INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF OPTICS A: PURE AND APPLIED OPTICS

J. Opt. A: Pure Appl. Opt. **6** (2004) 675–678 PII: S1464-4258(04)77273-6

Index formulae for singular lines of polarization

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Received 8 March 2004, accepted for publication 6 May 2004
Published 26 May 2004
Online at stacks.iop.org/JOptA/6/675
DOI: 10.1088/1464-4258/6/7/003

Abstract
Formulae are obtained for the indices (signed rotation numbers) of lines in space on which the polarization of a monochromatic light field is purely circular (C) or purely linear (L). The indices ($\pm 1/2$ for C lines and ± 1 for L lines) involve the electric or magnetic field and its derivatives on the line.

Keywords: singularities, topology, diffraction

1. Introduction

In a seminal paper [1], Nye and Hajnal introduced the singular lines of polarization optics for a general monochromatic light field, where the state of polarization depends on position in three-dimensional space. These are the C lines, on which the light is circularly polarized, and the L lines, on which the light is linearly polarized. The lines were soon detected experimentally [2]. Interest in these singularities has recently revived, both theoretically [3–6] and experimentally [7].

A line singularity is characterized by its index, globally defined as the signed number of rotations of the associated line or vector field around a circuit surrounding the singularity. For a C line, where the associated field is the major or minor axis of the polarization ellipse, projected onto a surface containing the circuit, the index is generically $\pm 1/2$. For an L line, where the associated field is the normal vector to the polarization ellipse, projected onto a surface containing the circuit, the index is generically ± 1. It is however also convenient to express each index locally, that is in terms of the electric or magnetic field and its derivatives on the line itself, and the aim of the present paper is to obtain such explicit index formulae, using elementary geometrical ideas.

We will present the theory in terms of the singularities of the electric field. The magnetic field has similar singularities, usually distinct from those of the electric field [2, 8]; however, all subsequent formulae apply, *mutatis mutandis*, to the magnetic singularities.

It is necessary to recapitulate the basic theory [1]. The field of polarization is represented by the complex electric field vector

$$\mathbf{E}(\mathbf{r}) = \mathbf{P}(\mathbf{r}) + \mathrm{i}\mathbf{Q}(\mathbf{r}) = \exp\{\mathrm{i}\gamma(\mathbf{r})\}(\mathbf{A}(\mathbf{r}) + \mathrm{i}\mathbf{B}(\mathbf{r})). \qquad (1)$$

Henceforth we will omit the dependence on position $\mathbf{r} = \{x, y, z\}$ except where it is necessary to indicate this explicitly. In (1), $\mathbf{P}$ and $\mathbf{Q}$ are real vectors, and $\mathbf{A}$ and $\mathbf{B}$ are the major and minor axes of the polarization ellipse, defined by

$$\mathbf{A} \cdot \mathbf{B} = 0, \qquad |\mathbf{A}| \geqslant |\mathbf{B}|. \qquad (2)$$

In the useful terminology of Dennis [6], γ is the rectifying phase, for which a short calculation gives

$$\gamma = \tfrac{1}{2} \arg \mathbf{E} \cdot \mathbf{E}, \qquad \text{i.e. } \mathbf{A} + \mathrm{i}\mathbf{B} = \frac{\sqrt{\mathbf{E}^* \cdot \mathbf{E}^*}}{|\sqrt{\mathbf{E} \cdot \mathbf{E}}|}\mathbf{E}, \qquad (3)$$

leading to the following explicit formulae for the ellipse axes, satisfying (2):

$$\mathbf{A} = \frac{1}{|\sqrt{\mathbf{E} \cdot \mathbf{E}}|} \mathrm{Re}\left\{\mathbf{E}\sqrt{\mathbf{E}^* \cdot \mathbf{E}^*}\right\}, \quad \mathbf{B} = \frac{1}{|\sqrt{\mathbf{E} \cdot \mathbf{E}}|} \mathrm{Im}\left\{\mathbf{E}\sqrt{\mathbf{E}^* \cdot \mathbf{E}^*}\right\}. \qquad (4)$$

