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Lateral and transverse shifts in reflected dipole radiation

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In-plane (lateral) and out-of-plane (transverse) shifts in the direction of arbitrarily polarized electromagnetic waves in a denser medium, reflected totally or partially at an interface with a rarer medium, are calculated exactly, in terms of the deviation of the Poynting vector from radial. The shifts are analogous to the Goos–Hänchen and Fedorov–Imbert shifts for beams. There is a transverse shift even for unreflected dipole radiation if the polarization is not linear. With reflection, there is a transverse shift for linear polarization, provided this is not pure transverse electric or transverse magnetic. The contributions from the geometrical ray, the lateral ray that interferes strongly with it, and the large peak at the Brewster angle (for transverse magnetic polarization), are calculated asymptotically far from the geometrical image. At the critical angle, the lowest order asymptotics is inadequate and a more sophisticated treatment is devised, reproducing the exact shifts accurately.

Keywords: Goos–Hänchen; Fedorov–Imbert; polarization; diffraction; asymptotics

1. Introduction

Beams reflected from surfaces can be slightly shifted, laterally (Goos–Hänchen shift (Lotsch 1970)) or transversely (Fedorov–Imbert shift (Costa de Beauregard & Imbert 1972)). These shifts have been extensively studied theoretically and experimentally, over many decades (Lotsch 1970), in light, sound, elastic and quantum (Hirschfelder & Christoph 1974) waves. For beams, such as Gaussian beams or the recently studied Bessel beams (Schilling 1965; Bliokh et al. 2010; Aiello & Woerdman 2011), the shifts are naturally described as displacements of the centre of gravity or system of vortices (Dennis & Götte in preparation) of the reflected beam.

My aim here is to study a different situation: shifts not in beams but in reflected light diverging from localized sources such as point dipoles. In this case, the shifts are naturally described as deflections: deviations in the direction of the Poynting vector far from the reflecting surface, measured with respect to the radial direction from the geometrical image of the source. Multiplying the angular shift by the distance to the geometrical image, the shifts appear equivalently as apparent displacements of the image in the plane perpendicular to the radial direction.

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These deviations and displacements are vectors with two components: ‘lateral’, in the plane containing the observation point and the normal to the surface; and, orthogonal to this, ‘transverse’.

Any general theory must incorporate the polarization of the incident wave, an aspect that requires some care, as was emphasized recently (Bliokh & Bliokh 2006, 2007). The treatment here makes extensive use of techniques described in a detailed account (Brekhovskikh 1960) of reflected electric and magnetic dipole radiation in terms of integrals over an angular superposition of plane waves travelling in different directions. The quantity to be calculated is the direction (unit vector) of the Poynting vector representing the energy flow in a monochromatic electromagnetic wave, represented by complex electric and magnetic fields \( \mathbf{E} \) and \( \mathbf{B} \), namely

\[
\mathbf{s} = \frac{\mathbf{S}}{|\mathbf{S}|}, \quad \text{where} \quad \mathbf{S} = \text{Re} \mathbf{E}^* \times \mathbf{B}.
\]  

\( \mathbf{S} \), and hence \( \mathbf{s} \), is calculated as a function of position, for a general class of fields, in §2. The shifts are represented by the components of \( \mathbf{s} \) perpendicular to the radial direction.

In §3, \( \mathbf{s} \) is calculated for dipole radiation without reflection. There is no lateral shift, but even in this simple case there is a transverse shift unless the polarization is purely linear.

The main work, beginning in §4, concerns reflection from a planar interface between two transparent dielectrics (figure 1), with the source and reflected wave in the denser medium, whose refractive index is conveniently defined as 1. The transmitted wave is in the rarer medium, with (relative) refractive index \( n < 1 \). In numerical illustrations, we will take \( n = 2/3 \), representing glass. Important roles are played by the critical angle \( \theta_c(n) \), and, for in-plane polarization, the Brewster angle \( \theta_B(n) \), defined by

\[
\theta_c(n) = \arcsin n \quad \text{and} \quad \theta_B(n) = \arctan n.
\]
The derivation in §4 leads to exact expressions, easily evaluated numerically, for the components of $S$, and in particular the components $s_\theta$ and $s_\phi$ representing the lateral and transverse shifts in polar coordinates centred on the geometrical image. There is a lateral shift for any polarization, and also a transverse shift for any polarization, in particular linear unless this is TE (electric field purely out-of-plane) or TM (magnetic field out-of-plane, i.e. electric field in-plane).

Different contributions to the shifts, associated with distinct physical processes, emerge with increasing distance $r$ from the geometrical image. Mathematically, these contributions arise from different plane waves in the superposition, and can be disentangled by asymptotic approximations to the angular integrals, in which the large parameter is $kr$, with $k = 2\pi/\lambda$, where $\lambda$ is the wavelength and $k$ the wavenumber, both in the denser medium.

The simplest of these contributions, derived in §5, is the geometrical optics-reflected wave, associated with the saddle-point of the integrals; the corresponding angular deviations are of the order of $1/kr$, so the apparent displacements of the source are of the order of $\lambda$. The lateral shift, non-zero only for observation directions $\theta > \theta_c$, is given by the well-known Artmann formula (Lotsch 1970). There is also a transverse geometrical shift, which for $\theta > \theta_c$ requires only that the incident wave is not purely TE or TM, and for $\theta < \theta_c$ requires the polarization to be not purely linear in any direction.

The next contribution, significant only for TM polarization, is a large peak in the lateral shift at the Brewster angle, whose precise height and shape are obtained in §6. This is also associated with the geometrical-reflected wave but requires more sophisticated asymptotics even though the peak represents a large angular deviation: of order $(kr)^3$ (approx. $8.5^\circ$ for $n = 2/3$). The Brewster angle is a zero of the in-plane reflection coefficient; but, on a different Riemann sheet, $\theta_B$ also corresponds to a pole, representing a surface wave in certain cases where $n$ is complex, for example, when one medium is conducting. The surface wave does not contribute directly to the reflected wave—unsurprising because the reflection coefficient is zero—but was nevertheless the subject of controversy for many years (Kahan & Eckart 1949, 1950; Wait 1998). The reason for a strong lateral shift at $\theta_B$ is the denominator $|S|$ in the normalization of the unit vector in equation (1.1): away from $\theta_B$, this is dominated by the geometrical wave, which vanishes at $\theta_B$ itself.

