Stream function for optical energy flow

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Abstract

A formula is presented for the stream function \( S \) corresponding to optical fields in the plane that can be represented as a superposition of plane waves with the same transverse wavenumbers, including evanescent waves. Energy flow lines (parallel to the Poynting vector) are easily computed as contour lines of \( S \). Optical vortices (including higher-order vortices) correspond to extrema of \( S \).

Keywords: Poynting, rays, flux, interference, polarization

1. Introduction

A useful picture of a wave-optical field is given by the trajectories of energy flow, that is, the set of curved current lines, tangent to the Poynting vector at each point. For a general three-dimensional field, computing these lines requires the numerical integration of trajectory equations. However, in many two-dimensional cases a simpler option is available: the flow lines are contours of a real-valued stream function, and our purpose here is to present a simple general formula for it and discuss its structure and implications.

The formula is (18) in section 2. Its generality lies in its applicability to fields containing evanescent as well as real waves. Each of us has used, without writing it explicitly, the special case ((21) of section 3) in which there are no evanescent waves. Each of us has used, without writing it explicitly, the special case ((21) of section 3) in which there are no evanescent waves [1, 2].

In the cases we are interested in, the optical field can be expressed in terms of a complex scalar field \( \psi(r) \) \((r = (x, y))\), satisfying the two-dimensional Helmholtz equation

\[
\nabla^2 \psi(r) + k^2 \psi(r) = 0, \tag{1}
\]

and the current vector, parallel to the gradient of the phase \( \arg \psi \), is

\[
J(r) = \text{Im}[\psi^*(r) \nabla \psi(r)] = |\psi(r)|^2 \nabla \arg \psi(r),
\tag{2}
\]

\( \nabla = e_x \partial_x + e_y \partial_y \).

It follows that

\[
\nabla \cdot J = 0, \tag{3}
\]

so \( J \) can be written as the curl of a vector perpendicular to the plane:

\[
J = \nabla \times S(r) \hat{e}_z : \quad J_x = \partial_y S, \quad J_y = -\partial_x S. \tag{4}
\]

\( S(r) \) is the stream function. Its existence as a scalar function relies on the divergencelessness property (3) which follows from (1). Since its gradient is orthogonal to \( J \), its contours are parallel to \( J \) at each point \( r \) and so give the energy flow lines. The direction of \( J \) at each point is the sense in which \( S(r) \) increases to the left (not the contrary as in [2], where an opposite convention was used). Because of (3), \( S \) can be determined by integration along any path ending at \( r \), for example

\[
S(r) = -\int_0^r \; dx' J_x(x', 0) + \int_0^r \; dy' J_y(x, y'). \tag{5}
\]

Interesting aspects of the geometry of \( S(r) \), regarded as the height function of a landscape and emphasizing the connection with optical vortices (including higher-order vortices), are discussed in the appendix.

Our formalism applies to a wide variety of cases, including:

(a) Electric fields linearly polarized perpendicular to the plane, i.e.

\[
E(r) = \psi(r) e_z. \tag{6}
\]

Then it follows from Maxwell’s equations that the time-averaged Poynting vector

\[
P = \text{Re} E^* \times H \tag{7}
\]

is proportional to \( J \) in (2).

(b) Any scalar field in three dimensions for which the \( z \) dependence separates, i.e.

\[
\Psi(r, z) = \exp(ikz) \psi(r). \tag{8}
\]

Such fields include nondiffracting beams, for example Bessel beams [3, 4] and isotropic random waves, e.g. optical speckle, with the ‘ring spectrum’ [5].
Thus, from (5), the stream function is
\[ S(r, z) = \exp(ik_z z)(\psi_+ (r)e_+ + \psi_- (r)e_-), \]
\[ e_+ = \frac{1}{\sqrt{2}} (e_x + ie_z), \quad e_- = \frac{1}{\sqrt{2}} (e_x - ie_z). \]

As shown elsewhere [2, 6], the Poynting vector for such fields is proportional to the sum of an orbital and a spin part, each of which is itself the sum of helicity contributions:
\[ P(r) = \text{Im} \{ \psi_+^* \nabla \psi_+ + \psi_-^* \nabla \psi_- \} + \frac{i}{2} \nabla \times (|\psi_+|^2 - |\psi_-|^2) e_z = P_{\text{orb}} + P_{\text{sp}}. \]

Thus, from (5), the stream function is
\[ S(r) = S_{\text{orb}, +} + S_{\text{orb}, -} + \frac{i}{2} (|\psi_+|^2 - |\psi_-|^2), \]
that is, the sum of two contributions of the form (5), plus a spin contribution that is easy to calculate.

