Typical weak and superweak values

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Abstract
Weak values, resulting from the action of an operator on a preselected state
when measured after postselection by a different state, can lie outside the
spectrum of eigenvalues of the operator: they can be 'superweak'. This
phenomenon can be quantified by averaging over an ensemble of the two
states, and calculating the probability distribution of the weak values. If there
are many eigenvalues, distributed within a finite range, this distribution takes a
simple universal generalized lorentzian form, and the 'superweak probablility',
of weak values outside the spectrum, can be as large as $1–1/\sqrt{2} = 0.293\ldots$.
By contrast, the familiar expectation values always lie within the spectral range,
and their distribution, although approximately gaussian for many eigenvalues,
is not universal.

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1. Introduction

In the familiar type of quantum measurement, the expectation value of an operator $\hat{A}$ in a
normalized preselected state $|\psi\rangle$ is

$$A_{\text{exp}} = \langle\psi|\hat{A}|\psi\rangle. \tag{1.1}$$

If the spectrum of $\hat{A}$ is bounded, with all eigenvalues A_n lying in the range $A_{\min} \leqslant A_n \leqslant A_{\max}$,
A_{exp} can never lie outside this range. The situation is different for the 'weak measurements'
introduced by Aharonov and his colleagues [1], in which the effect of $\hat{A}$ on $|\psi\rangle$ is detected,
by coupling to a system sensitive to $\hat{A}$, when a different state $|\phi\rangle$ is postselected. There, what
is detected is the 'weak value'

$$A_{\text{weak}} = \frac{\langle\phi|\hat{A}|\psi\rangle}{\langle\phi|\psi\rangle} \equiv A + iA'. \tag{1.2}$$

This is usually a complex number, with the following significance [2–4].

If the initial state of the detector is represented by a real wavepacket (e.g. a minimal
gaussian), the effect of the weak measurement is to shift the expectation value of $\hat{A}$ by a

multiple of Re A_{weak}. In a weak measurement, Re A_{weak} can lie outside the interval $A_{\min} \leqslant A_n \leqslant A_{\max}$: it can be 'superweak'. (The expectation of the dynamical variable canonically conjugate to $\hat{A}$ is shifted by an amount proportional to Im A_{weak}, with the constant of proportionality depending on the width of the initial packet.)

Here we ask: How are weak values typically distributed, if the states $|\psi\rangle$ and $|\phi\rangle$ are randomly distributed? Are superweak values common or rare? To answer these questions, we calculate the probability distribution $P(A)$ of Re A_{weak} over an ensemble of states $|\psi\rangle$ and $|\phi\rangle$, and hence the superweak probability

$$P_{\text{super}} = \int_{-\infty}^{A_{\min}} \mathrm{d}A\, P(A) + \int_{A_{\max}}^{\infty} \mathrm{d}A\, P(A) \tag{1.3}$$

of the weak value lying outside the spectrum of $\hat{A}$.

The calculation (section 2) concerns the spectrum of $\hat{A}$ containing N eigenvalues distributed in the interval $A_{\min} \leqslant A_n \leqslant A_{\max}$, when $N \gg 1$, that is for high-dimensional Hilbert spaces. The result is that the distribution $P(A)$, and therefore the superweak probability, is universal, in the sense of being independent of N for $N \gg 1$, of the statistics assumed for $|\psi\rangle$ and $|\phi\rangle$, and of whether the eigenvalues A_n are randomly distributed or regularly (rigidly) arranged. The distribution is different if the states $|\psi\rangle$ and $|\phi\rangle$ are assumed real in the $\hat{A}$ basis, but is still universal for this class of states (of course Im $A_{\text{weak}} = 0$ for real states).

In equation (1.2), the matrix element is divided by the overlap $\langle\phi|\psi\rangle$ between the pre- and post-selected states, leading to the interpretation of A_{weak} as a quantum conditional probability [2]. This denominator will play an important part in our calculations – indeed, the very existence of superweak values depends on it.

