Conical diffraction from an $N$-crystal cascade

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Received 21 May 2010, accepted for publication 14 June 2010
Published 7 July 2010
Online at stacks.iop.org/JOpt/12/075704

Abstract
The transmitted field is calculated for a cascade of $N$ biaxial crystals, with their optic axes parallel but with arbitrary orientations about this axis, and arbitrary strengths. The focused pattern is the superposition of $2^{N−1}$ single-crystal concentric conical diffraction patterns, whose ring radii are combinations of those from the individual crystals in the cascade. Explicit expressions are given for the general case of arbitrary incident polarization and for the mean intensity for a cascade with random orientations and strengths and arbitrary $N$. For circular incident polarization, the ring pattern is rotationally symmetric for all $N$, but for $N > 1$ and away from focus the patterns for left and right circular incident polarizations are different.

Keywords: biaxial, diffraction, conical intersection, polarization, crystal optics

1. Introduction
Recently, interest in conical refraction has revived, both theoretically [1–3] and experimentally [4–6]. The development of a conical refraction laser [7], whose resonator is a biaxial crystal within which light reflects back and forth, raises questions about the image that would result from a cascade of $N$ such crystals. My aim here is to devise a theory for an $N$-crystal cascade and explore its consequences.

The system to be investigated is shown in figure 1. $N$ biaxial crystals, denoted $n = 1, \ldots, N$, are arranged in series, with a common optic axis: the $z$ direction. The crystals can be of different materials and can have different lengths $l_n$ and different cone angles $\alpha_n$, and need not be in contact (though each must be wide enough to capture the light from its predecessor). The crystals are illuminated by a beam directed along the axis. The beam has arbitrary profile, with intensity $1/e$ radius $w$, and arbitrary polarization state. Each is characterized by its strength parameter $\rho_n = \alpha_n l_n / w$ [1, 8], that is, the radius of the emerging ring, in units of the beam width. The case where the crystals are identically oriented is trivial: it is intuitively obvious (and will follow from the general theory) that the cascade acts like a single long crystal, and the image focused onto a screen will consist of the familiar Hamilton–Lloyd pair of rings, separated by the Poggendorff dark ring. The interesting phenomena to be studied here arise when the crystals are differently oriented, that is, characterized by different rotation angles $\gamma_n$ about the common optic axis; for the $n$th crystal, $\gamma_n$ is defined as the angle between the space-fixed $x$ axis and the principal direction perpendicular to the axis.

We seek the two-component complex electric field vector $E_N(\rho, \zeta)$. Here $\rho = (\rho, \theta)$ is the transverse position, with $\rho$ measured in units of $w$, and the longitudinal coordinate is

$$\zeta = \frac{1}{kw^2} \left( z + \sum_{n=1}^{N} l_n \left( \frac{1}{\mu_n} - 1 \right) \right),$$

(1.1)

whose interpretation [1, 8] is as follows. $\zeta$ is the scaled distance from the focal image plane, that is, the plane in which the incident beam waist would be focused if the crystals were isotropic with mean refractive indices $\mu_n$, and the unscaled distance $z$ is measured from the beam waist.

The general formalism is developed in section 2, and its main consequences derived in section 3. Explicit formulae for $N = 1−4$ are presented in section 4. The theory of the average image intensity for a cascade of $N$ random crystals, that is, where the orientations $\gamma_n$ and the strengths $\rho_n$ are random variables, is studied in section 5. Concluding remarks are given in section 6.

2. Cascade formalism
Let $U_{\text{tot}}(\kappa)$ denote the $2 \times 2$ unitary matrix describing the transformation of the plane wave with transverse wavenumber $\kappa = (\kappa, \phi)$ (in units of $1/w$) by the crystal cascade. This is the
simplest starting point, because there is translation symmetry perpendicular to \( z \), so \( \kappa \) is conserved when entering and leaving the cascade and from crystal to crystal within it. The electric field \( E_N(\rho, \zeta) \) in the plane at scaled distance \( \zeta \) is obtained, after Fourier transformation, by letting this matrix act on the incident Gaussian beam, whose Fourier amplitude (in terms of scaled wavenumber) is the complex two-component vector \( \bar{E}_0(\kappa) \).

\[
E_N(\rho, \zeta) = \frac{1}{2\pi}d\kappa \exp\left\{i\left(\kappa \cdot \rho - \frac{1}{2}i\kappa^2 \zeta\right)\right\} U_{\text{tot}}(\kappa) \bar{E}_0(\kappa).
\]  

We can neglect multiple reflections between crystals in the cascade, for reasons explained at the end of section 6. Thus \( U_{\text{tot}}(\kappa) \) is the product of the unitary matrices \( U_n(\kappa) \) for the individual crystals:

\[
U_{\text{tot}}(\kappa) = U_N(\kappa)U_{N-1}(\kappa)\cdots U_2(\kappa)U_1(\kappa).
\]  

For \( U_n(\kappa) \), we use existing paraxial theory [1, 9, 10], slightly modified in an obvious way to incorporate the orientation \( \gamma_n \), which was tacitly assumed zero in previous treatments but is now essential. We will need the three Pauli matrices and the identity matrix:

\[
s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]  

Then

\[
U_n(\kappa) = \exp\{-i\rho_0(\kappa)\sin(\phi - \gamma_n)s_1 + \cos(\phi - \gamma_n)s_3).\]
\]  

A technical point: the origin of transverse position \( \rho \) is offset from the incident beam axis by \( \sum_{n=1}^{N} \rho_0(\cos \gamma_n, \sin \gamma_n) \), incorporating the cumulative effect of the shifts associated with the fact that the cones in the crystals are slanted [1, 11].