A stream function exists even if the wavenumber \( k \) in (1) is a function of position. However, the corresponding fields cannot be expressed as superpositions of plane waves and we do not consider them further here.

Stream functions originated in fluid mechanics, where \( J \) represents the flow velocity. The difference from optics is that in fluid mechanics the divergencelessness of \( J \) arises from incompressibility, not from an underlying complex scalar field as in (2) or (10).

As emphasized elsewhere [2, 7], the lines of energy flow are very different from the rays of geometrical optics for fields consisting of more than one plane wave. This reflects interference: the phase gradient lines of the superposition, as in (2), are very different from those of the superposition of the phase gradients.

2. Stream function calculation

Any solution of (1) can be written as a (discrete or continuous) superposition of waves travelling in different directions in the plane, all with the same wavenumber \( k \):
\[ \psi(r) = \sum_n c_n \exp(ik_n \cdot r), \quad k_n \cdot k_n = k^2_{nx} + k^2_{ny} = k^2. \]

Thus the wavenumbers can be written in the form
\[ k_n = k \cos \theta_n e_x + k \sin \theta_n e_y. \]

It is important to note that, in the general superpositions we are considering, the directions \( \theta_n \) can be complex, representing evanescent waves. The familiar special case in which all the \( k_n \) are real will be considered in section 3.

From (2), the current is
\[ J(r) = \text{Im} \sum_m \sum_n c_m^* c_n \exp(i(k_n - k_m^\ast) \cdot r) \]
\[ = \sum_n |c_n|^2 \text{Re} (k_n \exp(-2 \text{Im} k_n \cdot r) \]
\[ + \text{Re} \sum_m \sum_{n \neq m} c_m^* c_n (k_n + k_m^\ast) \exp(i(k_n - k_m^\ast) \cdot r), \]

in which the second equality follows from separating the double sum into its diagonal and off-diagonal parts and, in the off-diagonal sum, using symmetry under interchange of \( m \) and \( n \) and complex conjugation. Now, from (5), the stream function is obtained by elementary integration:
\[ S_{\text{diag}}(r) = \frac{1}{2} \sum_n |c_n|^2 \left\{ \frac{\text{Re} k_{nx}}{\text{Im} k_{ny}} \exp \left(-2 \text{Im} k_{nx} \cdot r \right) - 1 \right\} \]
\[ - \text{Re} \frac{k_{nx}}{k_{ny}} \left[ \exp \left(-2 \text{Im} k_{nx} \cdot r \right) - \exp \left(-2 \text{Im} k_{nx} \cdot r \right) \right]. \]

This can be simplified. First we note that the term involving \(-1\) can be discarded, because it is independent of \( r \) and so does not affect the streamlines. Second, we note that the terms involving \( k \cdot x \) cancel, because of the identity
\[ \text{Re} \frac{k_{nx}}{\text{Im} k_{ny}} + \text{Re} \frac{k_{nx}}{\text{Im} k_{ny}} = \frac{\text{Re} k_{nx} \text{Im} k_{ny} + \text{Re} k_{nx} \text{Im} k_{ny}}{\text{Im} k_{nx} \text{Im} k_{ny}} = 0. \]

Third, we note that the coefficients in the remaining diagonal and off-diagonal sums can be written in terms of the plane-wave directions:
\[ \frac{\text{Re} k_{nx}}{\text{Im} k_{ny}} = \text{coth} \theta_n, \quad \frac{k_{nx} + k_{nx}^*}{k_{ny} - k_{ny}^*} = \frac{\cos \theta_n + \cos \theta_n^*}{\sin \theta_n - \sin \theta_n^*} = \cot \left( \frac{1}{2} (\theta_n - \theta_n^*) \right). \]