The sign ambiguity associated with the square roots reflects the fact that $\mathbf{A}$ and $\mathbf{B}$ are not vector fields but line fields, since each polarization ellipse is invariant under π rotation in its plane. The normal to the polarization ellipse is

$$\mathbf{N} = \mathrm{Im}\,\mathbf{E}^* \times \mathbf{E} = 2\mathbf{P} \times \mathbf{Q} = 2\mathbf{A} \times \mathbf{B}. \qquad (5)$$

Unlike $\mathbf{A}$ and $\mathbf{B}$, which are direction lines without sense, $\mathbf{N}$ is a vector, whose sense indicates the rotation direction of the physical electric field $\mathrm{Re}[\mathbf{E}\exp(-\mathrm{i}\omega t)]$ as it describes the polarization ellipse.

It is worth emphasizing that this paper concerns polarization lines in three dimensions. Formulae for two-dimensional (e.g. paraxial) fields were given by Dennis [6], with interesting interpretations in terms of Stokes parameters.

2. C lines

When the polarization ellipse is a circle, the axes **A** and **B** are undefined and the expressions (3) and (4) are degenerate. The condition for this is the vanishing of the complex scalar field

$$\psi(\mathbf{r}) = \mathbf{E}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r}), \tag{6}$$

where, from (3), $\arg\psi = 2\gamma$. Therefore the C lines of **E** are the dislocation lines (phase singularities, optical vortices) of ψ. We seek the index I_C of the C line, and expect this to be related to the strength S of the dislocation. A subtlety is that S depends on the sense in which the circuit is described [9, 10], whereas I_C does not [1, 5]: it is an intrinsic property of the **E** field [1]. In fact, as will soon be shown, $I_C = S/2$, where S is defined for a circuit whose sense is positive relative to **N**. Since S is always +1 for a circuit chosen to lie along the vorticity (direction of the C line [3]), defined as

$$\omega = \mathrm{Im}\,\nabla\psi^*\times\nabla\psi, \tag{7}$$

the dislocation strength relative to **N** is $S = \mathrm{sgn}\,\omega\cdot\mathbf{N}/2$, so the C index is [11]

$$I_C = \tfrac{1}{2}\,\mathrm{sgn}\,\omega\cdot\mathbf{N} \tag{8}$$

references [1, 5]. (The length $|\omega|$ is the Jacobian needed to calculate the density of C lines in random waves [3].)

To justify (8), it is necessary to calculate the rotation of the line field **A** around the C line. From (4) and (6),

$$\mathbf{A}(\mathbf{r}) = \mathrm{Re}[\mathbf{E}(\mathbf{r})\exp(-\tfrac{1}{2}\mathrm{i}\arg\psi(\mathbf{r}))]. \tag{9}$$

Now set up local coordinates with $\mathbf{r} = 0$ on the C line and the positive z axis along **N**. Around an infinitesimal circuit of the C line, the field is circularly polarized, and with the coordinates just defined has, as a consequence of (5), the form

$$\mathbf{E}_0 = \exp(\mathrm{i}\mu)\{1, \mathrm{i}, 0\} \tag{10}$$

with μ arbitrary. Thus, locally,

$$\mathbf{A}(\mathbf{r}) = \{\cos(\tfrac{1}{2}\arg\psi(\mathbf{r})-\mu), \cos(\tfrac{1}{2}\arg\psi(\mathbf{r})-\mu), 0\}. \tag{11}$$