The matrix product (2.2) can be evaluated in terms of the eigenbasis for each matrix. Using Dirac notation, we write each matrix as a superposition of the two orthogonal eigenstates:

\[
U_n(\kappa) = \exp(-i\rho_0(\kappa))|n, +\rangle\langle n, +| + \exp(i\rho_0(\kappa))|n, -\rangle\langle n, -|.
\]  

where the eigenvectors are

\[
|n, +\rangle = \begin{pmatrix} \cos(\frac{1}{2}(\phi - \gamma_n)) \\ \sin(\frac{1}{2}(\phi - \gamma_n)) \end{pmatrix},
|n, -\rangle = \begin{pmatrix} \sin(\frac{1}{2}(\phi - \gamma_n)) \\ \cos(\frac{1}{2}(\phi - \gamma_n)) \end{pmatrix}.
\]  

We will need the overlaps

\[
\langle m, +|n, +\rangle = \langle m, -|n, -\rangle = \langle n, +|m, +\rangle = \langle n, -|m, -\rangle = \langle n, +|m, +\rangle = \langle n, -|m, -\rangle = \langle n, +|m, -\rangle
\]

\[
= \langle n, -|m, +\rangle = \sin(\Gamma_{mn}),
\]

where

\[
\Gamma_{mn} = \frac{i}{2}(\gamma_m - \gamma_n), \quad \gamma_{mn} = \frac{i}{2}(\gamma_m + \gamma_n).
\]

including \( \gamma_{mn} \) for later reference. The fact that these overlaps are independent of the direction \( \phi \) of the vector \( \kappa \) will greatly facilitate the integration in (2.1).

According to (2.5), the product (2.2) is a sum over sequences of overlaps, represented as strings of eigenvalue indices \( + \) and \( - \). It is convenient to refer to these as ‘paths’, and use the notations

\[
\varepsilon \equiv \{\varepsilon_N \cdots \varepsilon_1\}, \quad \rho_{\text{tot}}(\varepsilon) \equiv \sum_{n=1}^{N} \varepsilon_n \rho_0(\varepsilon_n)(\varepsilon_n = \pm 1).
\]

Then (2.2) can be written

\[
U_{\text{tot}}(\kappa) = A_{++}|N, +\rangle|1, +\rangle + A_{+-}|N, +\rangle|1, -\rangle
\]

\[
+ A_{-+}|N, -\rangle|1, +\rangle + A_{--}|N, -\rangle|1, -\rangle,
\]

in which the coefficients are

\[
A_{\varepsilon_N \varepsilon_1} = \sum_{\varepsilon} \exp\{-i\kappa\rho_{\text{tot}}(\varepsilon)\} F_{\varepsilon_N \varepsilon_1}(\varepsilon),
\]

where

\[
F_{\varepsilon_N \varepsilon_1}(\varepsilon) \equiv \langle N, \varepsilon_N|N = 1, \varepsilon_{N-1}\rangle \cdots \{2, \varepsilon_2|1, \varepsilon_1\}.
\]

(This notation contains a slight redundancy, because the signs at the ends of the paths are fixed, but this should
cause no confusion.) There are $2^{N-2}$ sequences $\varepsilon$ in (2.11), each representing a product of the exponential containing the cylinder radii $\rho_0$, and $N - 1$ overlaps $(n, \varepsilon_0|n-1, \varepsilon_{n-1})$.

A short calculation gives, for the operators in (2.10),

$$|N, +\rangle = \frac{1}{2}(\cos\gamma_{1N})|1\rangle + i\frac{1}{2}i\sin\gamma_{1N}|s_2\rangle$$

$$|N, -\rangle = \frac{1}{2}(\cos\gamma_{1N})|1\rangle - i\frac{1}{2}i\sin\gamma_{1N}|s_2\rangle$$