Thus we arrive at our main result:
\[ S(r) = -\frac{1}{2} \sum_n |c_n|^2 \exp(-2 \text{Im} k_n \cdot r) \text{coth} \theta_n \]
\[ + \text{Im} \sum_m \sum_{n \neq m} c_m^* c_n \cot \left( \frac{1}{2} (\theta_n - \theta_n^*) \right) \]
\[ \times \exp(i(k_n - k_m^\ast) \cdot r). \]

Figure 1(a) shows the energy flow lines (and also the streamlines) of the field including evanescent waves, and displays rich sub-wavelength structure: twenty vortices in a region only two wavelengths square.

3. Two special cases

The first is the case of no evanescent waves: all directions \( \theta_n \) in (13) are real, that is
\[ \text{Im} k_n \rightarrow 0. \]

The limit of the off-diagonal sum in (18) is straightforward, but the diagonal contribution needs some care. For each contributing plane wave, the required limit is
\[ -\frac{1}{2} |c_n|^2 \exp(-2 \text{Im} k_n \cdot r) \text{coth} \theta_n \]
\[ = \text{const} + |c_n|^2 (k_n \sin \theta_n + k_n \cos \theta_n). \]
form (12) with six plane waves and coefficients $c_n$ chosen as complex numbers with moduli randomly distributed on the range $0 \leq |c_n| \leq 1$ and phases randomly distributed on $0 \leq \theta_n \leq 2\pi$, and $\kappa = 1$. Dashed black curves: wavefronts, that is contours of $|\psi|$ crossing at vortices where the energy flow lines are locally circular. The intensity $|\psi|$ is indicated by the underlying greyscale shading. (b) 3D plot of stream function landscape for the same field as in (a), with contours (flow lines) superimposed in black. In both pictures, the region plotted is two wavelengths square.

The constant is irrelevant and can be discarded, leaving the stream function

$$S(\mathbf{r}) = r \times c_2 \cdot \sum_n |c_n|^2 k_n$$

$$+ \text{Im} \sum_{n \neq m} |c_n|^2 c_m \cot \left( \frac{1}{2} (\theta_n - \theta_m) \right)$$

$$\times \exp \left( i \left( k_{n} r - k_{m} r \right) \right).$$

(21)

Thus when all plane waves are propagating, rather than evanescent, the diagonal terms are linear in $r$, rather than exponential as in the general formula (18). The linear contribution represents a mean flow parallel to the mean momentum $\sum_n |c_n|^2 k_n$.

The second special case is when the directions of two plane waves (real or evanescent) coincide, that is, if the waves correspond to $n = 1$ and 2,

$$k_1 \rightarrow k_2.$$ 

(22)

Then their off-diagonal contributions can be simplified using the limit

$$\cot \left( \frac{1}{2} (\theta_1 - \theta_2) \right) \rightarrow -i \coth \theta_1,$$

(23)

so the total contribution from waves 1 and 2, including the diagonal elements, is, from (18),

$$S_{12}(r) = \coth \theta_1 \exp(-2 \text{Im} k_1 \cdot r)$$

$$\times \left( -\frac{1}{2} |c_1|^2 - \frac{1}{2} |c_2|^2 - \text{Re}(c_1^* c_2^*) \right)$$

$$= -\frac{1}{2} |c_1 + c_2|^2 \coth \theta_1 \exp(-2 \text{Im} k_1 \cdot r).$$

(24)

Thus the strength of the combined contribution from the coinciding waves is proportional to $|c_1 + c_2|^2$, as it must be.

Appendix. The stream function landscape

If $S(\mathbf{r})$ is regarded as the smooth height function of a landscape (figure 1(b)), its significant features are the critical points, where $\nabla S = 0$. From (4), these points satisfy $J = 0$, that is

$$\text{Im} \psi^* \partial_x \psi = 0, \quad \text{Im} \psi^* \partial_y \psi = 0.$$ 

(A.1)

The critical points are of two kinds: extrema and saddles.