The remaining contribution is the ‘lateral wave’ (Tamir & Oliner 1969; Lai et al. 1986), associated with the branch-point singularity (Brekhovskikh 1960) of the angular integral from the plane wave in the critical direction $\theta_c$. For observation directions away from $\theta_c$, the leading-order lateral wave, derived in §7, exists only for $\theta > \theta_c$. Although the contribution of the lateral wave to the field strengths is smaller than that of the geometrical wave, its contribution to the Poynting direction $s$ is of the same order, because Maxwell’s equations imply that $S$ involves derivatives of the fields. This causes powerful interference oscillations, as a function of observation direction, between the lateral and geometrical waves.

For observation at the critical direction itself (§8), the geometrical and lateral waves coincide. In the integrals, the saddle-point coincides with the branch-point, and in the leading-order approximation the shifts are of the order of $1/(kr)^{3/4}$—that is, larger than geometrical by a factor $(kr)^{1/4}$. However, this asymptotic
regime is approached extremely slowly, especially for the TM polarization that requires $kr \gg 1000$. Approximating the shifts accurately for more modest (but still large) values of $kr$ requires a more sophisticated asymptotics (§8).

A differently defined shift, implicit in the exact fields calculated here but not captured by the Poynting-based analysis of the fields, is associated with the modulation of the intensity of the reflected wave caused by the angle-dependence, below $\theta_c$ and in particular near $\theta_B$, of the reflection coefficients of the constituent plane waves. The analogous effect for Gaussian beams has been observed and is fully understood (Aiello et al. 2009; Merano et al. 2009).

2. General theory

We will study reflected fields $\mathbf{E}$, $\mathbf{B}$ at a point $\mathbf{R}$, using spherical polar coordinates centred on the geometrical image of the source (figure 1), with the $z$ direction chosen normal to the interface, and $\mathbf{r}$ representing position in planes with constant azimuth $\phi$:

$$\mathbf{R} = \{r, \theta, \phi\} = \{r, \phi\}, \quad \mathbf{r} = \{r, \theta\}.$$  \hfill (2.1)

The fields will be written as a general superposition of orthogonally linearly polarized fields, labelled $i$, where,

$$i = 1, 2 \quad 1 = \text{TE and } 2 = \text{TM},$$  \hfill (2.2)

with ‘transverse’ in TE and TM referring to the constant $\phi$ planes. Thus, with complex polarization coefficients $\alpha_1$, $\alpha_2$, the fields are

$$\mathbf{E}(\mathbf{R}) = \alpha_1 \mathbf{E}_1(\mathbf{R}) + \alpha_2 \mathbf{E}_2(\mathbf{R})$$

and

$$\mathbf{B}(\mathbf{R}) = \alpha_1 \mathbf{B}_1(\mathbf{R}) + \alpha_2 \mathbf{B}_2(\mathbf{R}) \quad (|\alpha_1|^2 + |\alpha_2|^2 = 1).$$  \hfill (2.3)

It is convenient to represent the fields as scalar waves $\psi_i$, generated from azimuth-independent electric and magnetic Hertz potentials $\Pi_i$ (Brekhovskikh 1960) associated with the sources, that is, for the TE and TM cases, and with unit direction and polarization vectors here and hereafter represented by $\mathbf{e}$,

$$\mathbf{E}_1(\mathbf{R}) = i\nabla \times \Pi_1(\mathbf{R}) = \psi_1(\mathbf{r}) \mathbf{e}_\phi,$$

and

$$\mathbf{B}_2(\mathbf{R}) = \frac{i}{c} \nabla \times \Pi_2(\mathbf{R}) = \frac{\psi_2(\mathbf{r})}{c} \mathbf{e}_\phi,$$  \hfill (2.4)

where

$$\Pi_i(\mathbf{R}) = f_i(\mathbf{r}) \mathbf{e}_z.$$  \hfill (2.5)

The scalar $f(\mathbf{r})$ satisfies the scalar Helmholtz equation with wavenumber $k$. Thus, the scalar fields are obtained from the potentials, as

$$\psi_i(\mathbf{r}) = i\mathbf{e}_\phi \left( -\sin \theta \partial_r f_i(\mathbf{r}) + \frac{\cos \theta}{r} \partial_\theta f_i(\mathbf{r}) \right).$$  \hfill (2.6)

In §4, we will calculate the reflected scalar fields $\psi_1$ and $\psi_2$, which are different because the Fresnel reflection coefficients differ for TE and TM polarizations.
A simple illustration of this formulation, to which we will return in the next section, is the unreflected field from a point dipole:

\[ f(r) = \frac{\exp(ikr)}{r} \Rightarrow \psi(r) = kr \sin \theta \frac{\exp(ikr)}{r} \left( 1 + \frac{i}{k} \right), \quad (2.7) \]

in which \( r \) and \( \theta \) are (temporarily) coordinates whose origin is the dipole source rather than its geometrical image.

Returning to the general case, the magnetic field \( B_1 \) corresponding to \( E_1 \), and the electric field \( E_2 \) corresponding to \( B_2 \), are given by Maxwell’s equations:

\[
\begin{align*}
B_1(R) &= -\frac{i}{ck} \nabla \times E_1 = -\frac{i}{ckr^2 \sin \theta} (e_r r \partial_\theta (\sin \theta \psi_1) - e_\theta r \sin \theta \partial_r (\psi_1)) \\
E_2(R) &= \frac{ic}{k} \nabla \times B_2 = \frac{i}{kr^2 \sin \theta} (e_r r \partial_\theta (\sin \theta \psi_2) - e_\theta r \sin \theta \partial_r (\psi_2))
\end{align*}
\] (2.8)