The type of statistical calculation we perform here was pioneered by Botero [5], who obtained a different result by calculating the distribution of weak values for a different situation: when the preselected state $|\psi\rangle$ is fixed and only the postselected state $|\phi\rangle$ is random. In another anticipation, Dennis [6] calculated the probability distribution of local wavenumbers $|\boldsymbol{k}(\boldsymbol{r})|$ (gradient of phase of the wavefunction) for random monochromatic waves (the preselected states $|\psi\rangle$) in the plane, satisfying the Helmholtz equation with wavenumber k_0. $\boldsymbol{k}(\boldsymbol{r})$ is the real part of the weak value of $\hat{A} = $ momentum$/\hbar$, and the postselected state is a position eigenstate, that is $|\phi\rangle = |\boldsymbol{r}\rangle$. Superweak values of $\boldsymbol{k}(\boldsymbol{r})$ correspond to superoscillations [7, 8], where the local wave oscillates faster than the wavenumber k_0 of all the fourier components of the wave. The resulting superoscillation probability is $1/3$. Later, this two-dimensional calculation was extended to waves in spaces of any dimension [9].

In section 3 we calculate the very different probability distribution of expectation values (1.1). This is not universal: it depends on N, and is different for randomly spaced and regularly arranged eigenvalues.

2. Universal weak value distributions

In terms of the spectrum of $\hat{A}$, defined by

$$\hat{A}|\alpha_n\rangle = A_n|\alpha_n\rangle, \quad 1 \leqslant n \leqslant N, \quad A_{\min} \leqslant A_n \leqslant A_{\max}, \tag{2.1}$$

we can expand the states $|\psi\rangle$ and $|\phi\rangle$:

$$|\psi\rangle = \sum_{n=1}^{N} \psi_n|\alpha_n\rangle, \qquad |\phi\rangle = \sum_{n=1}^{N} \phi_n|\alpha_n\rangle. \tag{2.2}$$

Thus the weak value (1.2) can be written

$$A = \operatorname{Re} \frac{\sum_{n=1}^{N} \phi_n^* \psi_n A_n}{\sum_{n=1}^{N} \phi_n^* \psi_n}. \tag{2.3}$$

We seek the distribution of values of A that are typical, in the sense that $|\psi\rangle$ and $|\phi\rangle$ are regarded as random states, and for $N \gg 1$. To implement this, we write

$$\phi_n^* \psi_n \equiv b_n = b_{n1} + \mathrm{i} b_{n2}, \tag{2.4}$$

and treat the numbers b_{n1} and b_{n2}, as well as the eigenvalues A_n, as independent random variables with zero mean. Thus the distribution of A, given by (2.3), is

$$P(A) = \left\langle \delta \left(A - \operatorname{Re} \sum_{n=1}^{N} b_n A_n \Big/ \sum_{n=1}^{N} b_n \right) \right\rangle$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}s \, \exp(-\mathrm{i} A s) \left\langle \exp \left(\mathrm{i} s \operatorname{Re} \sum_{n=1}^{N} b_n A_n \Big/ \sum_{n=1}^{N} b_n \right) \right\rangle, \tag{2.5}$$

in which $\langle \cdots \rangle$ denotes an average over the distribution of the random variables. The denominator in the exponential makes direct averaging awkward, so we treat

$$\sum_{n=1}^{N} b_n \equiv c = c_1 + \mathrm{i} c_2 \tag{2.6}$$

as a constraint, implemented by a double δ-function which can then be written as fourier integrals:

$$P(A) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}c_1 \int_{-\infty}^{\infty} \mathrm{d}c_2 \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2 \, \exp\left\{ -\mathrm{i} \left(A s + c_1 t_1 + c_2 t_2 \right) \right\}$$

$$\times \left\langle \exp \left\{ \mathrm{i} \sum_{n=1}^{N} \left[s A_n \operatorname{Re} \left(\frac{b_{n1} + \mathrm{i} b_{n2}}{c_1 + \mathrm{i} c_2} \right) + t_1 b_{n1} + t_2 b_{n2} \right] \right\} \right\rangle. \tag{2.7}$$

The central limit theorem implies that for $N \gg 1$ the exponent, denoted X, is a gaussian random variable with zero mean, because it is a sum over many random numbers, irrespective of the distribution of b_{n1} and b_{n2}. Thus we can use the relation

$$\langle \exp(\mathrm{i} X) \rangle = \exp\left(-\tfrac{1}{2} \langle X^2 \rangle \right). \tag{2.8}$$

