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Superluminal speeds for relativistic random waves

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Abstract

For Klein–Gordon and Dirac waves representing massive quantum particles, the local group velocity v (weak value of the velocity operator) can exceed c. If the waves consist of superpositions of many plane waves, with different (but subluminal) group velocities u, the superluminal probability P_{super}, i.e. that $|v| > c$ for a randomly selected state, can be calculated explicitly. P_{super} depends on two parameters describing the distribution (power spectrum) of u in the superpositions, and lies between 0 and 1/2 for Klein–Gordon waves and $1\text{–}1/\sqrt{2}$ and 1/2 for Dirac waves. Numerical simulations display the superluminal intervals in space and regions in spacetime, and support the theoretical predictions for P_{super}.

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1. Introduction

Several phenomena are known in which features of physical systems can travel faster than the light speed c without violating relativistic causality [1, 2]. Well-known examples are the bright patch made by a laser beam illuminating a cloud, which could travel superluminally if the laser is rotated fast enough, and Moiré fringes between gratings being rotated relatively to each other. The phase velocity of light and matter waves can exceed c. And in the early days of relativity it was realized that for light traversing a medium with frequencies close to an absorption band the group velocity can also exceed c, leading to detailed studies [3, 4] of the corresponding superluminal reshaping of the light pulse while its front travels at c. The effect of absorption can be mimicked by post-selecting the polarization states of light in an optical fibre, leading to another superluminal effect that has been observed [5]—this and other phenomena being interpreted in terms of weak measurements [6–9]. Analogous apparent superluminal reshaping can occur in relativistic massive particle waves propagating causally in free space [10].

The aim here is to add to this growing list by exploring the local group velocity $v(x, t)$ of waves satisfying the one-dimensional Klein–Gordon and Dirac equations in empty space. The waves to be considered are superpositions of propagating plane waves with a range of wavenumbers, each corresponding to a subluminal group velocity. With a natural definition, $v(x, t)$ can exceed c in substantial intervals of x and t.

A convenient theoretical framework is the weak measurement/weak value formalism, involving a state $|\psi\rangle$—the wave—and a velocity operator $\hat{v}$ which is measured after post-selection with a state $|\text{post}\rangle$ incorporating the position where the measurement is made. The 'weak value' to be studied, which as will be explained later is the local group velocity, is [6, 11–13]

$$v = \mathrm{Re}\frac{\langle \text{post}|\hat{v}|\psi\rangle}{\langle \text{post}|\psi\rangle}. \tag{1.1}$$

(The corresponding imaginary part is interesting [14, 15] but will not be discussed here.)

When the superposition contains many waves, it is possible to calculate the measure of these intervals, namely the superluminal probability that $v > c$ for a randomly-sampled event. These calculations comprise the main results of this paper. They complement and extend other recent statistical calculations of 'superweak' probabilities [16–20], that is, probabilities that weak values lie outside the spectrum of the operator being measured.

The plan of the paper is as follows. In section 2 the local group velocity formalism is developed for Klein–Gordon waves, and the connection with the weak value is confirmed by an separate argument. The corresponding formalism for Dirac waves is developed in section 3. Section 4 contains the statistical calculation of the superluminal probability for Klein–Gordon waves, and section 5 gives the more elaborate corresponding calculation for Dirac waves. Section 6 contains numerical simulations illustrating these superluminal phenomena, and section 7 suggests directions for further study.

In what follows, we use units such that Planck's constant $\hbar = 1$, $c = 1$, and the particle mass $m = 1$, so that the superluminal group velocities in which we are interested correspond to $|v| > 1$. And we will denote probabilities and probability densities by the generic symbol P, using arguments and subscripts to avoid ambiguity. If the probability density for finding a local group velocity is $P(v)$, then the required superluminal probability is

$$P_{\text{super}} = \int_{-\infty}^{-1} \mathrm{d}v P(v) + \int_{1}^{\infty} \mathrm{d}v P(v). \tag{1.2}$$

$P(v)$ will be calculated over ensembles to be defined later.

2. Klein–Gordon waves: group velocity formalism

The one-dimensional Klein–Gordon equation is

$$\partial_x^2 \psi - \partial_t^2 \psi = \psi, \tag{2.1}$$

and a class of solutions can be written as a sum over plane waves:

$$\psi(x,t) = \sum_k c_k \exp\{\mathrm{i}\gamma_k(x,t)\}, \tag{2.2}$$

where the phases are

$$\gamma_k(x,t) = \mu_k + kx - t\sqrt{k^2+1}. \tag{2.3}$$

To define a given superposition of this type, it is necessary to specify the set of contributing wavevectors (signed wavenumbers) k, their real excitation amplitudes c_k and the phases μ_k; in later sections, we will regard the power spectrum $|c_k|^2$ as determined, and the μ_k as random. We consider only real wavevectors k (positive or negative), corresponding to propagating waves; interactions with boundaries can generate evanescent waves, for which k is complex, but in the present context this is an unnecessary complication.