A direct calculation now gives the desired Poynting vector in terms of the TE and TM scalar fields:

\[
S(R) = \frac{1}{kc} \left[ |\alpha_1|^2 \text{Im} \psi_1^* \nabla \psi_1 + |\alpha_2|^2 \text{Im} \psi_2^* \nabla \psi_2 \right.
\]
\[
+ \frac{1}{kr^2 \sin^2 \theta} \text{Re} \left( \alpha_1^* \alpha_2 \nabla (r \sin \theta \psi_1^*) \times \nabla (r \sin \theta \psi_2) \right) \left. \right]. \quad (2.9)
\]

Explicitly, the separate components are

\[
S_r = S \cdot e_r = \frac{1}{kc} (|\alpha_1|^2 \text{ Im} \psi_1^* \partial_r \psi_1 + |\alpha_2|^2 \text{ Im} \psi_2^* \partial_r \psi_2)
\]
\[
S_\theta = S \cdot e_\theta = \frac{1}{kcr} (|\alpha_1|^2 \text{ Im} \psi_1^* \partial_\theta \psi_1 + |\alpha_2|^2 \text{ Im} \psi_2^* \partial_\theta \psi_2)
\]
\[
S_\phi = S \cdot e_\phi = \frac{1}{kcr^2 \sin \theta} \text{Re} \left( \alpha_1^* \alpha_2 (\partial_r (r \psi_1^*)) \partial_\theta (\sin \theta \psi_2) \right.
\]
\[
- \partial_\theta (\sin \theta \psi_1^*) \partial_r (r \psi_2)) \left. \right], \quad (2.10)
\]

from which the unit Poynting direction \( s = s_r e_r + s_\theta e_\theta + s_\phi e_\phi \) can be obtained by normalization. The shifts are the component \( s_\theta \), which is the lateral shift, making an angle \( \sin^{-1} s_\theta \) with the radial direction, and the component \( s_\phi \), which is the transverse shift, whose corresponding angle is \( \sin^{-1} s_\phi \). The dependence on polarization is clear: in the lateral shift, the TE and TM fields contribute separately, and the transverse shift vanishes if the polarization is pure linear TE (\( \alpha_2 = 0 \)) or pure linear TM (\( \alpha_1 = 0 \)).

It is tempting in equation (2.9) to associate the lateral shift (the terms in \( |\alpha_1|^2 \) and \( |\alpha_2|^2 \)) with the orbital current, and the transverse shift (the terms in \( \alpha_1^* \alpha_2 \)) with the spin current. But this is not compatible with a recently identified natural separation between orbital and spin current (Bekshaev & Soskin 2007; Berry 2009; Bliokh et al. 2010). One way to see this is to note that there can be a transverse shift for oblique linear polarizations, i.e. neither pure TE nor pure TM.
3. Dipole wave

Before considering the reflected wave, it is worth pointing out that there is a shift even in the simple case of the incident wave from a dipole source with general polarization. Then $\psi_1(\mathbf{r}) = \psi_2(\mathbf{r}) = \psi(\mathbf{r})$, with $\psi(\mathbf{r})$ given by equation (2.7). Equation (2.10) gives the Poynting vector as

$$S = \frac{k^2 c^2 r^2}{2} \left[ \sin^2 \theta e_r + \frac{4}{kr} \text{Im}(\alpha_1^* \alpha_2) \sin \theta \cos \theta e_\phi \right],$$  

(3.1)

and hence the energy flow direction

$$s = \frac{e_r + (4/kr) \text{Im}(\alpha_1^* \alpha_2) \cot \theta e_\phi}{\sqrt{1 + ((4/kr) \text{Im}(\alpha_1^* \alpha_2) \cot \theta)^2}} \approx e_r + \frac{4}{kr} \text{Im}(\alpha_1^* \alpha_2) \cot \theta e_\phi.$$  

(3.2)

A similar expression has been found before (Schilling 1965) for the transverse shift of an incident beam.

This shows that there is always a transverse shift for the dipole wave, unless the polarization is purely linear. Therefore, it makes sense to interpret the shift as an example (possibly the simplest) of the spin Hall effect of light (Bliokh & Bliokh 2006, 2007), that is, a manifestation of spin–orbit coupling. As mentioned earlier, this interpretation fails for the general transverse shifts of §2, and also for the transverse shifts in reflected waves to be discussed in §4, because these can exists even for linear polarization provided this is oblique.

For $kr \gg 1$, the energy flow is mostly radial, i.e. diverging from the source, but with a gentle swirl around the dipole axis (figure 2). The swirl gets stronger as $kr$.
gets smaller, and the $z$-axis is a singularity of the unit vector field $\mathbf{s}$ (though not of the Poynting vector $\mathbf{S}$). The sense of swirl is opposite in the two hemispheres, so there is no torque on the dipole source. The region of swirl corresponds to the interior of a paraboloid, a measure of whose radius is the distance from the axis for which $s_r \sin \theta = s_\phi$:

$$r_{\text{swirl}} = \sqrt{\frac{z \Im \alpha_1^* \alpha_2}{k}}. \quad (3.3)$$

It is worth noticing that the helicity (Moffatt & Ricca 1992) of the Poynting field (equation (3.1)), that is,

$$\mathbf{S} \cdot \nabla \times \mathbf{S} = -\frac{4k^3}{c^2 r^6} \sin^4 \theta \Im (\alpha_1^* \alpha_2), \quad (3.4)$$

is non-zero. This implies that $\mathbf{S}$ cannot be written in the form $\mathbf{S} = a(\mathbf{R}) \nabla_{\mathbf{R}} b(\mathbf{R})$, so there are no exact ‘wavefronts’, which are contour surfaces of a smooth function normal to $\mathbf{S}$.