Thus, using a natural ordering of the matrices,

$$U_{totN}(\kappa) = U_{totN}(\kappa)I + U_{2totN}(\kappa)\mathbf{s}_2$$

where

$$U_{totN}(\kappa) = \frac{1}{2}[A_{++} + A_{--}]\cos\gamma_{1N}$$

$$U_{2totN}(\kappa) = \frac{1}{2}[A_{++} + A_{--}]\sin\gamma_{1N}$$

$$U_{totN}(\kappa) = \frac{1}{2}[A_{++} - A_{--}]\cos(\phi - \gamma_{1N})$$

$$U_{1totN}(\kappa) = \frac{1}{2}[A_{++} - A_{--}]\sin(\phi - \gamma_{1N})$$

It is convenient to separate the index paths into pairs with $\varepsilon$ and $-\varepsilon$, and note two useful consequences of (2.7):

$$F_{++}(\varepsilon) = F_{--}(-\varepsilon), \quad F_{+-}(\varepsilon) = -F_{-+}(-\varepsilon).$$

Thus

$$A_{++} + A_{--} = \sum_{\varepsilon} F_{++}(\varepsilon)\cos(\kappa\rho_{ot}(\varepsilon))$$

$$A_{+-} + A_{-+} = -2i\sum_{\varepsilon} F_{++}(\varepsilon)\sin(\kappa\rho_{ot}(\varepsilon))$$

$$A_{++} - A_{--} = \sum_{\varepsilon} F_{+-}(\varepsilon)\cos(\kappa\rho_{ot}(\varepsilon))$$

$$A_{+-} - A_{-+} = \sum_{\varepsilon} F_{++}(\varepsilon)\cos(\kappa\rho_{ot}(\varepsilon)).$$

Now we can evaluate the integrals (2.1) for any incident beam profile $E_0(\kappa)$. From now on, we write explicit formulae for a Gaussian beam with uniform polarization $e_0$, for which

$$E_0(\kappa) = \exp(-\frac{1}{2}\kappa^2)e_0.$$

Then (2.1) can be written in the form

$$E_N(\rho, \zeta) = M_N(\rho, \zeta)e_0,$$
obvious if the coordinate representation had been used to describe the transformations of the wave by the individual crystals. The ring amplitudes are given by the coefficients $F_{++}(\varepsilon)$ and $F_{+−}(\varepsilon)$, which depend on the relative orientations $\Gamma_{n,n−1}$ of successive crystal pairs. Because the matrices (2.4) in the product (2.2) do not commute, their order matters: the patterns would be different if the crystals were permuted. The polarization structure depends on the incident polarization $e_0$, and is generally elliptical, with the axes of the polarization ellipse turning by $\pi$ as $\theta$ increases by $2\pi$, that is, during a circuit of the beam axis $\rho = 0$.

For paths $\varepsilon$ representing rings with large radii, that is, $|\rho_{\text{tot}}(\varepsilon)| \gg 1$, and for small $\zeta$, each ring is actually a pair, separated by the Poggendorff dark ring. In this regime [8]

$$B_1(\rho, \zeta; |\rho_0|) \approx B_0(\rho, \zeta; \rho_0),$$

and the rings are linearly polarized, independently of $e_0$. In fact, the field for each of the rings is

$$E(\varepsilon) = 2 \left( \begin{array}{c} \cos \left( \frac{1}{2}(\theta - \gamma) \right) \\ \sin \left( \frac{1}{2}(\theta - \gamma) \right) \end{array} \right) \frac{B_0(\rho, \zeta; \rho_{\text{tot}}(\varepsilon))}{|\rho_{\text{tot}}(\varepsilon)|} \times [F_{++}(\varepsilon) \left( \cos \left( \frac{1}{2}(\theta - \gamma_1) \right) e_{01} + \sin \left( \frac{1}{2}(\theta - \gamma_1) \right) e_{02} \right) + F_{+−}(\varepsilon) \left( \sin \left( \frac{1}{2}(\theta - \gamma_1) \right) e_{01} - \cos \left( \frac{1}{2}(\theta - \gamma_1) \right) e_{02} \right)].$$

(3.3)

As $\zeta$ increases, the outer ring in each pair fades, and the inner shrinks onto the axial Raman spot, in a way that is now familiar for a single crystal [1, 8]; in an $N$-crystal cascade, this happens synchronously for all the rings, because of the common longitudinal variable $\zeta$.

Cancellations in (2.9) can generate zero-radius rings, the simplest example being $N = 2$ with $\rho_0 = \rho_2$ and $\varepsilon_1/\varepsilon_2 = -1$; then the corresponding rings in the cascade pattern simply

Figure 2. Intensity patterns for a three-crystal cascade with strengths $\rho_0 = 5$, $\rho_0 = 10$ and $\rho_0 = 20$, giving rings at $|\rho_{\text{tot}}| = 5$ 15, 25 and 35 (cf (2.9)). (a)–(d) In the focal plane $\zeta = 0$, for orientations (a) $\gamma_1 = 0$, $\gamma_2 = \pi/4$, $\gamma_3 = \pi/2$; (b) $\gamma_1 = 0$, $\gamma_2 = \pi/3$, $\gamma_3 = \pi/4$; (c) $\gamma_1 = 0$, $\gamma_2 = \pi/4$, $\gamma_3 = \pi/4$ and (d) $\gamma_1 = 0$, $\gamma_2 = 0$, $\gamma_3 = \pi/2$. The rings in cases (c) and (d), for which two of the orientations are the same, correspond to those from a two-crystal cascade, with $\rho_0 = 5$ and $\rho_0 = 30$ (case (c)) and $\rho_0 = 15$ and $\rho_0 = 20$ (case (d)). (e) is a simulation of the rings for case (a) but with $\zeta = 2$ so that the rings are broader and therefore easier to discern.
circularly polarized, that is, in the simplest case, the incident beam is right- or left-circularly polarized. Without loss of generality, we always choose $N$ to reproduce the $\zeta$-broadened incident Gaussian beam, because the integrals \ref{eq:220} reduce to

