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Asymptotic dominance by subdominant exponentials

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A prevalent though unexpected asymptotic phenomenon occurs near anti-Stokes lines, on which two exponentials contributing to a function have the same absolute value: the subdominant exponential contribution can be larger than that from the dominant exponential. The phenomenon arises because the factors multiplying the two exponentials have different asymptotic forms. The boundary of the region of dominance by the subdominant exponential (DSE) is a line, for which an explicit general form is given; this shows that the region of DSE is asymptotically infinitely wide. The DSE line contains the zeros of the function, resulting from complete destructive interference between the two exponential contributions. Several examples are given; two have a physical origin in diffraction physics, and illustrate the fact that DSE can explain observed optical phenomena.

Keywords: anti-Stokes lines; zeros; asymptotic oscillations

1. Introduction

The asymptotic behaviour of functions involving a large parameter is frequently determined by two or more exponentials (Olver 1997), each multiplying a power series that is commonly divergent. The exponential with the largest absolute value is called dominant; the others are subdominant. When the exponentials have almost the same absolute value, interference can occur, and functions oscillate. In standard asymptotics terminology, this is the situation near an anti-Stokes line (Fröman & Fröman 1965; Berry & Mount 1972). It will suffice to consider just two exponentials. Then it is natural to expect that the subdominant exponential is a perturbation of the dominant one, giving rise to a weak oscillatory decoration.

My purpose here is to point out the existence of precisely the contrary phenomenon, in which it is the subdominant exponential that controls the leading behaviour of the function, with the smaller contribution, giving rise to interference oscillations, coming from the dominant exponential. This dominance of the subdominant exponential (DSE) seems paradoxical at first but, as explained in § 2, it is actually a universal aspect of asymptotics, as prevalent as the expected behaviour, i.e. dominance of the dominant exponential (DDE).

Of course, the numerical significance of exponentially subdominant terms is familiar (see, for example, Balian *et al.* 1978) and was emphasized by Stokes himself (Stokes 1864). And, in particular applications (for example, in matched asymptotics

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associated with boundary layers), it may well have been noticed that the subdominant exponential gives the larger contribution. Here I aim to draw attention to the generality of the DSE phenomenon and the fact that it can be described analytically in a universal way.

In the common case where the function depends on a single variable, the crossover between DDE and DSE is a line in the complex plane of this variable. Along this line, the zeros of the function occur, representing complete destructive interference between the dominant and subdominant exponentials.

Section 3 contains four examples of DSE in functions defined by integrals. As well as simple cases, contrived to illustrate DSE, these include observed physical phenomena whose explanation requires DSE.

DSE should be distinguished from the quite different behaviour near a Stokes line (Dingle 1973), where the absolute values of the two exponentials are maximally different, corresponding to extreme DDE. Across a Stokes line, subdominant exponentials can appear and disappear, this is the Stokes phenomenon, in which the divergence of the asymptotic series plays a central role (Berry 1989). DSE has no bearing on the analytic structure of the asymptotic series, the positions of the Stokes and anti-Stokes lines, or the connection formulae.

2. General theory

Consider a function f, depending on a complex variable z, whose asymptotic behaviour, representing f accurately when a parameter (which we will not need to write explicitly) becomes large, is determined by two exponentials:

$$f(z) \approx a_+(z)\exp\{\mathrm{i}\phi_+(z)\} + a_-(z)\exp\{\mathrm{i}\phi_-(z)\}. \tag{2.1}$$

Here '+' and '−' denote the dominant and subdominant exponentials, so

$$\operatorname{Im}\phi_+ < \operatorname{Im}\phi_-. \tag{2.2}$$

Equation (2.1) represents the leading-order behaviour. Usually the prefactors $a_\pm$ are multiplied by an asymptotic series, but DSE does not depend on this otherwise important aspect of asymptotics and it will not be considered further.

To bring this class of functions to a common form, it is convenient to extract the subdominant term, and write, exactly,

$$f(z) = a_-(z)\exp\{\mathrm{i}\phi_-(z)\}g(\Phi(z)), \tag{2.3}$$

where

$$\Phi(z) = \phi_-(z) - \phi_+(z). \tag{2.4}$$

The anti-Stokes line is the real axis of the variable Φ, where both exponentials are equal in absolute value, and the region where ϕ_+ dominates is the upper half-plane $\operatorname{Im}\Phi > 0$. (The variable Φ is related to the 'singulant' F (Dingle 1973) that plays an important role near Stokes lines (Berry 1989), by $\Phi = \mathrm{i}F$.)