With no essential loss of generality, we can shift the spectral range to $-A_{\max} \leqslant A_n \leqslant A_{\max}$, and we will make the simplification that the distribution of the A_n is symmetric about $A_n = 0$, with variance $\langle A_n^2 \rangle$, and define the common variance of b_{n1} and b_{n2}:

$$\langle b_{1n}^2 \rangle = \langle b_{2n}^2 \rangle \equiv B^2. \tag{2.9}$$

Thus the average in (2.7) becomes

$$\left\langle \exp \left\{ \mathrm{i} \sum_{n=1}^{N} \left[s A_n \operatorname{Re} \left(\frac{b_{n1} + \mathrm{i} b_{n2}}{c_1 + \mathrm{i} c_2} \right) + t_1 b_{n1} + t_2 b_{n2} \right] \right\} \right\rangle$$

$$= \exp \left\{ -\frac{1}{2} N \left[\frac{s^2 B^2 \langle A_n^2 \rangle}{(c_1^2 + c_2^2)} + B^2 \left(t_1^2 + t_2^2 \right) \right] \right\}. \tag{2.10}$$

In the integration (2.7), N and B can now be eliminated by the following scaling of the integration variables:

$$B\sqrt{N} \{t_1, t_2\} \to \{t_1, t_2\}, \qquad (B\sqrt{N})^{-1} \{c_1, c_2\} \to \{c_1, c_2\}, \tag{2.11}$$

leaving an expression free from parameters, i.e. universal. The integrations over s, t_1 and t_2 are gaussian, leaving

$$P(A) = \frac{1}{(2\pi)^{3/2}\sqrt{\langle A_n^2 \rangle}} \int_{-\infty}^{\infty} dc_1 \int_{-\infty}^{\infty} dc_2 \sqrt{(c_1^2 + c_2^2)} \exp\left\{-\frac{1}{2}(c_1^2 + c_2^2)\left(1 + \frac{A^2}{\langle A_n^2 \rangle}\right)\right\}. \quad (2.12)$$

This is also an elementary integral, giving our main result

$$P(A) = \frac{\langle A_n^2 \rangle}{2\left(\langle A_n^2 \rangle + A^2\right)^{3/2}}. \quad (2.13)$$

Note that this does not involve the spectrum's boundary value A_{max}. The cauchy (lorentz)-like, rather than gaussian, form (2.13) has been generated by the integrations over the denominator variables c_1 and c_2, reflecting the essential role of the overlap $\langle \phi | \psi \rangle$ in the weak value formula (1.3).

For the probability of superweak values, (1.3) gives

$$P_{super} = 2 \int_{A_{max}}^{\infty} dA \, P(A) = 1 - \frac{A_{max}}{\sqrt{\langle A_n^2 \rangle + A_{max}^2}} \quad (2.14)$$

Interesting special eigenvalue distributions are

(a) uniform:

$$\langle A^2 \rangle = \frac{1}{3} A_{max}^2 \Rightarrow P_{super} = 1 - \frac{\sqrt{3}}{2} = 0.133\,98\ldots$$

(b) random-matrix (semicircle):

$$\langle A^2 \rangle = \frac{1}{4} A_{max}^2 \Rightarrow P_{super} = 1 - \frac{2}{\sqrt{5}} = 0.105\,57\ldots \quad (2.15)$$

(c) concentrated at $\pm A_{max}$:

$$\langle A^2 \rangle = A_{max}^2 \Rightarrow P_{super} = 1 - \frac{1}{\sqrt{2}} = 0.292\,89\ldots$$

The largest superweak probability occurs for case (c), where $N/2$ eigenvalues are concentrated near $-A_{max}$ and $N/2$ near $+A_{max}$.

To test the formula for $P(A)$, we simulated (2.3) by sampling A_n from a uniform distribution (case (2.15a)), and b_{n1} and b_{n2} from gaussian distributions, over the interval $\{-1,1\}$, for different values of N. As figure 1 illustrates, the fit is excellent, even for $N = 5$. The fit is equally good if b_{n1} and b_{n2} are sampled from a uniform distribution; only the variance B^2 is changed, but this quantity has been scaled away.