$$B_0(\rho, \zeta; 0) = \frac{\exp\left\{ \frac{1}{2} (1 + i\zeta) \rho^2 \right\}}{1 + i\zeta}, \quad B_1(\rho, \zeta; 0) = 0. \tag{3.4}$$

The intensity ring pattern beyond the $N$-crystal is

$$I_N(\rho, \zeta) = E_N(\rho, \zeta) \cdot E_N(\rho, \zeta) = e_0^* M_N(\rho, \zeta) M_N(\rho, \zeta) e_0. \tag{3.5}$$

In the simplest case, the incident beam is right- or left-circularly polarized, that is

$$e_0 = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ \pm i \end{array} \right). \tag{3.6}$$

Then a short calculation shows that the emerging intensity is independent of the azimuth $\theta$, and

$$I_N(\rho, \zeta) = \sum_{\epsilon} B_0(\rho, \zeta; \rho_{0\epsilon}) F_{++}(\epsilon)$$

$$\pm i \sum_{\epsilon} B_0(\rho, \zeta; \rho_{0\epsilon}) F_{--}(\epsilon)$$

$$+ \sum_{\epsilon} B_1(\rho, \zeta; \rho_{0\epsilon}) F_{++}(\epsilon)$$

$$\pm i \sum_{\epsilon} B_1(\rho, \zeta; \rho_{0\epsilon}) F_{--}(\epsilon). \tag{3.7}$$

An interesting consequence of this intensity formula follows from the fact that $B_0$ and $B_1$ are real for $\zeta = 0$ and complex for $\zeta \neq 0$. For $\zeta = 0$, the circularly symmetric ring patterns are identical for left- and right-circularly polarized incident light, but for $\zeta \neq 0$ they are different. Ultimately, this is a consequence of the already-mentioned noncommutativity of the matrices in \ref{eq:222}. Figure 3 shows an example.

**4. Explicit formulae for small $N$**

Without loss of generality, we always choose $\gamma_1 = 0$.

For one crystal, the formulae of section 3 reproduce existing theory \cite{8–10}:

$$F_{++}(+) = 1, \quad F_{--}(+) = 0, \quad \Gamma_{11} = 0, \tag{4.1}$$

so

$$M_1(\rho, \zeta) = B_0(\rho, \zeta; \rho_{01}) \mathbf{I} + B_1(\rho, \zeta; \rho_{01}) \times [\cos \phi s_3 + \sin \phi s_2]. \tag{4.2}$$

For two crystals, \ref{eq:211}, \ref{eq:27} and \ref{eq:215} give

$$F_{++}(++) = \cos \frac{1}{2} \gamma_2, \quad F_{--}(++) = \sin \frac{1}{2} \gamma_2. \tag{4.3}$$

Together with \ref{eq:37} these formulae give the intensity

$$I_2(\rho, \zeta) = (|B_0(\rho, \zeta; \rho_{01} + \rho_{02})|^2$$

$$+ |B_1(\rho, \zeta; \rho_{01} + \rho_{02})|^2 \cos^2 \frac{1}{2} \gamma_2$$

$$+ (|B_0(\rho, \zeta; \rho_{01} - \rho_{02})|^2$$

$$- |B_1(\rho, \zeta; \rho_{01} - \rho_{02})|^2 \sin^2 \frac{1}{2} \gamma_2$$

$$\pm 2 \text{Im}[B_0(\rho, \zeta; \rho_{01} + \rho_{02}) B_1(\rho, \zeta; \rho_{01} - \rho_{02})]$$

$$+ B_1(\rho, \zeta; \rho_{01} + \rho_{02})^2 B_1(\rho, \zeta; \rho_{01} - \rho_{02})]$$

$$\times \cos^2 \frac{1}{2} \gamma_2 \sin^2 \frac{1}{2} \gamma_2. \tag{4.4}$$

Thus the larger ring, with radius $\rho_{01} + \rho_{02}$, and the smaller ring, with radius $|\rho_{01} - \rho_{02}|$, wax and wane alternately as one crystal is rotated relative to the other, with orientation dependences $\cos^2(\gamma_2/2)$ and $\sin^2(\gamma_2/2)$, exactly as has been observed by Abdolvand and Rafailov \cite{16}.