### 4. Reflected wave

The standard way to obtain the reflected (also transmitted) fields is to represent the Hertz potential (equation (2.7)) as a superposition of plane waves travelling in directions $\theta'$, that is (after integrating over azimuth directions $\phi$ (Brekhovskikh 1960)),

$$\frac{\exp(ikr)}{r} = ik \int_{C_+} d\theta' \sin \theta' J_0(kr \sin \theta \sin \theta') \exp(ikr \cos \theta \cos \theta')$$

$$= \frac{ik}{2} \int_{C} d\theta' \sin \theta' H_0^{(1)}(kr \sin \theta \sin \theta') \exp(ikr \cos \theta \cos \theta'), \quad (4.1)$$

where $J$ and $H$ denote standard Bessel functions (DLMF 2010). The contours are shown in figure 3, with the complex segments corresponding to the unavoidable evanescent waves in the superposition. Each constituent plane wave is reflected with its Fresnel reflection coefficient $R_i(\theta')$ corresponding to direction $\theta'$ and polarization $i = 1$ or $i = 2$. In cartesian coordinates with $z = 0$ representing the interface, with the source at $z = h$ (figure 1), the plane waves reflect according to

$$\exp \left\{ i \left( k_x x + k_y y - (z - h) \right) \sqrt{k^2 - k_x^2 - k_y^2} \right\}$$

$$\Rightarrow R_i(\theta) \exp \left\{ i \left( k_x x + k_y y + (h + z) \right) \sqrt{k^2 - k_x^2 - k_y^2} \right\}, \quad (4.2)$$

where here and hereafter square roots are interpreted as

$$\sqrt{F} = \sqrt{|F|} \times \begin{cases} +1 & (F \geq 0) \\ +i & (F < 0) \end{cases}. \quad (4.3)$$
Figure 3. Integration contours and branch cuts in the $\theta'$ plane.

The Fresnel coefficients are (Born & Wolf 2005)

\[
R_1(\theta) = \frac{\cos \theta - \sqrt{n^2 - \sin^2 \theta}}{\cos \theta + \sqrt{n^2 - \sin^2 \theta}} \quad \text{and} \quad R_2(\theta) = \frac{n^2 \cos \theta - \sqrt{n^2 - \sin^2 \theta}}{n^2 \cos \theta + \sqrt{n^2 - \sin^2 \theta}},
\]

with branch-points at the critical angles $\pm \theta_c(n)$ (equation (1.2)) and, for $R_2$, a zero at the Brewster angle $\theta_B(n)$. In the regime $\theta > \theta_c$ of total reflection, $|R_i| = 1$ and it is convenient to write

\[
R_i(\theta) = \exp\{i\mu_i(\theta)\},
\]

\[
\mu_1(\theta) = -2 \arctan \left( \sqrt{\tan^2 \theta - n^2 \sec^2 \theta} \right)
\]

and

\[
\mu_2(\theta) = -2 \arctan \left( \frac{1}{n^2} \sqrt{\tan^2 \theta - n^2 \sec^2 \theta} \right).
\]

With the notation

\[
s = \sin \theta, \, c = \cos \theta, \, s' = \sin \theta' \quad \text{and} \quad c' = \cos \theta',
\]

(for this section only, and with $s$ not to be confused with the Poynting unit vector $s$), the Hertz potentials are

\[
f_i(r, \theta) = ik \int_{C_+} d\theta' s' R_i(\theta') J_0(krss') \exp(ikrc')
\]

\[
= \frac{ik}{2} \int_C d\theta' s' R_i(\theta') H_0^{(1)}(krss') \exp(ikrc').
\]
Bessel identities (DLMF 2010) and equation (2.6) now lead to the following expressions for the scalar waves representing the two fields and the derivatives involved in $S$:

$$
\psi_i = -k^2 K_i, \quad \partial_r r \psi_1 = -k^3 r (s I_{0,1} + i c I_{1,1}) \quad \text{and} \quad \partial_\theta s \psi_i = -k^3 r s (c I_{0,1} - i s I_{1,1}),
$$

where the integrals are

$$
K_i = \int_{C_+} d\theta' s'^2 R_i (\theta') J_1 (k r s'') \exp (i k r c')
$$

$$
I_{0,i} = \int_{C_+} d\theta' s'^3 R_i (\theta') J_0 (k r s'') \exp (i k r c')
$$

and

$$
I_{1,i} = \int_{C_+} d\theta' s'^2 c' R_i (\theta') J_1 (k r s'') \exp (i k r c').
$$

Because of the branch-points, there are cuts in the plane of the integration variable $\theta'$; the cuts shown in figure 3 are consistent with the specification (equation (4.3)) of the square roots. Thus, we obtain the exact expressions for the components (equation (2.10)) of $S$:

$$
S_r = \frac{k^4}{\mu_0 c} (|\alpha|^2 \text{Im} K_1^* (s I_{0,1} + i c I_{1,1}) + |\beta|^2 \text{Im} K_2^* (s I_{0,2} + i c I_{1,2}))
$$

$$
S_\theta = \frac{k^4}{\mu_0 c} (|\alpha|^2 \text{Im} K_1^* (c I_{0,1} - i s I_{1,1}) + |\beta|^2 \text{Im} K_2^* (c I_{0,2} - i s I_{1,2}))
$$

and

$$
S_\phi = \frac{k^4}{\mu_0 c} \text{Im} \alpha^* \beta (I_{0,2}^* I_{1,2} + I_{1,1}^* I_{0,2}).
$$

As already noticed (Brekhovskikh 1960), these exact formulae do not involve the distance $h$ between the source and the interface. This implies that the reflected fields and shifts depend only on the location of the field point relative to the geometrical image, not where the source and interface are. Of course, if $z = r \cos \theta - h$ is negative the reflected field is virtual rather than real.

The integrals $I$ and $K$ are oscillatory, but not difficult to evaluate numerically, even up to $kr \sim 1000$. Figure 4 shows the lateral and transverse shifts for four representative polarizations that will be used in all subsequent figures, corresponding to TE polarization ($\alpha_1 = 1$, $\alpha_2 = 0$), TM polarization ($\alpha_1 = 0$, $\alpha_2 = 1$), polarization at $45^\circ$ to the plane of incidence ($\alpha_1 = 1/\sqrt{2}$, $\alpha_2 = 1/\sqrt{2}$) and circular polarization ($\alpha_1 = 1/\sqrt{2}$, $\alpha_2 = i/\sqrt{2}$). The main features, to be explained in subsequent sections, are: strong oscillations ($\S 7$) about a non-zero mean ($\S 5$) for $\theta > \theta_c$, increasing in amplitude towards finite values at $\theta = \theta_c$ ($\S 8$); very small values of the lateral shifts for $\theta < \theta_c$ ($\S 5$), except for a large peak at $\theta = \theta_B$ for TM polarization ($\S 6$). (The increasing strengths of the transverse shifts as $\theta$ approaches zero for polarizations that are neither
Figure 4. Lateral direction shift $s_\phi$ (blue) and transverse shift $s_\theta$ (red) for $n = 2/3$ and $kr = 100$, as functions of observation angle, computed exactly from (4.10), for (a) TE polarization ($\alpha_1 = 1, \alpha_2 = 0$); (b) TM polarization ($\alpha_1 = 0, \alpha_2 = 1$), (c) $45^\circ$ linear polarization ($\alpha_1 = 1/\sqrt{2}, \alpha_2 = 1/\sqrt{2}$), and (d) circular polarization ($\alpha_1 = 1/\sqrt{2}, \alpha_2 = i/\sqrt{2}$). (Online version in colour.)