The foregoing argument survives almost unchanged if the eigenvalues are regularly arranged on $\{-A_{max}, A_{max}\}$, rather than randomly distributed, that is (corresponding to case (2.15a)

$$A_n = \left(\frac{2(n-1)}{N-1} - 1\right) A_{max}, \qquad \{1 \leqslant n \leqslant N\}. \quad (2.16)$$

The only changes in $P(A)$ and P_{super} arise from the variance of the eigenvalues, which is now

$$\langle A_n^2 \rangle = \frac{(N+1)}{3(N-1)} A_{max}^2. \quad (2.17)$$
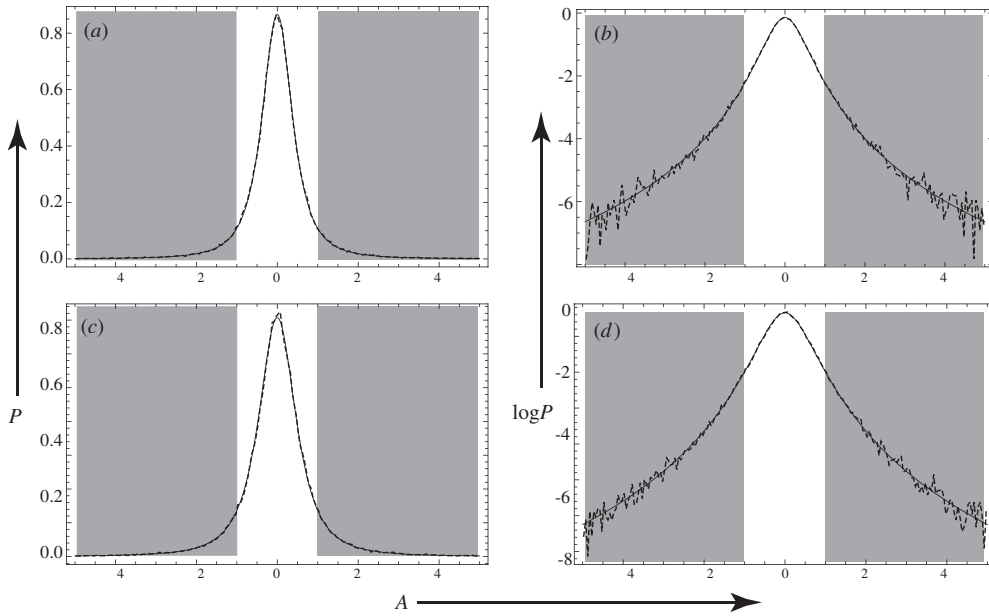
Figure 1. (*a*), (*c*) full curves: theoretical probability distribution $P(A)$ of weak values (equation (2.13)) for eigenvalues uniformly distributed (case 2.15*a*) on the range $\{-1,1\}$; dashed curves: $P(A)$ computed from (2.3) with 10^5 sample spectra with (a) 100 eigenvalues, (c) 5 eigenvalues. (b), (d) as (a), (c) for logP(A). In the shaded regions the weak values are superweak, lying outside the spectrum of $\hat{A}$.

An intriguing observation is that for $A' = \text{Im } A_{\text{weak}}$ the probability distribution $P(A')$ is identical to that for $A = \text{Re } A_{\text{weak}}$. This is surprising at first, but is obvious from the analysis leading to (2.13), and we have confirmed it by numerical simulation.

However, the distribution of weak values is different if the eigenstates $|\alpha_n\rangle$, and the states $|\psi\rangle$ and $|\phi\rangle$, can be represented as real, for example if there is time-reversal symmetry. Then b_2 and c_2 are zero and the calculation is simpler. We do not give the details but only the formulas that replace (2.13) and (2.14) for this different universality class:

$$P_{\text{real}}(A) = \frac{\sqrt{\langle A_n^2 \rangle}}{\pi \left(\langle A_n^2 \rangle + A^2 \right)}, \qquad P_{\text{super, real}} = 1 - \frac{2}{\pi} \tan^{-1} \frac{A_{\text{max}}}{\sqrt{\langle A_n^2 \rangle}}. \qquad (2.18)$$

As in the more general complex case (illustrated in figure 1), simulation gives good agreement. Of course $A' = \text{Im } A_{\text{weak}} = 0$ for this case.