The matrices in \ref{eq:222} are

$$M_{02}(\rho, \zeta) = B_0(\rho, \zeta; \rho_{02} + \rho_{01}) \cos \frac{1}{2} \gamma_2$$

$$+ B_0(\rho, \zeta; \rho_{02} - \rho_{01}) \sin \frac{1}{2} \gamma_2$$

$$M_{22}(\rho, \zeta) = [B_0(\rho, \zeta; \rho_{02} + \rho_{01})$$

$$- B_0(\rho, \zeta; \rho_{02} - \rho_{01})] \cos \frac{1}{2} \gamma_2 \sin \frac{1}{2} \gamma_2$$

$$M_{12}(\rho, \zeta) = B_1(\rho, \zeta; \rho_{01} + \rho_{02}) \cos \left( \phi - \frac{1}{2} \gamma_2 \right)$$

$$- B_1(\rho, \zeta; \rho_{02} - \rho_{01}) \sin \frac{1}{2} \gamma_2 \cos \left( \phi - \frac{1}{2} \gamma_2 \right)$$

$$M_{32}(\rho, \zeta) = B_1(\rho, \zeta; \rho_{01} + \rho_{02}) \cos \left( \phi + \frac{1}{2} \gamma_2 \right)$$

$$+ B_1(\rho, \zeta; \rho_{02} - \rho_{01}) \sin \frac{1}{2} \gamma_2 \sin \left( \phi + \frac{1}{2} \gamma_2 \right). \tag{4.5}$$

If the two crystals are aligned, that is, $\gamma_2 = 0$, these formulae reduce to

$$M_{02}(\rho, \zeta) = B_0(\rho, \zeta; \rho_{02} + \rho_{01}), \quad M_{22}(\rho, \zeta) = 0$$

$$M_{12}(\rho, \zeta) = B_1(\rho, \zeta; \rho_{02} + \rho_{01}) \sin \phi$$

$$M_{32}(\rho, \zeta) = B_1(\rho, \zeta; \rho_{02} + \rho_{01}) \cos \phi, \tag{4.6}$$

confirming that, as expected, the two crystals act like a single crystal with strength $\rho_{01} + \rho_{02}$. If they are anti-aligned and of equal strength, that is, $\gamma_2 = \pi$ and $\rho_{01} = \rho_{02}$:

$$M_{02}(\rho, \zeta) = B_0(\rho, \zeta; 0),$$

$$M_{22}(\rho, \zeta) = M_{12}(\rho, \zeta) = M_{32}(\rho, \zeta) = 0, \tag{4.7}$$

$$M_{02}(\rho, \zeta) = B_0(\rho, \zeta; \rho_{01} + \rho_{02}),$$
indicating that for this case the second crystal reverses the effect of the first, reproducing the incident beam, as was described in section 3 (cf (3.4)), and as was previously predicted and observed [7].

For three crystals, (2.11), (2.7) and (2.15) give

\[
\begin{align*}
F_{++}(+++) &= \cos \Gamma_{32} \cos \frac{1}{2} \gamma_2, \\
F_{++}(+-+) &= -\sin \Gamma_{32} \sin \frac{1}{2} \gamma_2, \\
F_{+-}(++--) &= \cos \Gamma_{32} \sin \frac{1}{2} \gamma_2, \\
F_{+-}(+--) &= \sin \Gamma_{32} \cos \frac{1}{2} \gamma_2, \\
F_{-+}(---+) &= \cos \Gamma_{32} \cos \frac{1}{2} \gamma_2, \\
F_{-+}(---) &= -\sin \Gamma_{32} \sin \frac{1}{2} \gamma_2.
\end{align*}
\]

(4.8)

so

\[
M_{03}(\rho, \zeta) =
\begin{align*}
B_0(\rho, \zeta; \rho_03 + \rho_2 + \rho_01) \cos \Gamma_{32} \cos \frac{1}{2} \gamma_2 \cos \frac{1}{2} \gamma_3 \\
- B_0(\rho, \zeta; \rho_03 - \rho_2 + \rho_01) \sin \Gamma_{32} \sin \frac{1}{2} \gamma_2 \cos \frac{1}{2} \gamma_3 \\
+ B_0(\rho, \zeta; \rho_03 + \rho_2 - \rho_01) \cos \Gamma_{32} \sin \frac{1}{2} \gamma_2 \sin \frac{1}{2} \gamma_3 \\
+ B_0(\rho, \zeta; \rho_03 - \rho_2 - \rho_01) \sin \Gamma_{32} \cos \frac{1}{2} \gamma_2 \sin \frac{1}{2} \gamma_3
\end{align*}
\]

(4.9)

These expressions were used in the computations for figure 2.

