameter β (with $\beta = 1$, 2 and 4 for the orthogonal, unitary and symplectic ensembles respectively), the small-spacings behaviour of P_p is the connected product

$$P_p(x_1, \ldots, x_p) \approx C \prod_{m=1}^{p-1} \prod_{n=m+1}^{p} |x_m - x_n|^\beta. \quad (10)$$

This is obtained by taking the contribution of the cluster in question to the full joint probability density of the eigenvalues. Use of this local formula is justified because the resulting integrals (9) (and (11) below) converge for the relevant values of k and p, as we will show later.

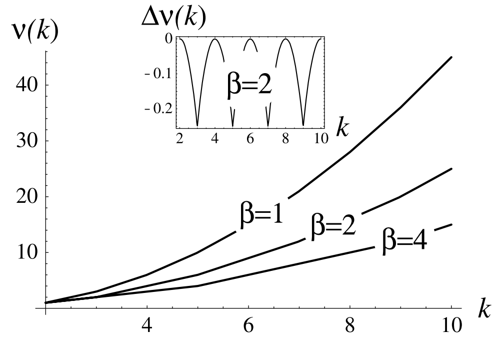

Figure 2. Exponents $\nu(k)$ in the dominant power-law divergence (12) of the negative spectral moments, for the orthogonal ($\beta = 1$), unitary ($\beta = 2$) and symplectic ($\beta = 4$) random-matrix spectra; inset: $\Delta\nu(k) = \nu(k) - k^2/4$ for $\beta = 2$.

The next step is to scale δ from (9) with the change of variables $x_m = \delta u_m$. To implement this, we note that the number of factors in the product (10) is $p(p-1)/2$. Thus

$$M_p(-k,\delta) \approx \frac{C\delta^{p+p(p-1)\beta/2}}{\delta^{pk}} \int_{-\infty}^{\infty} \mathrm{d}u_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}u_p \, \frac{|u_1-u_2|^\beta \cdots |u_{p-1}-u_p|^\beta}{[(u_1^2+1)\cdots(u_p^2+1)]^{k/2}}. \tag{11}$$

The exponents in these power-law divergences depend on p. The moment (3) will be dominated by the cluster with the largest negative exponent. Thus we obtain

$$M(-k,\delta) \sim \frac{C}{\delta^{\nu(k)}} \qquad \text{as } \delta \to 0 \tag{12}$$

where

$$\nu(k) = \max_p \left(p(k-1+\tfrac{1}{2}\beta) - \tfrac{1}{2}\beta p^2\right). \tag{13}$$

The dominating cluster size $p(k)$ is

$$p(k) = \mathrm{int}[(k-1)/\beta + 1] \tag{14}$$

giving the divergence exponent

$$\nu(k) = \mathrm{int}[(k-1)/\beta+1]\left((k-1+\tfrac{1}{2}\beta) - \tfrac{1}{2}\beta\,\mathrm{int}[(k-1)/\beta+1]\right). \tag{15}$$

This is our main result. The underlying argument is non-rigorous, so we present it as a conjecture. Rigorous mathematical investigation of it, and even numerical exploration, seem challenging.

Figure 2 shows the exponents for the three random-matrix ensembles. The formula (15) holds for any real $k \geqslant 1$, not necessarily integer. For the integers $k = 1 + n\beta$ (n integer) the dominating cluster size switches from $p = n$ to $p = n + 1$; $\nu(k)$ is continuous, with value $\beta n(n+1)/2$, but the slope of the graph of $\nu(k)$ is discontinuous. At these k-values where $p(k)$ switches, it is possible that the leading-order power-law behaviour (12) is multiplied by a power of $\log\delta$. Halfway between the switching values, that is where $k = 1 + (n - 1/2)\beta$, $\nu(k) = \beta n^2/2$.

For the unitary ensemble ($\beta = 2$), $\nu(k) = (k^2 - 1)/4$ at odd integers (the switching values), and $\nu(k) = k^2/4$ at even integers. This is close to the exponent $k^2/4$ obtained by a rigorous argument by Hughes *et al* (2001) for the case of random matrices of large dimension N but where the energies are not scaled with N, so the small-δ limit does not probe the details of the degeneracies. There is no obvious way to adapt the rigorous argument to be sensitive to the clusters of degeneracies we are studying here.