Corresponding to (2.2) is the dispersion relation (Hamiltonian)

$$\omega(k) = \sqrt{k^2+1}. \tag{2.4}$$

This leads to a natural definition of the velocity operator in terms of the wavenumber operator $\hat{k} = -\mathrm{i}\partial_x$: from the first Hamilton equation,

$$\hat{v} = \partial_{\hat{k}}\omega(\hat{k}) = \frac{\hat{k}}{\sqrt{\hat{k}^2+1}}. \tag{2.5}$$

For the post-selected state in (1.1) we choose the position eigenstate corresponding to the point x being considered:

$$\langle \mathrm{post}| = \langle x|, \tag{2.6}$$

leading to the weak value

$$v(x,t) = \mathrm{Re}\frac{\sum_k c_k \frac{k}{\sqrt{k^2+1}}\exp\{\mathrm{i}\gamma_k(x,t)\}}{\sum_k c_k \exp\{\mathrm{i}\gamma_k(x,t)\}}. \tag{2.7}$$

The same result follows from a more intuitive argument. The natural definition of a local velocity operator is the symmetrized product

$$\hat{v}(x) = \tfrac{1}{2}(\delta(x-\hat{x})\hat{v} + \hat{v}\delta(x-\hat{x})). \tag{2.8}$$

Its normalized local expectation value is

$$\begin{aligned} v(x,t) &= \frac{\langle\psi|\hat{v}(x)|\psi\rangle}{\langle\psi|\delta(x-\hat{x})|\psi\rangle} \\ &= \frac{\sum_k\sum_{k'} c_k c_{k'}\frac{1}{2}\left(\frac{k}{\sqrt{k^2+1}} + \frac{k'}{\sqrt{k'^2+1}}\right)\exp\{\mathrm{i}(\gamma_k(x,t)-\gamma_{k'}(x,t))\}}{\sum_k\sum_{k'} c_k c_{k'}\exp\{\mathrm{i}(\gamma_k(x,t)-\gamma_{k'}(x,t))\}}, \end{aligned} \tag{2.9}$$

after which a short calculation reproduces (2.7) exactly. (The analogue of (2.8) with $\hat{v}$ replaced by the local wavevector (momentum) $\hat{k} = -\mathrm{i}\partial_x$ reproduces the local phase gradient $k\,(x,t) = \partial_x \mathrm{Im}\log\psi\,(x,t)$.)

We now write (2.7) in a form that is more convenient for later development, by introducing the group velocity u of the plane-wave contribution with wavenumber k. Defining

$$\begin{aligned} &u \equiv \frac{k}{\sqrt{k^2+1}} \quad (|u|<1), \\ &c_k \equiv d_u, \quad \gamma_k(x,t) \equiv \delta_u(x,t) = \mu_k + \frac{ux-t}{\sqrt{1-u^2}}, \end{aligned} \tag{2.10}$$

the local group velocity becomes

$$v(x,t) = \mathrm{Re}\frac{\sum_u d_u u \exp\{\mathrm{i}\delta_u(x,t)\}}{\sum_u d_u \exp\{\mathrm{i}\delta_u(x,t)\}}. \tag{2.11}$$

This expresses $v(x,\,t)$ as a weighted sum over the contributing group velocities, according to the weak value prescription. And although the contributing values lie in the range $-1 < u < +1$, corresponding to the bounded spectrum of the velocity operator (2.5), the denominator in (2.11) allows $v(x,\,t)$ to take values outside this range: in the present context, superweak = superluminal.

3. Dirac waves: group velocity formalism

For one-dimensional waves, the Dirac equation can be written in terms of a two-component spinor [21], and the following form is convenient:

$$\begin{aligned} &\mathrm{i}\partial_t|\psi\rangle = \hat{H}|\psi\rangle = (\sigma_3\hat{k}+\sigma_1)|\psi\rangle, \ \text{i.e.} \\ &\begin{pmatrix}\mathrm{i}\partial_t\psi_+\\ \mathrm{i}\partial_t\psi_-\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}-\mathrm{i}\partial_x\psi_+\\ -\mathrm{i}\partial_x\psi_-\end{pmatrix} + \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}. \end{aligned} \tag{3.1}$$