For four crystals, we give only the amplitude factors for the contributing paths:

\[
\begin{align*}
F_{++}(++++) &= \cos \Gamma_{43} \cos \Gamma_{32} \cos \frac{1}{2} \gamma_4, \\
F_{++}(+-++) &= -\cos \Gamma_{43} \sin \Gamma_{32} \sin \frac{1}{2} \gamma_4, \\
F_{+-}(++--) &= -\sin \Gamma_{43} \sin \Gamma_{32} \cos \frac{1}{2} \gamma_4, \\
F_{+-}(+--) &= \sin \Gamma_{43} \cos \Gamma_{32} \sin \frac{1}{2} \gamma_4, \\
F_{-+}(---+) &= \cos \Gamma_{43} \cos \Gamma_{32} \cos \frac{1}{2} \gamma_4, \\
F_{-+}(---) &= -\sin \Gamma_{43} \sin \Gamma_{32} \sin \frac{1}{2} \gamma_4.
\end{align*}
\]

5. Random cascade and $N \gg 1$

It is clear that the ring patterns will get increasingly complicated, and increasingly sensitive to the crystal parameters, as $N$ increases, so it is interesting to calculate the statistics of the intensity $I_{\text{tot}}(\rho, \zeta)$, and this is what will be calculated here. First we regard the crystal strengths $\rho_{\text{tot}}$ as fixed, and treat the rotation angles as random and uncorrelated in the range $0 \leq \gamma_n < 2\pi$. In an experiment this average could be implemented with $N$ crystals that are spun independently about their optic axes.

According to (3.7), the intensity consists of double sums over paths $\varepsilon$ and $\varepsilon'$. Averaging over orientations eliminates the off-diagonal terms. This is most easily seen by transforming the integrations over the $\gamma_n$ to variables

\[
\gamma_1, u_n = (\gamma_n - \gamma_{n-1}), \quad (2 \leq n \leq N),
\]

and noting (cf (2.7)) that any pair of off-diagonal paths must contain at least one factor cos $\gamma_n$, $\sin \gamma_n = \frac{1}{2} \sin u_n$, and therefore vanishes on integration. For the diagonal terms, we use $|\cos \frac{1}{2} u_n|_av = [\sin^2 \frac{1}{2} u_n]_av = \frac{1}{2}$, so each of the products $F$ in (3.7) contains $N - 1$ factors $\frac{1}{2}$. Then, combining the two identical contributions from $F_{++}$ and $F_{--}$ gives

\[
I_{\text{av}}(\rho, \zeta) = \frac{1}{2N-1} \sum_\varepsilon (|B_0(\rho, \zeta; \rho_{\text{tot}}(\varepsilon))|^2 + |B_1(\rho, \zeta; \rho_{\text{tot}}(\varepsilon))|^2).
\]

(5.2)

This shows that the average intensity is the incoherent superposition of the $2^{N-1}$ individual ring patterns (recall that $\varepsilon$ and $-\varepsilon$ are counted as one path).

The $2^{N-1}$ rings are localized within an interval $|\rho - \rho_{\text{tot}}(\varepsilon)| \sim 1$, so for sufficiently large $N$ the rings will overlap and be individually indiscernible. The criterion for this is $2^N/N \gg 2 \rho_{\text{tot}}$, where $\rho_{\text{tot}}$ is the rms strength of the crystals in the cascade. Then it is justified to average (5.2) over the distribution of ring radii $\rho_{\text{tot}}(\varepsilon)$ (2.9). By the central limit theorem, this is Gaussian, the mean square radius being $2\rho_{\text{tot}}^2$. The same average applies for any $N$ if the strengths $\rho_{\text{tot}}$ are random and Gauss-distributed. Thus the average intensity becomes

\[
I_{\text{av}}(\rho, \zeta) = \frac{2}{\rho_{\text{tot}} \sqrt{2\pi N}} \int_0^\infty d\rho_{\text{tot}} \exp\left(-\frac{\rho_{\text{tot}}^2}{2N\rho_{\text{tot}}^2}\right) \times (|B_0(\rho, \zeta; \rho_{\text{tot}})|^2 + |B_1(\rho, \zeta; \rho_{\text{tot}})|^2).
\]

(5.3)

The integral over $\rho_{\text{tot}}$ can be performed exactly by using the integral representations (2.20), leading to

\[
I_{\text{av}}(\rho, \zeta) = \frac{1}{2} \int_0^\infty dk_1 \int_0^\infty dk_1 \kappa_1 \kappa_2 \\
\times \exp\left[-\frac{1}{2}(k_1^2(1+i\zeta) + k_2^2(1-i\zeta))\right] \\
\times [J_0(k_1\rho)J_0(k_2\rho)] \exp\left[-\frac{1}{2}N\rho_{\text{tot}}^2(k_1 - k_2)^2\right] \\
\times \exp\left[-\frac{1}{2}N\rho_{\text{tot}}^2(k_1 + k_2)^2\right] \\
\times J_1(k_1\rho)J_1(k_2\rho) \exp\left[-\frac{1}{2}N\rho_{\text{tot}}^2(k_1 + k_2)^2\right] \times \exp\left[-\frac{1}{2}N\rho_{\text{tot}}^2(k_1 - k_2)^2\right].
\]

(5.4)
Next, we transform to polar coordinates $\kappa$, $\phi/2$ in $\kappa_1$, $\kappa_2$ space and use the following consequences of formula (10.22.67) in [12]:

$$
\int_0^\infty \mathrm{d} \kappa^3 \exp\left(-\frac{\kappa^2}{2}\right) J_0(\kappa \rho \cos \frac{\phi}{2}) J_0(\kappa \rho \sin \frac{\phi}{2})
$$

$$
= -2 \frac{\partial}{\partial \alpha} \left[ I_0 \left( \frac{\rho^2 \sin \phi}{2 \alpha} \right) \right],
$$

(5.5)
in which now and in (5.6) to follow $I_{0,1}$ denotes modified Bessel functions.