TE nor TM, and which occur even for the unreflected dipole field (cf. §3), are associated with very small values of the fields and therefore of the unnormalized energy flow $S$.

5. Geometrical optics

For $kr \gg 1$, the Hankel function in equation (4.7) can be replaced by its large-argument asymptotic approximation, leading to the wave function

$$\psi_i \approx \frac{k^{3/2} \exp((1/4)i\pi)}{\sqrt{2\pi r \sin \theta}} \int_C d\theta' s^{3/2} R_i(\theta') \exp\{ikr \cos(\theta' - \theta)\}. \quad (5.1)$$

The main contribution to this integral comes from the saddle-point at $\theta' = \theta$, and straightforward application of the method of stationary phase (Wong 1989) gives

$$\psi_{i,sp} = k \sin \theta R_i(\theta) \frac{\exp(ikr)}{r}. \quad (5.2)$$

This is the crudest approximation, giving the reflected field as the incident field modulated by the reflection coefficient.
For the Poynting vector, equation (2.10) now gives

\[
Sr,sp = \left( \frac{k^2 \sin^2 \theta}{c r^2} \right) (|\alpha_1|^2 |R_1|^2 + |\alpha_2|^2 |R_2|^2)
\]

\[
S\theta,sp = \left( \frac{k^2 \sin^2 \theta}{c r^2} \right) \frac{1}{kr} (|\alpha_1|^2 \partial_\theta \mu_1 + |\alpha_2|^2 \partial_\theta \mu_2) \Theta(\theta - \theta_c)
\]

\[
S\phi,sp = \left( \frac{k^2 \sin^2 \theta}{c r^2} \right) \frac{|R_1 R_2|}{kr} \left( 4 \cot \theta \Im(\alpha_1^* \alpha_2 \exp[i(\mu_2 - \mu_1)]) \right)
\]

\[
- \Theta(\theta - \theta_c) \Re(\alpha_1^* \alpha_2 \exp[i(\mu_2 - \mu_1)]) (\partial_\theta \mu_2 - \partial_\theta \mu_1)
\]

in which (here and hereafter) \(\Theta\) denotes the unit step. Normalizing, we obtain the shifts:

\[
sr,sp \approx 1, \quad s\theta,sp = \frac{1}{kr} (|\alpha_1|^2 \partial_\theta \mu_1 + |\alpha_2|^2 \partial_\theta \mu_2) \Theta(\theta - \theta_c)
\]

and

\[
s\phi,sp = \frac{1}{kr} \Theta(\theta_c - \theta) \frac{4 \cot \theta |R_1 R_2 \Im(\alpha_1^* \alpha_2)}{(|\alpha_1|^2 |R_1|^2 + |\alpha_2|^2 |R_2|^2)}
\]

\[
+ \frac{1}{kr} \Theta(\theta - \theta_c) [4 \cot \theta \Im(\alpha_1^* \alpha_2 \exp[i(\mu_2 - \mu_1)])]
\]

\[
- \Re(\alpha_1^* \alpha_2 \exp[i(\mu_2 - \mu_1)]) (\partial_\theta \mu_2 - \partial_\theta \mu_1)
\]

At this level of approximation, similar shifts have been found before for bounded (e.g. Gaussian) beams. The expression for \(s\theta,sp\), giving the lateral shift in terms of the derivatives of the phases of the reflection coefficients, is familiar as the Artmann formula (Schilling 1965; Lotsch 1970) for the Goos–Hänchen shift, and the transverse shift \(s\phi,sp\) has also been calculated by several authors (Schilling 1965; Bliokh & Bliokh 2006, 2007).

Figure 5 shows that these geometrical shifts reproduce the mean of the oscillations in the exactly computed shifts for \(\theta > \theta_c\), and also the transverse shift for circular polarization. But they fail to give the oscillations, the smooth behaviour at \(\theta = \theta_c\) or the peak at \(\theta = \theta_B\) for TM polarization.

### 6. Brewster peak

In passing from \(S\) in equation (5.3) to the normalized \(s\) in equation (5.4), one point was glossed over. If \(\alpha_1 = 0\), that is for TM polarization, the zero in \(R_2\) at \(\theta = \theta_B\) implies a zero in the geometrical approximation to \(S_r\), so the geometrical approximation to \(S_\theta\) is undefined \((0/0)\). To resolve the behaviour near the Brewster angle, it is necessary to examine \(R_2\) more closely near \(\theta_B\).
The required local approximation is

\[ R_2(\theta) = A(\theta - \theta_B) + B(\theta - \theta_B)^2 + \ldots, \quad \text{where} \quad A = \frac{1 - n^4}{2n^6} \]

and

\[ B = \frac{2 + n^2 - n^4 - n^6 - n^8}{4n^6}. \]  