As can be confirmed by substitution, solutions in the form of plane-wave superpositions, now expressed directly in terms of contributing group velocities u (cf (2.10) and (2.11)), are

$$\psi_{\pm}(x,t)=\sum_u d_u\sqrt{1\pm u}\exp\{\mathrm{i}\delta_u(x,t)\}. \tag{3.2}$$

The velocity operator, analogous to (2.5), is the matrix

$$\hat{v}=\partial_{\hat{k}}\hat{H}=\sigma_3. \tag{3.3}$$

As is well known [21], this has only the two eigenvalues ± 1, and values lying between these limits correspond to superpositions of the corresponding eigenstates. Values outside the limits, that is, the superluminal group velocities of interest here, require post-selection. As well as position x, it is necessary to specify a vector, which we define as

$$\langle\mathrm{post}|=\left(\cos\tfrac{1}{2}\theta,\ \exp(-\mathrm{i}\phi)\sin\tfrac{1}{2}\theta\right). \tag{3.4}$$

The angles θ, ϕ represent position on the Bloch sphere (= Poincaré polarization sphere in optics = Riemann sphere in mathematics). Then (1.1) defines the local post-selected group velocity

$$\begin{aligned} v\,(x,t) &= \mathrm{Re}\frac{\psi_+\cos\frac{1}{2}\theta-\psi_-\exp\,(-\mathrm{i}\phi)\sin\frac{1}{2}\theta}{\psi_+\cos\frac{1}{2}\theta+\psi_-\exp\,(-\mathrm{i}\phi)\sin\frac{1}{2}\theta}\\ &=\mathrm{Re}\frac{1-\tan\frac{1}{2}\theta\exp(-\mathrm{i}\phi')r}{1+\tan\frac{1}{2}\theta\exp(-\mathrm{i}\phi')r}\\ &=\frac{1-\tan^2\frac{1}{2}\theta\,r^2}{1+\tan^2\frac{1}{2}\theta\,r^2+2\tan\frac{1}{2}\theta\,r\cos\,(\phi')}, \end{aligned} \tag{3.5}$$

in which

$$r=r(x,t)\equiv\left|\frac{\psi_-(x,t)}{\psi_+(x,t)}\right|,\qquad \phi'\equiv\phi-\arg\frac{\psi_-(x,t)}{\psi_+(x,t)}. \tag{3.6}$$

The formula (3.5) reveals a natural Bloch-sphere anisotropy. For post-selection with one of the eigenstates of $\hat{v}$, the local group velocity is either $v(x,t)=+1$ (for the choice $\theta=0$, i.e. $\langle\mathrm{post}|=(1,0)$), or $v(x,t)=-1$ (for the choice $\theta=\pi$, i.e. $\langle\mathrm{post}|=(0,1)$). This implies that superluminal local group velocities require post-selection to be a superposition of the two eigenstates of $\hat{v}$.

4. Klein–Gordon equation group velocity statistics

To calculate the superluminal probability according to (1.2), we need the probability density of the local group velocity $v(x,t)$. We can write this as

$$P(v)=\langle\langle\delta(v-v(x,t))\rangle\rangle_{\mu_u}, \tag{4.1}$$

using the notation $\langle\langle\cdots\rangle\rangle$ to distinguish ensemble averages from quantum expectation values, and where for Klein–Gordon waves the average is over the random phases μ_k in (2.10) and (2.11). If the number N of contributing waves is large, we anticipate ergodicity, in which this ensemble average, in which x, t are held fixed, will give the same result as for almost all choices of fixed μ_k and with averaging over x or t; numerical experiments support this expectation.

To calculate the average, we use another consequence of $N\gg 1$: by the central limit theorem, the two sums in the numerator and denominator of (2.11) are Gauss-distributed. Thus the quotient in (2.11) can be written as

$$v(x,t)=\mathrm{Re}\frac{G_1+\mathrm{i}G_2}{G_3+\mathrm{i}G_4}=\frac{G_1G_3+G_2G_4}{G_3^2+G_4^2}, \tag{4.2}$$

in which the G_i are Gaussian random functions. The relevant averages and correlations are

$$\begin{aligned}
&\langle\langle G_i\rangle\rangle = 0 \quad (i = 1, 2, 3, 4)\,,\\
&\langle\langle G_1G_2\rangle\rangle = \langle\langle G_3G_4\rangle\rangle = \langle\langle G_1G_4\rangle\rangle = \langle\langle G_2G_3\rangle\rangle = 0\\
&\left\langle\left\langle G_3^2\right\rangle\right\rangle = \left\langle\left\langle G_4^2\right\rangle\right\rangle = 2\sum_u d_u^2 \equiv 1,\\
&\langle\langle G_1G_3\rangle\rangle = \langle\langle G_2G_4\rangle\rangle = 2\sum_u d_u^2 u = \langle\langle u\rangle\rangle \equiv \bar{u}\\
&\left\langle\left\langle G_1^2\right\rangle\right\rangle = \left\langle\left\langle G_2^2\right\rangle\right\rangle = 2\sum_u d_u^2 u^2 = \langle\langle u^2\rangle\rangle \equiv u_2 \equiv (\bar{u})^2 + \Delta^2,
\end{aligned} \tag{4.3}$$