The crucial observation leading to the prediction of DSE is that the prefactors a_+ and a_- commonly have different asymptotic behaviour. Here we will assume that

$$\frac{a_+(z)}{a_-(z)} = \frac{K}{\Phi^\mu}, \quad \mu \text{ real}, \ \mu > 0, \tag{2.5}$$

where K is a constant. Powers of $\log\Phi$ can occur, and also complex μ, but (2.5) is sufficiently general to illustrate DSE. It is not necessary to consider $\mu < 0$, because then DSE occurs in the lower half-plane of Φ rather than the upper.

Thus the asymptotic representation (2.1) becomes

$$g(\Phi) \approx \frac{K}{\Phi^\mu}\exp(-\mathrm{i}\Phi) + 1, \tag{2.6}$$

with the first term representing the dominant exponential and the second term (unity) representing the subdominant exponential. DSE occurs when $\operatorname{Im}\Phi > 0$ and

$$\frac{|K|\exp(\operatorname{Im}\Phi)}{|\Phi|^\mu} < 1, \tag{2.7}$$

that is

$$\operatorname{Re}\Phi > \sqrt{|K|^{2/\mu}\exp((2/\mu)\operatorname{Im}\Phi) - \operatorname{Im}^2\Phi}. \tag{2.8}$$

This equation defines a region where the square root is positive, bounded by a line in the upper half-plane of Φ, with DSE occurring in the region between this line and the real axis $\operatorname{Im}\Phi = 0$. Outside this region, the familiar DDE occurs.

To leading order, the zeros of $f(z)$ are given by the approximate solution of $g(\Phi) = 0$ when $\arg\Phi \ll 1$. A little calculation leads to

$$\Phi_m \approx (2m-1)\pi + \arg K + \mathrm{i}\log\left\{\frac{((2m-1)\pi + \arg K)^\mu}{|K|}\right\}, \quad m = 1, 2, \ldots. \tag{2.9}$$

The results (2.8) and (2.9) show that as $\operatorname{Re}\Phi$ increases the DSE region gets wider, extending up to $\operatorname{Im}\Phi \sim \log(\operatorname{Re}\Phi)$. Therefore, the DSE region is asymptotically infinitely wide, rather than being near the anti-Stokes line as might have been anticipated. (However, the angular width of the DSE region shrinks, since the boundary is $\arg\Phi \sim \log(\operatorname{Re}\Phi)/\operatorname{Re}\Phi$.)

The boundary of the DSE region, and the positions of the zeros, are qualitatively the same for all values of K and μ, so the DSE phenomenon is essentially universal.

3. Examples

(a) *Saddle point and endpoint*

A frequent situation where a_+ and a_- have different asymptotic forms occurs in oscillatory integrals over a finite interval containing a saddle point, i.e. a point of stationary phase. Then the asymptotics consists of contributions from the saddle point and from either or both endpoints. The simplest example is the integral

$$f_1(z) = \int_0^z \mathrm{d}t\, \exp(-\tfrac{1}{2}\mathrm{i}\pi t^2) = C(z) - \mathrm{i}S(z), \tag{3.1}$$

where C and S denote the Fresnel integrals (eqns 7.3.1 and 7.3.2 in Abramowitz & Stegun (1972)); we are interested in z nearly real and $\operatorname{Re} z \gg 1$. There is a saddle point at $t = 0$, and the phase $-\operatorname{Re}(\pi t^2/2)$ varies linearly at the upper endpoint $t = z$, which contributes the dominant exponential when $\operatorname{Im} z > 0$. Standard techniques (Olver 1997; Dingle 1973) give

$$f_1(z) \approx -\frac{1}{\mathrm{i}\pi z}\exp(-\tfrac{1}{2}\mathrm{i}\pi z^2) + \frac{1}{\sqrt{2}}\exp(-\tfrac{1}{4}\mathrm{i}\pi). \tag{3.2}$$