When inserted into equation (5.1), this gives, when expanded for \( \theta \) near \( \theta_B \),

\[ \psi_{2,B} = \frac{k^{3/2} \exp((1/4)i\pi)}{\sqrt{2\pi r \sin \theta}} \int_{-\infty}^{\infty} d\alpha \sin^{3/2}(\theta + \alpha) R_2(\theta + \alpha) \exp[ikr \cos \alpha] \]

\[ \approx k^{3/2} \exp \left( \frac{1}{4}i\pi \right) \exp(ikr) \]

\[ \times \sqrt{\frac{\sin \theta}{2\pi r}} \int_{-\infty}^{\infty} d\alpha [A(\theta - \theta_B) + A\alpha + C\alpha^2] \exp \left\{ -\frac{1}{2}ikr\alpha^2 \right\}, \]

where

\[ C = B + \frac{3}{2} A \cot \theta_B = \frac{2 + 4n^2 - n^4 - 4n^6 - n^8}{4n^6}. \]
The integrals are elementary, leading to
\[
\psi_{2B} = k \sin \theta \frac{\exp(ikr)}{r} \left[ A(\theta - \theta_B) - \frac{iC}{kr} \right].
\] (6.3)

The correction proportional to \( C \) resolves the normalization indeterminacy, leading to the shifts
\[
s_{\theta,B}(\theta) = \frac{A}{\sqrt{A^2 + C^2 + 2A^2(\theta_B - \theta)^2 + (A^2/C^2)(\theta_B - \theta)^4}},
\] (6.4)
and
\[
s_{\phi,B}(\theta) = 0.
\]

This shows that the Brewster peak in the lateral shift resembles the square root of a Lorentzian function. As figure 5b shows, this is an accurate approximation to the exactly computed Brewster peak.

The height of the peak at \( \theta = \theta_B \) is
\[
s_{\theta,B}(\theta_B) = \frac{A}{\sqrt{A^2 + C^2}} = \frac{2n^3}{\sqrt{4 + 16n^2 + 20n^4 + 12n^6 + n^8}}.
\] (6.5)

This is independent of \( kr \), corresponding to a deflection angle that remains constant far from the surface, or, equivalently, to a lateral apparent image displacement that increases as \( kr \). For \( n = 2/3 \), \( s_{\theta,B}(\theta_B) = 24/\sqrt{26497} = 0.14739 \), corresponding to a deflection of about 8.5°. The \( 1/\sqrt{2} \) width of the peak, defined by angles \( \theta_w \) at which
\[
s_{\theta}(\theta_w) = \frac{1}{\sqrt{2}} s_{\theta}(\theta_B),
\] (6.6)
is
\[
\theta_w - \theta_B = \pm \frac{1}{krC} \sqrt{C(\sqrt{2C^2 + A^2} - C)}.
\] (6.7)

For \( n = 2/3 \), \( \theta_w - \theta_B = \pm 4.35809/kr \). The \( 1/kr \) dependence means that the spatial range over which the large lateral deviation occurs is proportional to the wavelength, independent of \( r \).

There is also a Brewster peak in the otherwise negligible lateral shift for incidence from the less-dense medium, where of course there is no critical angle; this peak is also described by equation (6.4), with \( n > 1 \). Its height is given by equation (6.5); for \( n = 3/2 \), representing glass, \( s_{\theta,B}(\theta_B) = 108/\sqrt{77713} = 0.38742 \), corresponding to a deflection of about 22°.

7. Lateral wave

For \( kr \gg 1 \), there is a contribution from the branch-point in the integral (equation (5.1)) as well as the saddle. In this section, we consider observation angles \( \theta \) that are not close to \( \theta_c \), so the branch-point is well separated from the saddle-point.
at $\theta' = \theta$. For the branch-point contribution, we need the local approximation to the reflection coefficient:

$$R_i(\theta_c + \alpha) \approx 1 - \frac{2\sqrt{-2 R_i}}{(1 - n^2)^{1/4}} \gamma_1 \gamma_2 = \frac{1}{n^2}. \tag{7.1}$$

Substituting into equation (5.1), with $\theta' = \theta_c + \alpha$ gives the contribution from the branch-point as

$$\psi_{i, \text{lat}}(\theta, \alpha) = -\frac{2k^{3/2}n^2 \exp((1/4)i\pi)}{\sqrt{\pi r} \sin(\theta(1 - n^2)^{1/4})} \gamma_i \exp \{ikr \cos(\theta - \theta_c)\} \times \int_{C_{\text{lat}}} d\alpha \sqrt{\alpha} \exp(i\pi) \exp \{ikr \alpha \sin(\theta - \theta_c)\}, \tag{7.2}$$

with the contour shown in figure 6. This integral represents the lateral wave: as is well known (Brekhovskikh 1960), the phase $kr \cos(\theta - \theta_c)$ can be interpreted in terms of a ray issuing from the source in the critical direction, travelling in the rarer medium just below the surface (figure 1), and then leaving, again in the critical direction, to reach the field point $(r, \theta)$. (In the three-leg calculation, the height $h$, representing the location of the surface if the source and image are fixed, cancels, as it does in the general formulae of §4, leaving only the distance $r$ from the geometrical image to the field point.)

The contribution (7.2) is the lateral wave. For $\theta < \theta_c$, deformation of the contour towards $\alpha = -i\infty$ shows that the integral vanishes. Therefore, the lateral wave contributes only for $\theta > \theta_c$, when use of

$$\int_{C_{\text{lat}}} d\alpha \sqrt{\alpha} \exp(i\pi) \exp(iu) = -\sqrt{\pi} \exp\left(\frac{1}{4}i\pi\right), \tag{7.3}$$

gives

$$\psi_{i, \text{lat}} = \frac{2in^2 \exp(-(1/4)i\pi) \exp \{ikr \cos(\theta - \theta_c)\} \gamma_i \Theta(\theta - \theta_c)}{\sqrt{\sin(\theta(1 - n^2)^{1/4}) r^2 \sin(\theta - \theta_c)^{3/2}}} \tag{7.4}$$
This is smaller than the geometrical wave (5.2), from the saddle-point, by a factor $1/kr$. However, the oscillatory factor $\exp\{ikr\cos(\theta - \theta_c)\}$, and the derivatives involved in the calculation of $S$, combine to amplify the lateral wave, leading to shifts of the same order as the geometrical:

$$s_{q,\text{lat}} = \frac{2\Theta(\theta - \theta_c)}{kr\sqrt{\sin^5 \theta \sin(\theta - \theta_c)(1 - n^2)^{1/4}}} \times \text{Im} \left[ \exp \left\{ -2ikr \sin^2 \frac{1}{2}(\theta - \theta_c) \right\} \right. \times \left( |\alpha_1|^2 n^2 \exp(-i\mu_1) + |\alpha_2|^2 \exp(-i\mu_2) \right) \right]$$

and

$$s_{\phi,\text{lat}} = \frac{2\sqrt{\sin \Theta(\theta - \theta_c)}}{kr\sqrt{\sin^3 \theta \sin(\theta - \theta_c)(1 - n^2)^{1/4}}} \times \text{Im} \left[ \alpha_1^* \alpha_2 \left( n^2 \exp \left\{ 2ikr \sin^2 \frac{1}{2}(\theta - \theta_c) + i\mu_1 \right\} \right) \right. + \left. \exp \left\{ -2ikr \sin^2 \frac{1}{2}(\theta - \theta_c) - i\mu_2 \right\} \right]$$