indicating that the only relevant properties of the plane-wave superposition are the mean $\bar{u}$ and variance Δ^2 of the contributing group velocities. Because $|u| < 1$, these satisfy

$$|\bar{u}| < 1, \quad (\bar{u})^2 + \Delta^2 < 1. \tag{4.4}$$

Thus the distribution of the G_i is a product of two bivariate Gaussians, so the probability distribution (4.1) can be calculated explicitly:

$$\begin{aligned}
P(v) = &\int_{-\infty}^{\infty} \mathrm{d}G_1 \int_{-\infty}^{\infty} \mathrm{d}G_2 \int_{-\infty}^{\infty} \mathrm{d}G_3 \int_{-\infty}^{\infty} \mathrm{d}G_4\\
&\times P\left(G_1, G_3\right) P\left(G_2, G_4\right) \delta\left(v - \frac{G_1G_3 + G_2G_4}{G_3^2 + G_4^2}\right).
\end{aligned} \tag{4.5}$$

These integrals have been previously evaluated in a calculation [18] of the weak-value probability for a general operator with $N \gg 1$ eigenstates over the ensemble of all pre- and post-selected states. Therefore we can use that result directly, after an easy generalization to eliminate the restriction that in [18] only symmetric eigenvalue distributions were considered.

Thus we obtain

$$P(v) = \frac{\Delta^2}{2((v - \bar{u})^2 + \Delta^2)^{3/2}}, \tag{4.6}$$

and hence, from (1.2), the Klein–Gordon superluminal probability which is the main result of this section (and generalizing equation (2.14) of [18]):

$$P_{\text{super}}\left(\bar{u}, \Delta\right) = 1 - \frac{1 + \bar{u}}{2\sqrt{(1 + \bar{u})^2 + \Delta^2}} - \frac{1 - \bar{u}}{2\sqrt{(1 - \bar{u})^2 + \Delta^2}}. \tag{4.7}$$

The surface representing $P_{\text{super}}\left(\bar{u}, \Delta\right)$ is shown in figure 1(*a*). The restrictions (4.4) imply

$$0 \leqslant P_{\text{super}}(\bar{u}, \Delta) \leqslant \tfrac{1}{2}. \tag{4.8}$$

The largest value $P_{\text{super}} = 1/2$ is approached as $\bar{u} \to \pm 1$, i.e. $\Delta \to 0$, corresponding to a superposition in which all plane waves are travelling in the same direction with group velocities u close to the speed of light (e.g. photons). For symmetric distributions, in which equally many plane waves are travelling forwards as backwards, i.e. $\bar{u} = 0$, the largest value, corresponding to $\Delta = 1$, that is with group velocities u again close to the speed of light, is $P_{\text{super}} = 1 - 1/\sqrt{2} = 0.293$ (as in [18]). This is the same as the 'superweak' probability calculated earlier [17], for the local wavenumber $k(x)$ of almost-monochromatic nonrelativistic one-dimensional waves with component wavevectors $\pm k_0$ to satisfy $k(x) > k_0$ (mathematically this is exactly the same problem, though the physics is different). In the nonrelativistic limit $\bar{u} \to 0$, $\Delta \to 0$, in which all plane waves in the superposition travel slowly, $P_{\text{super}} \to 0$, as might be anticipated.