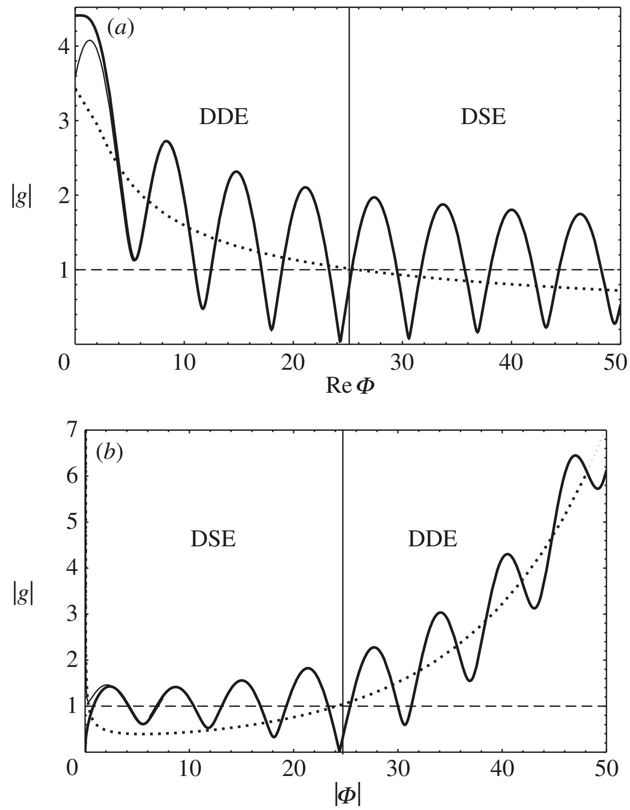

Figure 1. Modulus $|g(\Phi)|$, showing the transition between DSE and DDE for the Fresnel integral $g_1(\Phi)$ (equation (3.4)) of § 3 *a*: (*a*) on a line with $\operatorname{Im}\Phi = 2.2$; (*b*) on a ray with $\arg\Phi = \pi/35$. Thick curves, exact $|g|$; thin curves, the approximation (2.6); dashed lines, subdominant contribution (unity) to g; dotted curves, dominant contribution to g (first term of (2.6)).

This has the form (2.1), enabling the identifications

$$\Phi = \tfrac{1}{2}\pi z^2, \qquad K_1 = -\frac{1}{\sqrt{\pi}}\exp(-\tfrac{1}{4}\mathrm{i}\pi), \qquad \mu_1 = \tfrac{1}{2} \tag{3.3}$$

in (2.6), and

$$g_1(\Phi) = \sqrt{2}\exp(\tfrac{1}{4}\mathrm{i}\pi)\left[C\left(\sqrt{\frac{2\Phi}{\pi}}\right) - \mathrm{i}S\left(\sqrt{\frac{2\Phi}{\pi}}\right)\right] \tag{3.4}$$

in (2.3).

The transition between DSE and DDE appears in several different forms, depending on how the function is plotted. This is illustrated in figure 1, which shows the modulus $|g_1|$ along two lines: parallel to the real axis, that is with constant $\operatorname{Im}\Phi$ (figure 1*a*); and a ray with constant $\arg\Phi$ (figure 1*b*). The transitions are clear in both cases, as is the high accuracy of the approximation (2.6): the curves are indistinguishable except near the origin, and the small discrepancy could be further reduced by including higher orders in the asymptotics. Figure 2 shows $\arg g(\Phi)$ in the Φ-plane; the phase contours intersect at the zeros of g, which with high accuracy correspond to the zeros of the approximation (2.6).

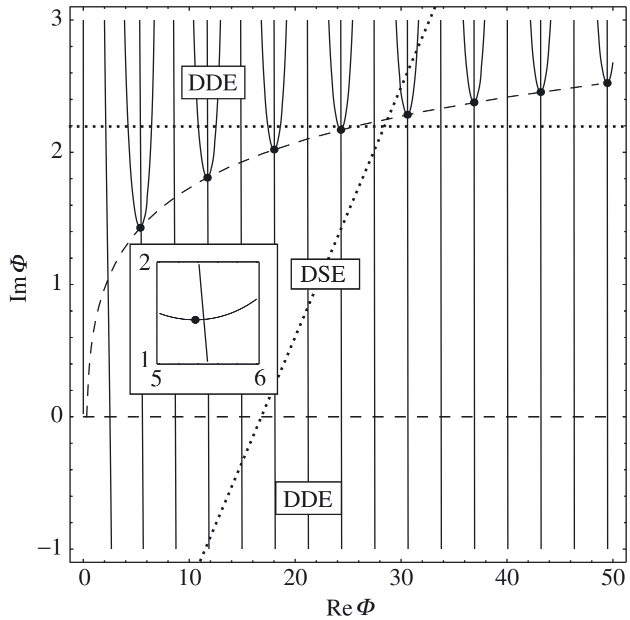

Figure 2. Contours of $\arg(g(\Phi))$ for the Fresnel integral $g_1(\Phi)$ (equation (3.4)) of § 3 *a* at intervals of $\pi/4$; the contours cross at the zeros of $g(\Phi)$. Filled circles, approximate zeros calculated from (2.6); dashed curves, the boundary between the regions of DSE and DDE, calculated from (2.8); dotted lines, lines along which the graphs of $|g|$ in figure 1 were calculated. The inset is a magnification near the smallest zero, illustrating the high accuracy of the approximation based on (2.6).