The nodal points $\psi = 0$ (i.e. optical vortices, phase singularities or wave dislocations), at which (A.1) obviously holds, are the extrema, because, from (4),

$$\partial_{xx} S = -\text{Im} \partial_x \psi^* \partial_y \psi - \text{Im} \psi^* \partial_{xy} \psi,$$

$$\partial_{yy} S = \text{Im} \partial_x \psi^* \partial_y \psi + \text{Im} \psi^* \partial_{xy} \psi,$$

$$\partial_{xy} S = \text{Im} \partial_x \psi^* \partial_x \psi + \text{Im} \psi^* \partial_x \psi = \text{Im} \psi^* \partial_{xx} \psi,$$

so, when $\psi = 0$,

$$\partial_{xx} S = \partial_{yy} S,$$

$$\partial_{xy} S = 0.$$ 

(A.3)

Thus, not only do the zeros of $\psi$ correspond to extrema of $S$, but these extrema have the special property that they are isotropic, reflecting the fact [1, 5, 8] that the flow lines are asymptotically circular around optical vortices. Maxima correspond to vortices with index +1 (arg $\psi$ increases by $+2\pi$ in an anticlockwise circuit of the zero), and minima to index −1. Isotropy of extrema implies a ‘local analyticity’ property of $S(\mathbf{r})$, originating in the fact that $\psi = 0$, equation (1) reduces to Laplace’s equation: close to an index $\pm 1$ point, $S$ can be written as $\pm$ (the square modulus of a function of $x + iy$).

As parameters vary, vortices with index $\pm 1$ can degenerate to form higher-order vortices, whose indices are encoded in the order of the extrema of $S(\mathbf{r})$. The easiest way to understand this is to start from the local form satisfying (1) for a vortex with index $m$: up to a constant,

$$\psi_m(\mathbf{r}) = (x + iy \text{sgn } m)^{|m|} = r^{|m|} \exp(imm\phi).$$ 

(A.4)

From (2), the current vector is

$$J = mr^{|m|-2} (-y, x) = mr^{|m|-1} e_\phi.$$ 

(A.5)
It now follows from (4) that the stream function is
\[ S(r) = -\frac{1}{2}m r^{2|m|}, \] (A.6)
which is indeed a higher-order extremum: a maximum for \( m > 0 \) and a minimum for \( m < 0 \).

The other critical points, where \( \psi \neq 0 \), correspond (cf (2)) to \( \nabla \arg \psi = 0 \) [9]. These are the saddle points of the stream function landscape, where the two principal curvatures have opposite sign, because, where \( \nabla \arg \psi = 0 \), it follows from (A.2) that

\[
\text{product of curvatures of } S = \det \left( \begin{array}{cc} \partial_{xx} S & \partial_{xy} S \\ \partial_{xy} S & \partial_{yy} S \end{array} \right) \\
= -|\psi|^4 ((\partial_{xy} \arg \psi)^2 + (\partial_{xx} \arg \psi)^2) \leq 0.
\] (A.7)

These saddles of \( S \) are also saddles of \( \arg \psi \), that is, phase saddles, because, from

\[
\partial_{xx} \arg \psi = -\partial_{yy} \arg \psi \quad \text{where } \nabla \arg \psi = 0 \quad \text{(A.8)}
\]
(which also follows from (A.2)),

\[
\text{product of curvatures of } \arg \psi \\
= \partial_{xx} \arg \psi \partial_{yy} \arg \psi - (\partial_{xy} \arg \psi)^2 \\
= - (\partial_{xx} \arg \psi)^2 - (\partial_{xy} \arg \psi)^2 \leq 0.
\] (A.9)

Finally, we note that \( S \) satisfies a Poisson equation whose source is the flow vorticity: from (4) and (2),

\[
\nabla^2 S = -\nabla \times J = -2 \text{Im} \partial_x \psi^* \partial_y \psi.
\] (A.10)

References