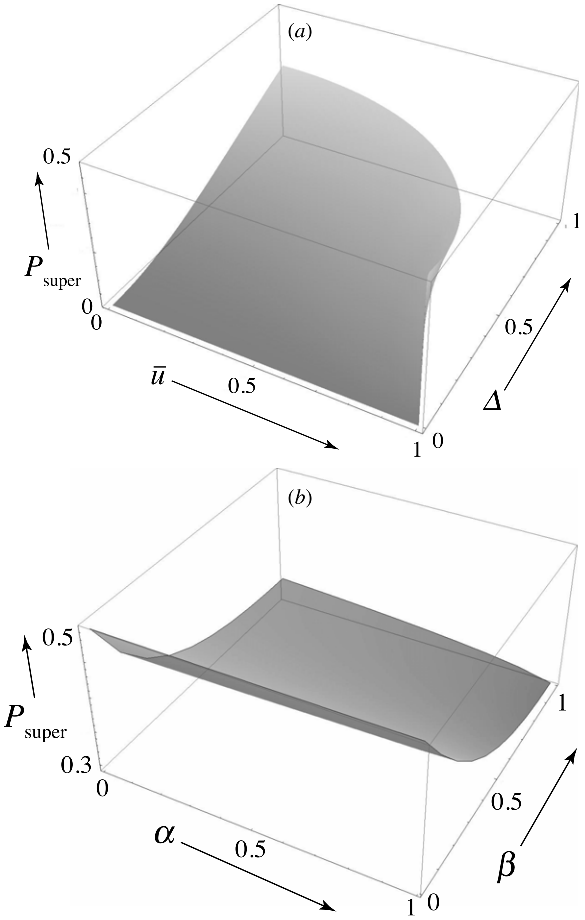


Figure 1. Theoretical superluminal probabilities (*a*) for Klein–Gordon waves (equation (4.7)), as functions of parameters $\bar{u}$ and Δ in (4.3); (*b*) for Dirac waves (equation (5.16)), as functions of parameters α and β in (5.6).

5. Dirac equation group velocity statistics

For Dirac wave superpositions (3.2), the post-selected group velocity is given by (3.5), involving the angles θ, ϕ and the ratio r in (3.6). Obtaining the probability distribution of the group velocities v is subtly different from a previous calculation [19] of the weak value probablility for general two-state systems, which involved averaging with both pre- and post-selected states uniformly distributed over the Bloch sphere, because in the present case it is only the post-selected states that are so distributed, with the pre-selected states being Gauss-distributed. Thus we must evaluate

$$P(v) = \frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\phi' \int_0^{\pi} \mathrm{d}\theta \, \sin\theta \langle\langle \delta(v - v(x,t))\rangle\rangle_{\mu_k}. \tag{5.1}$$

The average over phases μ_k involves the ratio of two complex Gaussians, that we write in the form

$$r = \left|\frac{\Gamma_1 + \mathrm{i}\Gamma_2}{\Gamma_3 + \mathrm{i}\Gamma_4}\right|. \tag{5.2}$$

By an analogous argument to that in the previous section, we can calculate the distribution of r as

$$P(r) = \int_{-\infty}^{\infty} \mathrm{d}\Gamma_1 \int_{-\infty}^{\infty} \mathrm{d}\Gamma_2 \int_{-\infty}^{\infty} \mathrm{d}\Gamma_3 \int_{-\infty}^{\infty} \mathrm{d}\Gamma_4 P(\Gamma_1, \Gamma_3) P(\Gamma_2, \Gamma_4) \delta\left(r - \sqrt{\frac{\Gamma_1^2+\Gamma_2^2}{\Gamma_3^2+\Gamma_4^2}}\right)$$
$$= \frac{2\left(V_+^2V_-^2 - V_{+-}^2\right) r \left(V_-^2 + r^2V_+^2\right)}{\left(\left(V_-^2 - r^2V_+^2\right)^2 + 4r^2\left(V_+^2V_-^2 - V_{+-}^2\right)\right)^{3/2}}, \tag{5.3}$$

in which the relevant variances, associated with the superpositions (3.2), are now

$$\begin{aligned} V_+^2 &= \left\langle\left\langle\Gamma_3^2\right\rangle\right\rangle = \left\langle\left\langle\Gamma_4^2\right\rangle\right\rangle = \langle\langle 1+u\rangle\rangle \\ V_-^2 &= \left\langle\left\langle\Gamma_1^2\right\rangle\right\rangle = \left\langle\left\langle\Gamma_1^2\right\rangle\right\rangle = \langle\langle 1-u\rangle\rangle \\ V_{+-} &= \langle\langle\Gamma_1\Gamma_3\rangle\rangle = \langle\langle\Gamma_2\Gamma_4\rangle\rangle = \langle\langle\sqrt{1-u^2}\rangle\rangle. \end{aligned} \tag{5.4}$$

The Bloch sphere average in (5.1) can now be evaluated. It is convenient to define the following new variables τ and x, replacing θ and r:

$$\tau \equiv r \tan\frac{1}{2}\theta, \qquad r \equiv \frac{V_-}{V_+}x^2. \tag{5.5}$$

It is also convenient to define new parameters α and β describing the spectrum of contributing group velocities u:

$$\alpha \equiv \frac{V_{+-}^2}{V_+^2V_-^2} = \frac{\langle\langle\sqrt{1-u^2}\rangle\rangle^2}{\langle\langle 1+u\rangle\rangle\,\langle\langle 1-u\rangle\rangle}, \qquad \beta \equiv \frac{V_-}{V_+} = \sqrt{\frac{\langle\langle 1-u\rangle\rangle}{\langle\langle 1+u\rangle\rangle}}. \tag{5.6}$$