When applied to the Riemann zeta function, the large values of negative moments associated with near-coalescences of zeros on the critical line can be regarded as a magnification of the Lehmer phenomenon (Edwards (1974)). The $\beta = 2$ exponents can be regarded as a quantification of the phenomenon. Our conjecture (5) agrees with one of Gonek (1989) over a limited range but not for all k. We incorporate contributions from clusters of zeros, whereas Gonek's heuristic argument includes only isolated zeros. (If δ is not allowed to tend to zero, the exponent is $\nu(k) = k^2/4$, in agreement with his conjecture in this case.)

For the above theory to be correct, each integral in (11) must converge for large u_n, for the dominating cluster size $p(k)$. The total number of factors in the numerator is $p(p-1)/2$, so the total degree of all these factors is $p(p-1)\beta/2$, and the degree of the individual factors is $(p-1)\beta/2$. Therefore the integral that must converge is

$$\int^{u} \frac{\mathrm{d}u_n\,(u_n)^{(p-1)\beta/2}}{(u_n)^k} \tag{16}$$

as $u \to \infty$, requiring

$$p < 2k/\beta + 1. \tag{17}$$

Comparison with (14) shows that this convergence condition is satisfied for all $k \geqslant 1$; further, it is satisfied not just for the dominant cluster size but for all p for which the exponent $p(k-1+\beta/2) - \beta p^2/2$ in (13) is positive.

For Poisson statistics (uncorrelated levels, with $\beta = 0$), the situation is different. Naive application of (15) for $\beta = 0$ would give the exponents $\nu = \infty$ for all k: the divergences in (11) get stronger with increasing p, so there is no dominant cluster size. This suggests that the negative moments diverge faster than in the cases previously considered, and we now confirm this by direct calculation. (A similar situation arises for the positive moments; cf. the $\beta = 0$ results in Keating and Snaith (2000).)

Consider N levels $x_1, \ldots, x_N$ randomly and independently distributed on the line $0 \leqslant x \leqslant N$. Corresponding to this is the spectral determinant (equation (2) with E suppressed)

$$Z(x,0) = \frac{6^{N/2}}{N^N} \prod_{n=1}^{N} (x - x_n). \tag{18}$$

(The normalization—which does not affect the δ-dependence—has been chosen so that the average $\langle Z^2(x) \rangle = 1$.) After incorporating the uniform probability distribution for the uncorrelated levels, the negative moments can be written as

$$\begin{aligned} M(-k,\delta) &= 6^{-Nk/2} N^{Nk} \int_0^N \frac{\mathrm{d}x}{N} \int_0^N \frac{\mathrm{d}x_1}{N} \cdots \int_0^N \frac{\mathrm{d}x_N}{N} \frac{1}{\prod_{n=1}^N ((x-x_n)^2 + \delta^2)^{k/2}} \\ &= 6^{-Nk/2} N^{Nk} \int_0^N \frac{\mathrm{d}x}{N} \left[\int_0^N \frac{\mathrm{d}y}{N} \frac{1}{((x-y)^2+\delta^2)^{k/2}} \right]^N \\ &= 6^{-Nk/2} \int_0^1 \mathrm{d}u \left[\int_0^1 \mathrm{d}v \, \frac{1}{((u-v)^2 + (\delta/N)^2)^{k/2}} \right]^N . \end{aligned} \tag{19}$$

For small δ/N, the limits of the v-integral can be replaced by $\pm\infty$, and then this integral becomes

$$\int_{-\infty}^{\infty} \mathrm{d}v \, \frac{1}{((u-v)^2 + (\delta/N)^2)^{k/2}} = A_k \left(\frac{N}{\delta} \right)^{k-1} (1 + \mathrm{o}(1)) \tag{20}$$

where A_k is given by (7). Thus (19) becomes

$$M(-k,\delta) = \frac{(6^{-k/2} A_k N^{k-1})^N}{\delta^{N(k-1)}} (1 + \mathrm{o}(1)). \tag{21}$$

A more careful analysis confirms that this result survives when corrections to the approximation (20) are incorporated into the integral in (19), even after this is raised to the Nth power.

The formula (21) has a very strong dependence on the size of the interval N, and in particular the δ-dependence itself depends on N. We conclude that for a Poisson spectrum the negative spectral moments have such powerful singularities for real energy that the simple regularization embodied in δ is inadequate to give finite averages that are local (that is, independent of N). This phenomenon is a consequence of the unrestricted clustering permitted by the lack of level repulsion.

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