The corresponding pictures for the examples to be presented in the next sections look almost identical to figures 1 and 2 and so will not be shown. The similarity is a consequence of the universality of the DSE phenomenon, as explained in the previous section.

(*b*) *Suppressed saddle*

Another case where a_+ and a_- have different asymptotic dependences is an integral with two saddle points, one of which is suppressed by a prefactor with a zero at the same location. The simplest example is

$$\begin{aligned} f_2(z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t\,(t-\sqrt{z})\exp\{\mathrm{i}(\tfrac{1}{3}t^3 - zt)\} \\ &= -\sqrt{z}\,\mathrm{Ai}(-z) - \mathrm{i}\,\mathrm{Ai}'(-z), \end{aligned} \tag{3.5}$$

in which the zeros are at $t = \pm\sqrt{z}$ and 'Ai' denotes the Airy function. Standard Airy asymptotics, or application of the method of stationary phase, including the first correction term for the suppressed saddle $t = +\sqrt{z}$, which is dominant for $\mathrm{Im}\, z > 0$, leads to

$$f_2(z) \approx \frac{z^{1/4}\exp(\frac{3}{4}\mathrm{i}\pi)}{\sqrt{\pi}}\left[-\frac{1}{8z^{3/2}}\exp\{-\tfrac{2}{3}z^{3/2}\} + \exp\{\tfrac{2}{3}z^{3/2}\}\right]. \tag{3.6}$$

This has the form (2.1), enabling the identifications

$$\Phi = \tfrac{4}{3}z^{3/2}, \qquad K_2 = -\tfrac{1}{6}, \qquad \mu_2 = 1 \tag{3.7}$$

in (2.6), and

$$g_2(\Phi) = \frac{\sqrt{\pi}}{(\frac{3}{4}\Phi)^{1/6}} \exp\{\mathrm{i}(\tfrac{1}{4}\pi - \tfrac{1}{2}\Phi)\}[(\tfrac{3}{4}\Phi)^{1/3}\,\mathrm{Ai}\{-(\tfrac{3}{4}\Phi)^{2/3}\} + \mathrm{i}\,\mathrm{Ai}'\{-(\tfrac{3}{4}\Phi)^{2/3}\}] \tag{3.8}$$

in (2.3). The factor $z^{-3/2} \sim \Phi^{-1}$, multiplying the dominant exponential, gives rise to the DSE phenomenon.

(c) *Suppressed endpoint (i)*

In the integral

$$\begin{aligned} f_3(z) &= \int_0^\infty \mathrm{d}t t \,\exp(-\tfrac{1}{2}\mathrm{i}\pi t^2)\cos(z\pi t) \\ &= -\frac{\mathrm{i}}{\pi} + z\exp(\tfrac{1}{2}\mathrm{i}\pi z^2)[C(z) - \mathrm{i}S(z)], \end{aligned} \tag{3.9}$$

there is a saddle at $t = z$, and the endpoint $t = 0$, at which the phase varies linearly and which contributes the dominant exponential when $\mathrm{Im}\, z > 0$, is suppressed by the prefactor t. Standard asymptotics gives

$$f_3(z) \approx -\frac{1}{\pi^2 z^2} + \frac{z}{\sqrt{2}}\exp\{\mathrm{i}\pi(\tfrac{1}{2}z^2 - \tfrac{1}{4})\}. \tag{3.10}$$

Again this has the form (2.1), enabling the identifications

$$\Phi = \tfrac{1}{2}\pi z^2, \qquad K_3 = -\frac{\exp(\frac{1}{4}\mathrm{i}\pi)}{2\sqrt{\pi}}, \qquad \mu_3 = \tfrac{3}{2} \tag{3.11}$$

in (2.6), and

$$g_3(\Phi) = \exp(\tfrac{1}{4}\mathrm{i}\pi)\left[\frac{1}{\mathrm{i}\sqrt{\pi\Phi}}\exp(-\mathrm{i}\Phi) + \sqrt{2}\left(C\left(\sqrt{\frac{2\Phi}{\pi}}\right) - \mathrm{i}S\left(\sqrt{\frac{2\Phi}{\pi}}\right)\right)\right] \tag{3.12}$$

in (2.3).