Now (5.1) becomes, after incorporating (5.3)

$$P(v) = \frac{(1-\alpha)}{\pi\beta^2}\int_0^{2\pi}\mathrm{d}\phi \int_0^\infty \mathrm{d}\tau\,\tau\,\delta\left(v - \frac{1-\tau^2}{1+\tau^2+2\tau\cos\phi}\right) g\left(\alpha, \frac{\tau^2}{\beta^2}\right), \tag{5.7}$$

in which

$$g(\alpha, b) \equiv \int_0^\infty \mathrm{d}x \frac{x(1+x)}{(x+b)^2((1+x)^2 - 4\alpha x)^{3/2}}$$
$$= \frac{1}{R^5}\left[\frac{R}{1-\alpha}(\alpha(3-2b+3b^2) - 2(b-1)^2)\right.$$
$$\left. + (1+b)((1-b)^2 - 2\alpha b)\log\left(\frac{1-b+2\alpha b+R}{b(1-b-2\alpha+R)}\right)\right], \tag{5.8}$$

where

$$R \equiv \sqrt{(1-b)^2 + 4\alpha b}. \tag{5.9}$$

The function g satisfies the reciprocity relation

$$g(\alpha, b) = \frac{1}{b^2} g\left(\alpha, \frac{1}{b}\right), \tag{5.10}$$

implying the following reflection relation for $P(v)$:

$$P(-v, \beta) = P\left(v, \frac{1}{\beta}\right). \tag{5.11}$$

Thus we can reduce the ranges of the integrals in (5.7):

$$P(v) = \frac{2(1-\alpha)}{\pi\beta^2}\int_0^{\pi}\mathrm{d}\phi \int_0^1 \mathrm{d}\tau\,\tau\,\delta\left(v - \frac{1-\tau^2}{1+\tau^2+2\tau\cos\phi}\right) g\left(\alpha, \frac{\tau^2}{\beta^2}\right) \quad (v \geqslant 0). \tag{5.12}$$

Now defining $\tau^2 \equiv u$ we can eliminate the δ function, to obtain the final form of the local group velocity distribution function:

$$P(v) = \frac{(1-\alpha)}{\pi v \beta^2} \int_{u_c}^{1} \mathrm{d}u \frac{\sqrt{1-u}}{\sqrt{u(v+1)^2 - (v-1)^2}} g\left(\alpha, \frac{u}{\beta^2}\right) \quad (v \geqslant 0), \tag{5.13}$$

in which

$$u_c = \left|\frac{v-1}{v+1}\right|^2. \tag{5.14}$$

The superweak probability (1.2) now involves a double integral, over u and v. Exchanging these variables, and using

$$\int_{1}^{(1-\sqrt{u})/(1+\sqrt{u})} \frac{\mathrm{d}v}{v\sqrt{u(v+1)^2 - (v-1)^2}} = \frac{\arctan\sqrt{\frac{1}{u} - 1}}{\sqrt{1-u}}, \tag{5.15}$$

leads to the superluminal probability for Dirac waves:

$$\begin{aligned} P_{\text{super}}(\alpha, \beta) &= P_{\text{super}}(\alpha, \beta^{-1}) \\ &= \frac{(1-\alpha)}{\pi} \int_0^1 \mathrm{d}u \arctan\left(\sqrt{\frac{1}{u} - 1}\right)\left(\frac{1}{\beta^2} g\left(\alpha, \frac{u}{\beta^2}\right) + \beta^2 g(\alpha, u\beta^2)\right). \end{aligned} \tag{5.16}$$

This is the main result of this section. I cannot see a way to evaluate the integral in closed form, but it is easy to compute numerically for any values of the parameters α and β defined by (5.6). The surface representing $P_{\text{super}}(\alpha, \beta)$ is shown in figure 1(b). The following special values can be evaluated analytically:

$$P_{\text{super}}(0, 1) = \frac{1}{3}, \quad P_{\text{super}}(1, 1) = 1 - \frac{1}{\sqrt{2}}, \quad P_{\text{super}}(\alpha, 0) = \frac{1}{2}. \tag{5.17}$$

The superluminal probability ranges from $1 - 1/\sqrt{2} = 0.293$ to $1/2$. The smallest value $1 - 1/\sqrt{2}$, for $\alpha = \beta = 1$, corresponds (cf (5.6)) to contributing group velocities symmetrically distributed (i.e. $\bar{u} = 0$) and concentrated near $u = 0$; this is the nonrelativistic limit, so in contrast to Klein–Gordon waves the superluminal probability is not zero in this case. This is the same as the value obtained before [18] for the *largest* superweak probability for generic many-state systems. The value $1/3$, for $\alpha = 0$, $\beta = 1$, corresponds to symmetrically distributed group velocities but now concentrated near the limiting speed $|u| = 1$. The largest value $1/2$, for $\beta = 0$ (or $\beta = \infty$), represents the extreme relativistic limit, with all plane waves travelling in the same direction with speeds close to that of light.