3. Expectation value distributions

In the eigenbasis (2.1) of $\hat{A}$, the expectation value (1.1) is

$$A_{\text{exp}} = \frac{\sum_{n=1}^{N} |\psi_n^2| A_n}{\sum_{n=1}^{N} |\psi_n^2|}. \qquad (3.1)$$

This form of writing is convenient for numerical simulations, because it is not necessary to normalize the states $|\psi\rangle$. But for $N \gg 1$ the denominator is self-averaging, so it is not

necessary to include it explicitly provided the probability distribution of the coefficients $|\psi_n|^2$ is chosen to satisfy

$$\langle |\psi_n|^2 \rangle = \frac{1}{N}. \tag{3.2}$$

Thus the probability distribution of the expectation values is

$$P(A_{\mathrm{exp}}) = \left\langle \delta \left(A_{\mathrm{exp}} - \sum_{n=1}^{N} |\psi_n^2| A_n \right) \right\rangle$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}s \, \exp(-\mathrm{i}s A_{\mathrm{exp}}) \left\langle \exp \left(\mathrm{i}s \sum_{n=1}^{N} |\psi_n^2| A_n \right) \right\rangle. \tag{3.3}$$

By the central limit theorem, the sum in the exponential is a gaussian variable, so, using (2.8),

$$P(A_{\mathrm{exp}}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}s \, \exp(-\mathrm{i}s A_{\mathrm{exp}}) \exp \left\{ -\frac{1}{2} s^2 \left\langle \left(\sum_{n=1}^{N} |\psi_n^2| A_n \right)^2 \right\rangle \right\}$$

$$= \frac{1}{\sqrt{2\pi \left\langle \left(\sum_{n=1}^{N} |\psi_n^2| A_n \right)^2 \right\rangle}} \exp \left\{ -\frac{1}{2} A_{\mathrm{exp}}^2 \Big/ \left\langle \left(\sum_{n=1}^{N} |\psi_n^2| A_n \right)^2 \right\rangle \right\}. \tag{3.4}$$

The average is

$$\left\langle \left(\sum_{n=1}^{N} |\psi_n^2| A_n \right)^2 \right\rangle = N \langle A_n^2 \rangle \langle |\psi_n|^4 \rangle. \tag{3.5}$$

For the coefficients $|\psi_n|^2$, the simplest distribution follows from choosing gauss-distributed $\mathrm{Re}\,\psi_n$ and $\mathrm{Im}\,\psi_n$:

$$P(|\psi_n|^2) = \tfrac{1}{2} \exp \left(-\tfrac{1}{2} |\psi_n|^2 \right). \tag{3.6}$$

Thus in (3.5)

$$\langle |\psi_n|^4 \rangle = \frac{2}{N^2}, \tag{3.7}$$

and (3.4) gives

$$P(A_{\mathrm{exp}}) = \frac{1}{2} \sqrt{\frac{N}{\pi \langle A_n^2 \rangle}} \exp \left(-\frac{1}{4} N \frac{A_{\mathrm{exp}}^2}{\langle A_n^2 \rangle} \right). \tag{3.8}$$

Figures 2(b)–(d), again for the uniform distribution (2.15a), show that this is an excellent approximation. It does not vanish outside the interval $-A_{\max} \leqslant A_{\mathrm{exp}} \leqslant A_{\max}$ as the exact $P(A_{\mathrm{exp}})$ must, but for $N \gg 1$ the outside contributions are minuscule. Unlike the distribution (2.13) of weak values, the formula (3.8) is not universal: $P(A_{\mathrm{exp}})$ gets narrower as N increases, and the width depends on the distribution of the coefficients $|\psi_n|^2$.