From this follows the C line index, obtained from the increment $\Delta\arg\psi$ around the circuit:

$$\begin{aligned} I_C &= \frac{\Delta\arg\psi}{4\pi} = \frac{1}{2}\,\mathrm{sgn}\oint|\psi|^2\nabla\arg\psi\cdot\mathrm{d}\mathbf{r} \\ &= \tfrac{1}{2}\,\mathrm{sgn}\,\mathrm{Im}\oint\psi^*\nabla\psi\cdot\mathrm{d}\mathbf{r} \\ &= \tfrac{1}{2}\,\mathrm{sgn}\,\mathrm{Im}\int\!\!\int\nabla\times\psi^*\nabla\psi\cdot\mathbf{N}\,\mathrm{d}A \\ &= \tfrac{1}{2}\,\mathrm{sgn}\,\omega\cdot\mathbf{N}, \end{aligned} \tag{12}$$

as claimed. This derivation relies on the fact that $|\psi|^2 > 0$ away from C (second equality), and Stokes theorem (fourth equality).

Substituting (5)–(7) and using elementary vector identities gives

$$\begin{aligned} \omega\cdot\mathbf{N} &= \mathrm{Im}(\nabla(\mathbf{E}^*\cdot\mathbf{E}^*)\times\nabla(\mathbf{E}\cdot\mathbf{E}))\cdot\mathrm{Im}\,\mathbf{E}^*\times\mathbf{E} \\ &= -\nabla(\mathbf{E}^*\cdot\mathbf{E}^*)\times\nabla(\mathbf{E}\cdot\mathbf{E})\cdot\mathbf{E}^*\times\mathbf{E} \\ &= |\mathbf{E}^*\cdot\nabla(\mathbf{E}\cdot\mathbf{E})|^2 - |\mathbf{E}\cdot\nabla(\mathbf{E}\cdot\mathbf{E})|^2. \end{aligned} \tag{13}$$

The formulae and (8) and (13) are the main results of this section. The C index switches at points where $\omega\cdot\mathbf{N} = 0$ [1]; at such points, $I_C = 0$.

A convenient illustration of these C index formulae is the field

$$\mathbf{E}(\mathbf{r}) = \begin{pmatrix} (1-x)\cos y + \mathrm{i}y\cos z \\ -(y + \mathrm{i}(1+z)\cos x) \\ z(\cos z + \mathrm{i}\cos y) \end{pmatrix} \tag{14}$$

devised by Nye and Hajnal [1]. A C line passes through the origin and has the local form

$$x = 2y, \qquad z = \tfrac{3}{2}y^2, \tag{15}$$

as confirmed by the expansion

$$|\psi(2y, y, \tfrac{3}{2}y^2)|^2 = 4y^6 + \mathrm{O}(y^7). \tag{16}$$

Application of any of the three expressions (12) leads, after some calculation, to

$$\omega\cdot\mathbf{N}(\mathbf{r} = \{2y, y, \tfrac{3}{2}y^2\}) = -48y + \mathrm{O}(y^2). \tag{17}$$

Therefore I_C switches from +1/2 to −1/2 as y increases through the origin, as claimed in [1]. Figure 1(a) illustrates these indices by showing a random swarm of **A** and **B** lines in a plane pierced twice by the C line.

The classification of patterns of axes of the polarization ellipses in any surface transverse to a C line requires more than the index I_C. As is well known [4, 6, 12], there are not two but three such patterns: the 'star', for index −1/2, and the 'monstar' and 'lemon' for index +1/2. (The classification originated [13] in the patterns of principal curvature directions on a curved surface.) Figure 1(a) shows a star and a lemon; for a smaller value of z, the lemon would transform into a monstar before annihilating with the star at $z = 0$.

3. L lines

The condition for linear polarization is that the minor axis $\mathbf{B} = 0$. This is equivalent to

$$\mathbf{N}(\mathbf{r}) = 0 \tag{18}$$

which from (5) implies that **P** and **Q** (and also **E** and $\mathbf{E}^*$) are parallel, so the condition defines a line [1]. To calculate the rotation of the vector field **N** around the L line, we choose an infinitesimal circuit in the plane, temporarily denoted xy, perpendicular to the major axis **A**.