By contrast with the nonzero values of P_{super}, the mean value of the local group velocity always lies between -1 and $+1$: it is always subuminal. A calculation based on (5.13) leads to

$$\begin{aligned} \bar{v}(\alpha, \beta) &= \int_{-\infty}^{\infty} \mathrm{d}v\, vP(v) \\ &= \frac{1-\beta^4}{R^2} + \frac{2(1-\alpha)(1-\beta^2)\beta^2}{R^3} \log\left(\frac{((1+\beta^2)(1+\beta^2-R) - 2\beta^2(1-\alpha))}{2\beta^2(1-\alpha)}\right), \end{aligned} \tag{5.18}$$

with R as defined in (5.8) with $b = \beta^2$. Special cases are

$$\begin{aligned} \bar{v}(0, \beta) &= \frac{1+\beta^2}{1-\beta^2} + \frac{2\beta^2}{(1-\beta^2)^2} \log\beta \\ \bar{v}(1, \beta) &= \frac{1-\beta^2}{1+\beta^2}, \quad \bar{v}(0, 0) = \bar{v}(1, 0) = 1, \quad \bar{v}(0, 1) = \bar{v}(1, 1) = 0. \end{aligned} \tag{5.19}$$

Figure 2 shows the surface corresponding to $\bar{v}(\alpha, \beta)$.

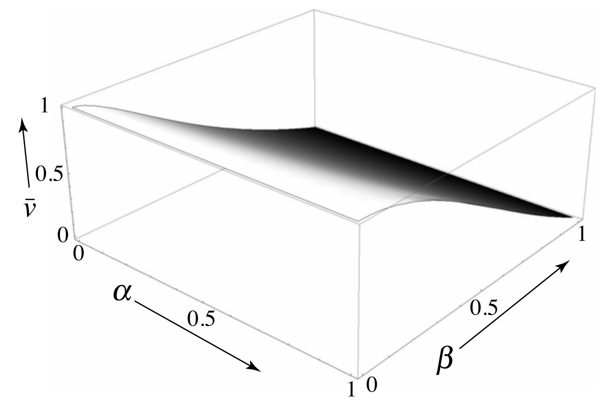


Figure 2. Theoretical mean group velocity $\bar{v}$ (5.18) for Dirac waves, as functions of parameters α and β in (5.6) (only values $\bar{v} \geqslant 0$ are shown; from (5.11), negative $\bar{v}$ correspond to $\beta > 1$).

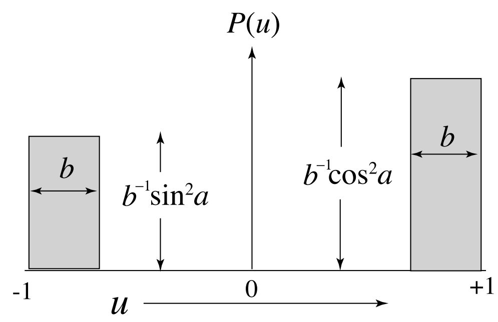


Figure 3. Two-parameter spectrum (6.1) of contributing plane-wave group velocities u, used in all the simulations.

6. Simulations

To illustrate the superluminal phenomena studied here, it is necessary to choose the distribution of group velocities in the superpositions contributing to the local group velocity formulas (2.11), (3.2) and (3.5). The theoretical interpretation of the computations that follow use the distribution

$$P(u) = \begin{cases} \dfrac{\sin^2 a}{b} & (-1 < u \leqslant -1 + b) \\ 0 & (-1 + b < u < 1 - b) \\ \dfrac{\cos^2 a}{b} & (1 - b \leqslant u < 1). \end{cases} \tag{6.1}$$

This involves two parameters as illustrated in figure 3: the angle a giving the symmetry-breaking between right- and left-moving plane waves, and b, giving the width of the ranges of contributing group velocities below light speed $|u| = 1$. It will be sufficiently general to choose the ranges $0 \leqslant a \leqslant \pi/4$, $0 \leqslant b \leqslant 1$. For the corresponding numerics, this is sampled discretely