As figure 5 illustrates, these expressions fit the exactly computed oscillations rather accurately. The fit appears less good for the $45^\circ$ linear polarization and $s_{\phi}$ (figure 5c), but it improves for larger values of $kr$.

8. Critical direction

At the critical angle, the lateral-wave formulae (7.4) and (7.5) diverge. This is because when $\theta = \theta_c$, the saddle-point and the branch-point coincide, and the argument of §7, which depended on them being well separated, is not valid. The version of equation (5.1) appropriate for this case is not equation (7.2) but

$$\psi_{i,\text{crit}} \approx \frac{k^{3/2} \exp\{i(kr + (1/4)\pi)\}}{\sqrt{2\pi r n}} \times \int_{C_{\text{crit}}} d\alpha \sin^{3/2}(\theta_c + \alpha) R_i(\theta_c + \alpha) \exp \left\{ -\frac{1}{2}ikr\alpha^2 \right\},$$

where the contour is shown in figure 6.

For a reason to be explained later, the approximation (7.1) for the reflection coefficients gives critical-angle shifts that are accurate only when $kr$ is extremely large. For more moderate values, the following formula, capturing the fact that $|R_i| = 1$ if $\theta > \theta_c$ (total reflection) is better:

$$R_i(\theta_c + \alpha) \approx \left( 1 - \frac{\sqrt{-2n\alpha \gamma_i}}{(1 - n^2)^{1/4}} \right) \left( 1 + \frac{\sqrt{-2n\alpha \gamma_i}}{(1 - n^2)^{1/4}} \right)^{-1}.$$
Then, equation (8.1) and the required derivatives can be written as

\[
\psi_i(\theta_c) \approx \frac{kn}{r \sqrt{2\pi r}} \exp \left\{ i \left( kr + \frac{1}{4} \pi \right) \right\} F_0(a_i(kr)), \quad \frac{1}{r} \partial_r (r \psi_i(\theta_c)) \approx ik \psi_i(\theta_c)
\]

and

\[
\partial_\theta (\sin \theta \psi_i(\theta_c))_{\theta=\theta_c} \approx -\frac{k^2 n^2}{\sqrt{2\pi}} \exp \left\{ i \left( kr - \frac{1}{4} \pi \right) \right\} \times \left[ \frac{-i\sqrt{1-n^2}}{2nkr} F_0(a_i(kr)) + \frac{1}{\sqrt{kr}} F_1(a_i(kr)) + \frac{3\sqrt{1-n^2}}{2nkr} F_2(a_i(kr)) \right]
\]

in which

\[
a_i(kr) = \sqrt{2n} \gamma_1 (kr)^{1/4}(1-n^2)^{1/4}
\]

and

\[
F_m(a) = \int_{C_{\text{crit}}} d\alpha \alpha^m \left( \frac{1-a\sqrt{\alpha \exp(i\pi)}}{1+a\sqrt{\exp(i\pi)}} \right) \exp \left( -\frac{1}{2} i\alpha^2 \right)
\]

Remarkably, the integrals can be evaluated analytically:

\[
F_0(a) = -\sqrt{2\pi} \exp \left( -\frac{1}{4} i\pi \right)
\]

\[
- \frac{i\sqrt{2}}{a^2} \exp \left( -\frac{i}{2a^4} \right) \left[ \pi \sqrt{2} \left( 1 + \text{erf} \left( \frac{\exp((3/4)i\pi)}{a^2\sqrt{2}} \right) \right) \right.
\]

\[
+ \Gamma \left( -\frac{1}{4}, -\frac{i}{2a^4} \right) \Gamma \left( \frac{5}{4} \right) - \Gamma \left( \frac{1}{4}, -\frac{i}{2a^4} \right) \Gamma \left( \frac{3}{4} \right) \right]
\]

\[
F_1(a) = \frac{2^{3/2}}{a^2} \sqrt{\pi} \exp \left( -\frac{1}{4} i\pi \right)
\]

\[
+ \frac{i\sqrt{2}}{a^4} \exp \left( -\frac{i}{2a^4} \right) \left[ \pi \sqrt{2} \left( 1 + \text{erf} \left( \frac{\exp((3/4)i\pi)}{a^2\sqrt{2}} \right) \right) \right.
\]

\[
+ \frac{1}{4} \left( 3\Gamma \left( -\frac{3}{4}, -\frac{i}{2a^4} \right) \Gamma \left( \frac{3}{4} \right) + \Gamma \left( \frac{1}{4}, -\frac{i}{2a^4} \right) \Gamma \left( \frac{1}{4} \right) \right) \right]
\]

and

\[
F_2(a) = -\frac{2^{3/2}}{a^4} \sqrt{\pi} \exp \left( -\frac{1}{4} i\pi \right) - \frac{2^{5/4}}{a^3} \Gamma \left( \frac{3}{4} \right) \exp \left( \frac{1}{8} i\pi \right)
\]

\[
- \frac{2^{7/4}}{a} \Gamma \left( \frac{5}{4} \right) \exp \left( -\frac{1}{8} i\pi \right) + \sqrt{2\pi} \exp \left( \frac{1}{4} i\pi \right)
\]

\[
- \frac{i\sqrt{2}}{a^6} \exp \left( -\frac{i}{2a^4} \right) \left[ \pi \sqrt{2} \left( 1 + \text{erf} \left( \frac{\exp((3/4)i\pi)}{a^2\sqrt{2}} \right) \right) \right.
\]

\[
+ \Gamma \left( -\frac{1}{4}, -\frac{i}{2a^4} \right) \Gamma \left( \frac{5}{4} \right) - \Gamma \left( \frac{1}{4}, -\frac{i}{2a^4} \right) \Gamma \left( \frac{3}{4} \right) \right]
\]
Shifts in reflected dipole radiation

Using equation (2.10), the Poynting vectors and lateral and transverse shifts can easily be calculated and, as figure 7 illustrates, they agree very well with the shifts computed exactly (equation (4.10)).