The index of the L line is the orientation of the map from position $\{x, y\}$ to the two-dimensional field $\{N_x, N_y\}$ (see the appendix), namely

$$\begin{aligned} I_L &= \mathrm{sgn}(\det\partial_i N_j)_{(i,j=x,y)} = \mathrm{sgn}(\partial_x N_x\partial_y N_y - \partial_x N_y\partial_y N_x) \\ &\equiv \mathrm{sgn}\,D(\mathbf{r}). \end{aligned} \tag{19}$$

Conversion of this expression to a three-dimensional form, involving the vector operator ∇ and the vector **N** in the plane perpendicular to **A**, gives

$$D(\mathbf{r}) = [(\mathbf{A}(\mathbf{r})\cdot\nabla_\mathbf{a}\times\nabla_\mathbf{b})(\mathbf{N}(\mathbf{a})\times\mathbf{N}(\mathbf{b})\cdot\mathbf{A}(\mathbf{r}))]_{(\mathbf{a}=\mathbf{b}=\mathbf{r})}, \tag{20}$$

where the subscripts **a** and **b** indicate the variables on which the gradients act. (Equation (20) can be written $D = \mathbf{A}\cdot\mathbf{N}_L$,

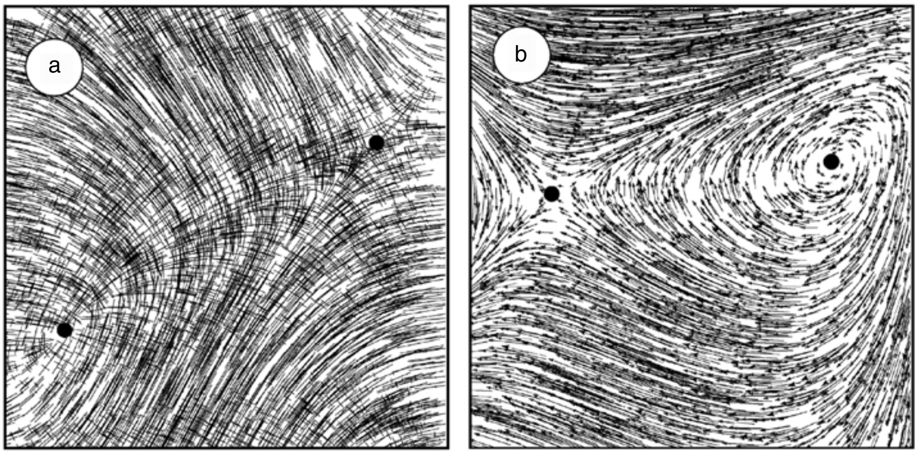

Figure 1. (a) Orthogonal major and minor axis directions $\mathbf{A}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ of the polarization ellipse for the field (14) [1], projected onto the xy plane, for $z = 0.5$; two C points (intersections of a C line with the plane $z = 0.5$) are indicated by dots; the left-hand C point has index $+1/2$ and the right-hand C point has index $-1/2$. (b) The ellipse normal vector $\mathbf{N}(\mathbf{r})$ for the field (25) [1], projected onto the xy plane, for $z = 0.2$; two L points (intersections of an L line with the plane $z = 1$) are indicated by dots; the left-hand L point has index -1 and the right-hand L point has index +1.

where $\mathbf{N}_\mathrm{L}$ is the direction of the L line, and the length $|\mathbf{N}_\mathrm{L}|$ is the Jacobian needed to calculate the density of L lines in random waves [3].)