Although the large N approximation (3.8) is good even for rather small N, as figures 2(b), (c) illustrate, it does fail for $N = 2$. Then $P(A_{\mathrm{exp}})$ can be evaluated exactly for the uniform distribution (2.15a), as shown in the appendix and illustrated in figure 2(a):
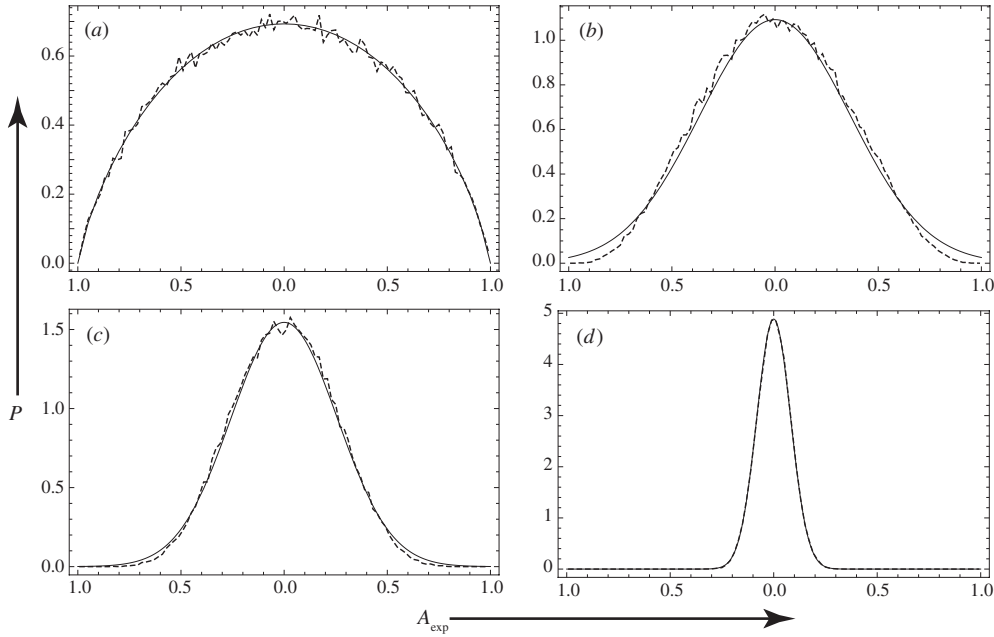
Figure 2. Probability distributions $P(A_{exp})$ of expectation values for eigenvalues uniformly distributed (case 2.15a) on the range $\{-1,1\}$, for (a) $N = 2$, (b) $N = 5$, (c) N = 10, (d) $N = 100$. Full curves: theoretical distributions, (3.8) for (b)–(d), and (3.9) for (a); dashed curves: $P(A_{exp})$ computed from (3.1) with 10^5 sample spectra.

$$P(A_{exp}) = \frac{1}{A_{max}} \left[\log 2 - \frac{1}{2} \left(\left(1 - \frac{A_{exp}}{A_{max}} \right) \log \left(1 - \frac{A_{exp}}{A_{max}} \right) \right. \right.$$
$$\left. \left. + \left(1 + \frac{A_{exp}}{A_{max}} \right) \log \left(1 + \frac{A_{exp}}{A_{max}} \right) \right) \right]. \tag{3.9}$$

In section 2 we found that if the eigenvalues are regularly arranged, as in (2.15), the distribution of weak values is slightly modified (cf (2.17). For $P(A_{exp})$ the effect is more significant. Instead of (3.5)–(3.7), the relevant average is

$$\left\langle \left(\sum_{n=1}^{N} |\psi_n^2| A_n \right)^2 \right\rangle = \sum_{m=1}^{N} \sum_{n=1}^{N} A_m A_n (\delta_{mn} \langle |\psi_n|^4 \rangle + (1 - \delta_{nm}) \langle |\psi_m|^2 \rangle \langle |\psi_n|^2 \rangle)$$
$$= \frac{2}{3N} \left(\frac{1 + 1/N}{1 - 1/N} \right) A_{max}^2 - \frac{1}{3N} \left(\frac{1 + 1/N}{1 - 1/N} \right) A_{max}^2$$
$$= \frac{1}{3N} \left(\frac{1 + 1/N}{1 - 1/N} \right) A_{max}^2, \tag{3.10}$$

so instead of (3.8) the distribution is

$$P(A_{exp}) = \frac{1}{A_{max}} \sqrt{\frac{3N(1 - 1/N)}{2\pi(1 + 1/N)}} \exp \left(-\frac{3}{2} N \left(\frac{1 - 1/N}{1 + 1/N} \right) \frac{A_{exp}^2}{A_{max}^2} \right). \tag{3.11}$$

This shows that even for $N \gg 1$ the width of the distribution is smaller by $\sqrt{2}$ for regularly arranged eigenvalues than for randomly distributed ones.