From equation (8.4), the limit \( kr \rightarrow \infty \) corresponds to \( a \rightarrow 0 \). Therefore, it is tempting to use not the exact expressions (8.5) for the integrals \( F_m(a) \), but their small-\( a \) two-term approximations

\[
F_0(a) = \sqrt{2\pi} \exp \left( -\frac{1}{4} i\pi \right) - 2^{5/4} \Gamma \left( \frac{3}{4} \right) \exp \left( -\frac{1}{8} i\pi \right) a + \mathcal{O}(a^2)
\]

\[
F_1(a) = -2^{-1/4} \Gamma \left( \frac{1}{4} \right) \exp \left( \frac{1}{8} i\pi \right) a - 2\sqrt{2\pi} \exp \left( \frac{1}{4} i\pi \right) a^2 + \mathcal{O}(a^3)
\]

and

\[
F_2(a) = -\sqrt{2\pi} \exp \left( \frac{1}{4} i\pi \right) + 2^{1/4} 3\Gamma \left( \frac{3}{4} \right) \exp \left( \frac{3}{8} i\pi \right) a + \mathcal{O}(a^2)
\]

This would be equivalent to using the simpler approximation (7.1) for the reflection coefficient, rather than equation (8.2), leading to the shifts

\[
s_r(r, \theta_c) \approx N, \quad s_\theta(r, \theta_c) \approx -N \frac{C}{(kr)^{3/4}} \left( |\alpha_1|^2 + \frac{|\alpha_2|^2}{n^2} \right)
\]

and

\[
\begin{align*}
& s_\theta(r, \theta_c) \approx -N \frac{C}{(kr)^{3/4}} \text{Re} \left[ \alpha_1^* \alpha_2 \left( \frac{\exp((3/8)i\pi)}{n^2} - \exp\left( -\frac{3}{8}i\pi \right) \right) \right], \quad (8.7)
\end{align*}
\]
where the normalization $N$ is given by

$$\frac{1}{N^2} = 1 + \frac{C^2}{(kr)^{3/2}} \left( \left| \alpha_1 \right|^2 + \frac{\left| \alpha_2 \right|^2}{n^2} \right)^2 + \frac{C^2}{(kr)^{3/2}} \text{Re}^2 \left[ \alpha_1^* \alpha_2 \left( \frac{\exp\left(\frac{3}{8}i\pi\right)}{n^2} - \exp\left(-\frac{3}{8}i\pi\right) \right) \right].$$

(8.8)

But although these formulae do give the leading-order shifts for $kr \gg 1$, the limiting regime is approached extremely slowly. To see why, notice that in equation (8.6), the second terms are smaller than the first if,

$$a \ll \frac{\sqrt{\pi}}{2^{3/4} \Gamma(3/4)} = 0.86 \ldots (F_0), \quad a \ll \frac{\Gamma(1/4)}{2^{1/4} \sqrt{\pi}} = 0.61 \ldots (F_1)$$

and

$$a \ll \frac{2^{1/4} \sqrt{\pi}}{3 \Gamma(3/4)} = 0.57 \ldots (F_2)$$

(8.9)

consistent with graphs of the exact and approximate $F_m(a)$ (figure 8). These inequalities imply that the condition

$$(kr)_i \gg \frac{64n^2\gamma_i}{(1 - n^2)}.$$  

(8.10)

must be satisfied if the leading-order approximations (8.7) are to accurately represent the shifts. For $n = 2/3$, this gives, for the two polarizations, $(kr)_1 \gg 51$ and $(kr)_2 \gg 1312$. Computer explorations (not shown) confirm that only for pure TE polarization does the leading-order formula accurately represent the lateral shift (there is no transverse shift for this special case) for $kr < 2000$—the largest value for which direct numerical integration is unproblematic.
9. Concluding remarks

It is becoming clear (Angelsky et al. 2011) that the Poynting vector can be explored experimentally for complicated fields with general polarization, raising the possibility of observing the shift phenomena identified here—if not in optics then surely for microwaves. These phenomena include: strong oscillations from the interference of the lateral wave with the geometrical wave; the Brewster peak for TM polarization, also for incidence from the rarer medium; existence of a transverse shift for linear polarizations that are not pure TE or TM; and the quantitative dependences of the shifts on $kr$: $1/kr$ away from the critical angle, $1/(kr)^{3/4}$ at the critical angle.

The large-$kr$ approximations obtained here could be unified into a uniform representation for the shifts, valid from the oscillatory regime $\theta > \theta_c$, through the critical angle $\theta = \theta_c$ itself and into the partial reflection regime $\theta < \theta_c$. This would be an application to general shifts of the uniform approximations already obtained (Brekhovskikh 1960; Lai et al. 1986) for the scalar wave function and the Goos–Hänchen shift for beams. It has not been presented here because it would be based on the simple approximation (7.1) to the reflection coefficients, not the more sophisticated approximation (8.2). Therefore, as the argument of §8 (supported by numerics) indicates, such a uniform approximation would be accurate only for extremely large $kr$, except for the special case of TE polarization.

Several extensions come to mind. (i) A parallel discussion of the transmitted wave would be fairly straightforward, (ii) complexifying the position of the source would be a model for shifts from Gaussian beams (Deschamps 1971), complementing existing treatments but not restricted to paraxial fields, and (iii) the relative refractive index $n$ could be complex or negative, leading to treatment of shifts from diverging waves reflected from absorbing or left-handed materials.

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References