Equation (20) can also be written in as the expectation value of a matrix:

$$D(\mathbf{r}) = \mathbf{A}(\mathbf{r}) \cdot \mathbf{M}(\mathbf{r}) \cdot \mathbf{A}(\mathbf{r}), \tag{21}$$

where $\mathbf{M}$ is the dyadic

$$\mathbf{M}(\mathbf{r}) = (\nabla_\mathbf{a} \times \nabla_\mathbf{b})(\mathbf{N_a} \times \mathbf{N_b})_{\mathbf{a}=\mathbf{b}=\mathbf{r}}. \tag{22}$$

The observation that the combination of derivatives in (19) is the cofactor of a three-dimensional determinant leads to an intriguing alternative form for $\mathbf{M}$:

$$\mathbf{M}(\mathbf{r}) = \det(\nabla\mathbf{N}(\mathbf{r}))(\nabla\mathbf{N}(\mathbf{r}))^{-1}. \tag{23}$$

Here $\nabla\mathbf{N}$ denotes the 3×3 matrix

$$(\nabla\mathbf{N})_{ij} = \partial_i N_j. \tag{24}$$

In (20) and (21) the outer vectors $\mathbf{A}$ can be replaced by $\mathbf{P}$ or $\mathbf{Q}$, since all three vectors are parallel on an L line. And if the matrix $\mathbf{M}$ is symmetrized the vectors $\mathbf{A}$ and $\mathbf{A}$ can also be replaced by $\mathbf{E}$ and $\mathbf{E}^*$, giving (21) the appearance of a quantum expectation value. Substitution of either of the expressions (5) for $\mathbf{N}$ leads to complicated formulae with no obvious interpretation. Equations (19)–(24) are the main results of this section.

The L index switches at points where $D = 0$ [1]; then $I_\mathrm{L} = 0$.

A convenient illustration of the L index formula is the field

$$\mathbf{E}(\mathbf{r}) = \begin{pmatrix} (x + \mathrm{i}z)\cos y \\ (-y + \mathrm{i}(z-y))\cos x \\ (1 + z + \mathrm{i}(z-y)\cos x) + (1 - z + \mathrm{i}x)\cos y \end{pmatrix} \tag{25}$$

devised by Nye and Hajnal [1]. An L line passes through the origin and has the local form

$$y = z = \tfrac{1}{2}x^2, \tag{26}$$

as confirmed by the expansion

$$\mathbf{N}(x, \tfrac{1}{2}x^2, \tfrac{1}{2}x^2) \cdot \mathbf{N}(x, \tfrac{1}{2}x^2, \tfrac{1}{2}x^2) = x^6 + \mathrm{O}(x^8). \tag{27}$$

Application of (20) leads after some calculation to

$$D(x, \tfrac{1}{2}x^2, \tfrac{1}{2}x^2) = 192x + \mathrm{O}(x^2). \tag{28}$$

Therefore I_L switches from -1 to 1 as x increases through the origin, as claimed in [1]. Figure 1(b) illustrates these indices by showing a random swarm of $\mathbf{N}$ vectors in a plane pierced twice by the L line.

Acknowledgments

I thank Professor Isaac Freund for correspondence that stimulated the work reported here, Dr Mark Dennis for many discussions, and Professor John Nye for his helpful comments on the manuscript. My research is supported by the Royal Society.

Appendix. Derivation of (19)

I could not find a convenient reference for the elementary and well-known result (19), and so provide one here for completeness. Near the L line, on which $\{N_x, N_y\} = 0$, we use the local expansion

$$\{N_x, N_y\} = \{ax + by, cx + dy\} + \cdots. \tag{A.1}$$

Denoting the direction of $\{N_x, N_y\}$ by θ, and introducing polar coordinates $x = r\cos\phi$, $y = r\sin\phi$, we have

$$\tan\theta = \frac{N_y}{N_x} = \frac{c\cos\theta + d\sin\theta}{a\cos\theta + b\sin\theta}. \tag{A.2}$$

The index I_L is the change in θ round an infinitesimal circuit, so

$$\begin{aligned} I_\mathrm{L} &= \frac{\Delta\theta}{2\pi} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \, \frac{\mathrm{d}\theta}{\mathrm{d}\phi} \\ &= \frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\phi \, \frac{(ad - bc)}{(a\cos\phi + b\sin\phi)^2 + (c\cos\phi + d\sin\phi)^2} \\ &= \mathrm{sgn}(ad - bc), \end{aligned} \tag{A.3}$$

which is the desired result (19).

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