Thus $I_{av}$ is reduced to an integral over $\phi$. Defining

$$
f_0(\rho, \alpha, \phi) = \frac{1}{2\alpha^3 \rho} \exp\left(-\frac{\rho^2}{2\alpha} \right)
$$

$$
\times \left[ I_0 \left( \frac{\rho^2 \sin \phi}{2\alpha} \right) (2\alpha - \rho^2) + I_1 \left( \frac{\rho^2 \sin \phi}{2\alpha} \right) \rho^2 \sin \phi \right],
$$

$$
f_1(\rho, \alpha, \phi) = \frac{1}{2\alpha^3 \rho} \exp\left(-\frac{\rho^2}{2\alpha} \right) \rho^2
$$

$$
\times \left[ I_0 \left( \frac{\rho^2 \sin \phi}{2\alpha} \right) \sin \phi - I_1 \left( \frac{\rho^2 \sin \phi}{2\alpha} \right) \right],
$$

(5.6)

$$
\alpha(\phi) \equiv 1 - i \xi \cos \phi + N\rho_{\text{lin}}^2 (1 \pm \sin \phi),
$$

we obtain

$$
I_{N,av}(\rho, \zeta) = \frac{1}{2} \text{Re} \int_0^\pi d\phi \sin \phi [f_0(\rho, \alpha(\phi), \phi)
$$

$$
+ f_1(\rho, \alpha(\phi), \phi) + f_0(\rho, \alpha_+(\phi), \phi) - f_1(\rho, \alpha_+(\phi), \phi)].
$$

(5.7)

The integral is easily evaluated numerically. As illustrated by figure 4, the average agrees well with simulations. A consistency check of the intensity formula is provided by normalization: the total power across the emergent beam is

$$
2\pi \int_0^\infty \mathrm{d}\rho \rho I_{N,av}(\rho, \zeta) = \pi,
$$

(5.8)

which is equal, as unitarity requires, to the power in the incident Gaussian beam which, in the variables we are using, has intensity $\exp(-\rho^2)$.

There are two simple limiting cases of the rather impenetrable formula (5.7). In the first, the individual rings are well delineated, that is, $N\rho_{\text{lin}} \gg 1$, and $\rho$ is not too small. Then in (5.3) the intensity $|B_0|^2 + |B_1|^2 \approx 2|B_0|^2$ (cf (3.2)) is localized near $\rho_{\text{tot}} = \rho$, so the exponential is slowly varying and we can replace $\rho_{\text{tot}}$ by $\rho$ therein. Thus

$$
I_{N,av}(\rho, \zeta) \approx \frac{4}{\rho_{\text{lin}} \sqrt{2\pi N}} \exp\left(-\frac{\rho^2}{2N\rho_{\text{lin}}^2} \right)
$$

$$
\times \int_0^\infty \mathrm{d}\rho_{\text{tot}} |B_0(\rho, \zeta; \rho_{\text{tot}})|^2.
$$

(5.9)

From (2.20), the large-$\rho$ asymptotics of the integral is

$$
\int_0^\infty \mathrm{d}\rho_{\text{tot}} |B_0(\rho, \zeta; \rho_{\text{tot}})|^2 = \frac{1}{2\pi} \int_0^\infty \mathrm{d} \kappa \kappa^3 J_0^2(\kappa \rho)
$$

$$
\times \exp(-\kappa^2) \approx \frac{1}{4\rho},
$$

(5.10)

leading to the intensity profile

$$
I_{N,av}(\rho, \zeta) \approx \frac{1}{\rho_{\text{lin}} \sqrt{2\pi N}} \exp\left(-\frac{\rho^2}{2N\rho_{\text{lin}}^2} \right),
$$

(5.11)

independent of $\zeta$. Like the exact profile (5.7), this satisfies the normalization (5.8).

The factor $1/\rho$ makes the distribution (5.11) non-Gaussian. The factor also generates a singularity at $\rho = 0$, but this is an artefact of the approximation in (5.9), which, unlike the exact average (5.7), breaks down for $\rho < 1$.