This example is not contrived, because $f_3(z)$ occurs in the theory of conical diffraction (Berry 2004), in which a Gaussian beam of light with width w and wavelength λ emerges from a slab of transparent biaxial crystal after traversing it parallel to an optic axis. The emergent beam is a hollow cylinder with radius R ($\gg w$) and an intensity spike along the axis. $f_3(z)$ governs the spike amplitude at distance d from the slab, with $z = R\sqrt{2\pi/(d\lambda - \mathrm{i}2\pi w^2)}$. The light intensity is dominated by the subdominant exponential in the region of significance, with DDE occurring only where the light is faint.

(*d*) *Suppressed endpoint (ii)*

The final example is the integral

$$\begin{aligned} f_4(z) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{d}t\, \sqrt{t} \exp(-\tfrac{1}{2}\mathrm{i}\pi t^2) \cos(tz + \tfrac{1}{4}\pi) \\ &= \tfrac{1}{8}\sqrt{\tfrac{1}{2}\pi} \exp(-\tfrac{3}{8}\mathrm{i}\pi) z^{3/2} \exp(\tfrac{1}{4}\mathrm{i}z^2) \\ &\quad \times [\mathrm{i}(1 - 2^{3/2}\exp(-\tfrac{1}{4}\mathrm{i}\pi)) J_{3/4}(\tfrac{1}{4}z^2) - Y_{3/4}(\tfrac{1}{4}z^2) \\ &\quad - \exp(-\tfrac{1}{4}\mathrm{i}\pi)[\mathrm{i}(1 - 2^{3/2}\exp(\tfrac{1}{4}\mathrm{i}\pi)) J_{1/4}(\tfrac{1}{4}z^2) - Y_{1/4}(\tfrac{1}{4}z^2)]], \end{aligned} \tag{3.13}$$

where J and Y denote Bessel functions of the first and second kinds and where the equality can be derived, for example, via parabolic cylinder functions (see §§ 19.5 and 19.19 of Abramowitz & Stegun (1972)). Again there is a saddle at $t = z$, and an endpoint at $t = 0$ that is dominant for $\mathrm{Im}\, z > 0$, but now the dominant exponential is suppressed by $\sqrt{z}$ rather than t as in (3.9). Straightforward Bessel asymptotics leads to

$$f_4(z) \approx \tfrac{1}{2} z^{1/2} \left[-\frac{1}{z^2\sqrt{2}} + \exp(\tfrac{1}{2}\mathrm{i}z^2) \right], \tag{3.14}$$

which has the same form as (2.1), enabling the identifications

$$\Phi = \tfrac{1}{2}z^2, \qquad K_4 = -\frac{1}{2\sqrt{2}}, \qquad \mu_4 = 1 \tag{3.15}$$

in (2.6), and

$$\begin{aligned} g_4(f) &= \tfrac{1}{4}\sqrt{\pi\Phi} \exp(-\tfrac{3}{8}\mathrm{i}\pi) \exp(-\tfrac{1}{2}\mathrm{i}\Phi) \\ &\quad \times [\mathrm{i}(1 - 2^{3/2}\exp(-\tfrac{1}{4}\mathrm{i}\pi)) J_{3/4}(\tfrac{1}{2}\Phi) - Y_{3/4}(\tfrac{1}{2}\Phi) \\ &\quad - \exp(-\tfrac{1}{4}\mathrm{i}\pi)[\mathrm{i}(1 - 2^{3/2}\exp(\tfrac{1}{4}\mathrm{i}\pi)) J_{1/4}(\tfrac{1}{2}\Phi) - Y_{1/4}(\tfrac{1}{2}\Phi)]] \end{aligned} \tag{3.16}$$

in (2.3).

The function $f_4(z)$ also plays an important role in conical diffraction, since it governs the intensity distribution across the cylinder of emergent light. The cylinder is split into two narrow 'Poggendorff rings' (Born & Wolf 1959). The profile of the inner ring, as a function of radial distance r with $r < R$, is governed (Berry 2004) by $f_4(z)$ with $z = (R - r)/\sqrt{d\lambda/2\pi - \mathrm{i}w^2}$. The ring profile is dominated by the subdominant exponential in the region of significant intensity, with the dominant exponential giving rise to interference oscillations in the form of the recently predicted 'secondary Poggendorff rings' (Warnick & Arnold 1997).

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