Finally, for $N = 2$, and eigenvalues $A_1 = -A_{\max}$, $A_2 = +A_{\max}$, the argument in appendix shows that the expectation values are uniformly distributed:

$$P(A_{\exp}) = \frac{1}{2A_{\max}}\Theta(A_{\max} - |A_{\exp}|), \tag{3.12}$$

in which Θ denotes the unit step. This particularly simple form is a consequence of the choice of the distribution (3.6) for $|\psi_n|^2$; for other distributions, $P(A_{\exp})$ is not uniform.

4. Concluding remarks

The main results (2.13) and (2.14) indicate an unanticipated universality in the distribution of weak and superweak values. Superweak values – results of weak measurements outside the spectral range of the operator being measured – are quite common: in the rather general situation considered here, there can be an almost 30% chance (for the case (2.15c)) of a weak value being superweak. We have considered two universality classes: the generic case, in which the overlap of the pre- and post-selected states with the eigenstates of the operator being measured are complex numbers, and the case of time-reversal symmetry, in which the overlaps are real. For the more familiar expectation values, the distribution is not universal; it depends on the number of states in the spectrum, and the probability distribution of the coefficients.

The weak value probability distribution (2.13) is similar to, but in general distinct from, those found recently for superoscillations in monochromatic waves in two [6] and D [9] dimensions. In these studies, a single eigenvalue was considered (the wavenumber k_0) which was degenerate (corresponding to the different directions of plane waves, i.e. momentum eigenstates). By contrast, we have considered discrete nondegenerate eigenvalues distributed over a finite range. The two cases coincide when $D = 1$ for superoscillations (section 3 of [9]) and our eigenvalue distribution is concentrated at the extremes (case (2.15c)).

Extensions of our analysis can be envisaged. For example, the spectrum of eigenvalues could be continuous, or the spectral range could be infinite (in which case there would be no superweak values); or the eigenstates could be degenerate. The ensemble of pre- and post-selected states that we have considered is equivalent to a density matrix, which in some applications could represent a thermal ensemble.

Acknowledgments

Appendix

This is the derivation of (3.9) and (3.12). With the notation

$$|\psi_1|^2 \equiv x, \qquad |\psi_2|^2 \equiv y, \tag{A.1}$$

the probability distribution of expectation values when there are just two eigenvalues, uniformly distributed between $-A_{\max}$ and $+A_{\max}$, is

$$\begin{aligned}
P(A_{\exp}) &= \int_{-\infty}^{\infty} \mathrm{d}x\, P(x) \int_{-\infty}^{\infty} \mathrm{d}y\, P(y) \left\langle \delta\left(A_{\exp} - \frac{A_1 x + A_2 y}{x + y}\right)\right\rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}x\, x\, P(x) \left\langle P\left(x \frac{(A_{\exp} - A_1)}{(A_2 - A_{\exp})}\right) \frac{(A_2 - A_1)}{(A_2 - A_{\exp})^2}\right\rangle,
\end{aligned} \tag{A.2}$$

in whch the average is

$$\langle \cdots \rangle = \frac{1}{2A_{\max}} \int_{-A_{\max}}^{A_{\max}} \mathrm{d}A_1 \int_{-A_{\max}}^{A_{\max}} \mathrm{d}A_2 \cdots \tag{A.3}$$

With the distribution (3.6) for $|\psi_n|^2$, the integral over x gives

$$P(A_{\exp}) = \left\langle \frac{1}{|A_2 - A_1|} \Theta \left(1 - \left| \frac{2A_{\exp} - A_1 - A_2}{A_1 - A_2} \right| \right) \right\rangle. \tag{A.4}$$

Evaluating the average (A.3) leads to (3.9). If the eigenvalues are fixed at $A_1 = -A_{\max}$, $A_2 = +A_{\max}$, there is no need to average, and (A.4) becomes (3.12).

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