The second special case is the approximation of $I_{N,av}$ at $\rho = 0$ for $N \gg 1$. From (5.6), $f_0(0, \alpha, \phi) = 1/\alpha^2$ and $f_1(0, \alpha, \phi) = 0$, so (5.7) becomes

$$
I_{N,av}(0, \zeta) = \frac{1}{4} \text{Re} \int_0^\pi d\phi \sin \phi \left( \frac{1}{\alpha(\phi)} \right)^2 + \frac{1}{\alpha(\phi)^2}).
$$

(5.12)

The dominant contribution comes from the term in $\alpha_-$ and from the neighbourhood of $\phi = \pi/2$. Expanding the integrand about this point leads, after a short calculation, to

$$
I_{N,av}(0, \zeta) \approx \frac{1}{4} \text{Re} \int_0^\infty \frac{\mathrm{d}u}{(1 + i\zeta u + \frac{1}{2} \rho_{\text{lin}}^2 N u^2)^2}
$$

$$
= \frac{\pi \rho_{\text{lin}}^2 N}{2(2\rho_{\text{lin}}^2 N + \zeta^2)^{3/2}}.
$$

(5.13)

For $N \gg 1$, the width $\rho_{\text{lin}} \sqrt{N}$ of the distribution (5.11) is large, so the cutoff for $\rho < 1$, with limiting value (5.13),

![Figure 4. Average intensity for six-crystal cascades with orientations $\gamma_1 \cdots \gamma_6$ random on $0-2\pi$ and strengths $\rho_1 \cdots \rho_6$ random on $0-2$ (i.e., $\rho_{\text{lin}} = 2/\sqrt{3}$), for (a) $\zeta = 0$ and (b) $\zeta = 5$. Full curves: intensities calculated exactly from (3.7) and averaged over 40 realizations; dashed curves: theoretical mean value, calculated from (5.7).](image)
represents a small correction: (5.11) represents the exact average (5.7) accurately over most of its range.

6. Concluding remarks

The \( N \)-crystal cascade patterns explored here are rich and intricate, and raise a number of possibilities for experimental investigation. The most interesting case is when the strengths \( \rho_n \) of the individual crystals are all large, so the ring patterns for each, consisting of the familiar close pair of bright rings separated by the Poggendorff dark ring, would be well defined. Then for the \( N \)-crystal cascade the theory predicts that there will be a common focal image plane \( \zeta = 0 \) in (1.1)), in which in general the intensity pattern (3.5) will exhibit a coherent superposition of \( 2^{N-1} \) concentric individual ring patterns (2.19) and (2.22), in which the radii \( \rho_{tot} \) are given by (2.9) and the coefficients by (2.12). Figure 2 shows some examples for \( N = 3 \). If one of the radii \( \rho_{tot} \) is zero, the corresponding ‘ring’ is a reproduction of the original Gaussian beam, centred on the axis.

Away from the focal image plane, that is, for \( |\zeta| > 0 \), the patterns for the left- and right-circularly polarized incident light are different, as figure 3 illustrates.

Explicit formulae for \( 1 \leq N \leq 4 \) are presented in section 4. For \( N \gg 1 \) a statistical theory is natural, in which the individual orientations and/or strengths are treated as random variables. This theory, which also applies for small \( N \), is developed in section 5, and leads to the formula (5.7) (see figure 4), which for well-delineated rings (all \( \rho_{tot} \) large and \( \rho \) not small can be approximated by the simpler formula (5.11).

The statistical theory involves the Fourier transform (2.1) of the product (2.2) of \( N \) random unitary matrices. It is worth explaining why the Furstenberg theorem [13] for random-matrix products, commonly employed to describe Anderson localization [14] in wave propagation through sequences of random transparent elements [15], does not apply to the conical diffraction cascades considered here. Mathematically, the theorem applies to \( 2 \times 2 \) transfer matrices, unimodular but otherwise unrestricted, whose entries describe forward-and backward travelling scalar waves; the theorem does not hold for unitary matrices such as those in crystal cascades, whose elements describe the polarization components of waves travelling forwards only.

Physically, the reflected waves whose coherent destructive interference [15] is responsible for localization are negligible for conical diffraction crystal cascades, because the individual reflection intensities between crystals in the cascade are of the order \((\text{difference between pairs of principal refractive indices})/(\text{sum of pairs of principal refractive indices}))^2\), which is at most a few parts in a thousand for existing experiments (see table 1 of [1]). This approximation would fail only for an at present physically unrealistic regime of extremely large \( N \), different from that considered here.

Acknowledgments

I thank Professor Edik Rafailov and Dr Amin Abdolvand for introducing me to cascaded conical diffraction, Professors John Donegan and James Lunney for showing me their unpublished related work, two referees for their helpful comments and suggestions, and the Physics Department of the Technion, Israel for hospitality while this work was completed. My research is supported by the Leverhulme Trust.

References

[16] Abdolvand A and Rafailov E U 2010 personal communication
Erratum

Conical diffraction from an N-crystal cascade
M V Berry 2010 J. Optics 12 075704

The previously published equation (2.1) should be corrected to include a double integral sign, to read

\[ E_N(\rho, \zeta) = \frac{1}{2\pi} \int d\kappa \exp \left\{ i \left( \kappa \cdot \rho - \frac{1}{2} i \kappa^2 \zeta \right) \right\} U_{totN}(\kappa) \bar{E}_0(\kappa) \]

(2.1)