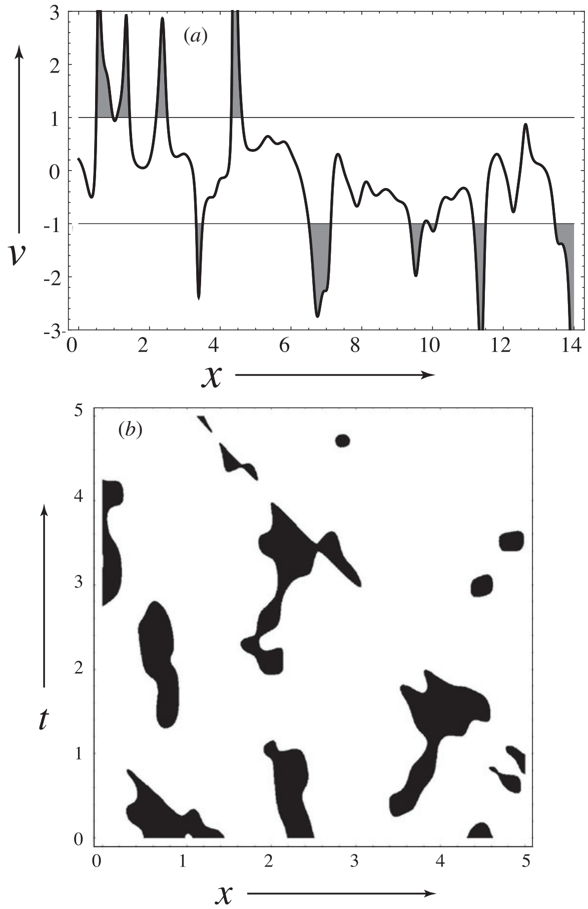


Figure 4. Sample local group velocity (2.11) for Klein–Gordon wave with $a = \pi/4$, $b = 0.5$ in the spectrum (6.1) and a choice of random phases, with the superluminal intervals $|v| > 1$ shaded; (*a*) $t = 0$, (*b*) in spacetime.

with an even number N of plane waves, with the following choices of group velocities and coefficients:

$$u_n = \begin{cases} 1 - \dfrac{2nb}{N} & \left(1 \leqslant n \leqslant \dfrac{1}{2}N\right) \\ -1 + \dfrac{2nb}{N} - b & \left(\dfrac{1}{2}N + 1 \leqslant n \leqslant N\right) \end{cases}$$
$$d_n = \begin{cases} \sqrt{\dfrac{2}{N}}\cos a & \left(1 \leqslant n \leqslant \dfrac{1}{2}N\right) \\ \sqrt{\dfrac{2}{N}}\sin a & \left(\dfrac{1}{2}N + 1 \leqslant n \leqslant N\right). \end{cases} \tag{6.2}$$

(Note the limits $n = 1$ and $n = N/2 + 1$, deliberately excluding light-speed group velocities $u = \pm 1$, which according to (2.10) would give infinitely fast spatial oscillations in the wavefunctions ψ.) In all the numerical illustrations to follow, $N = 200$, so the superpositions

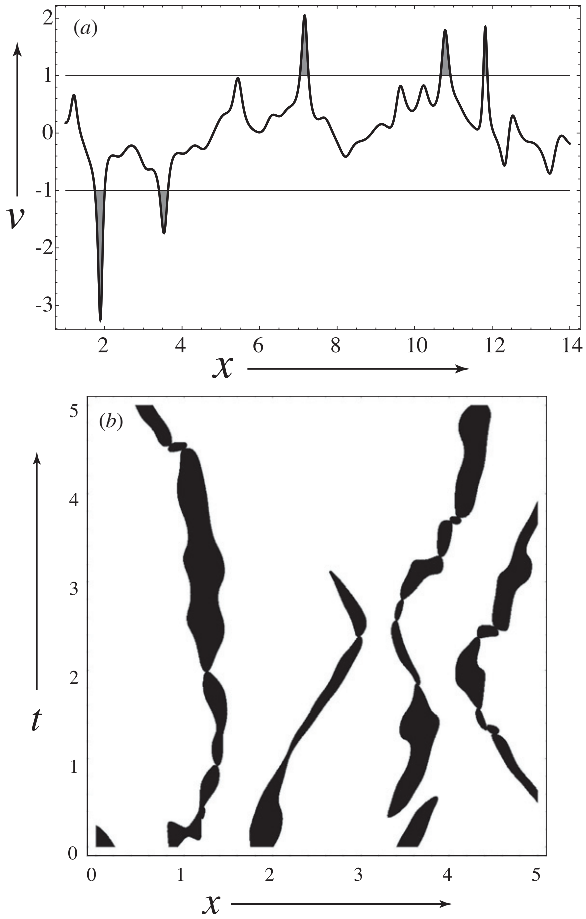
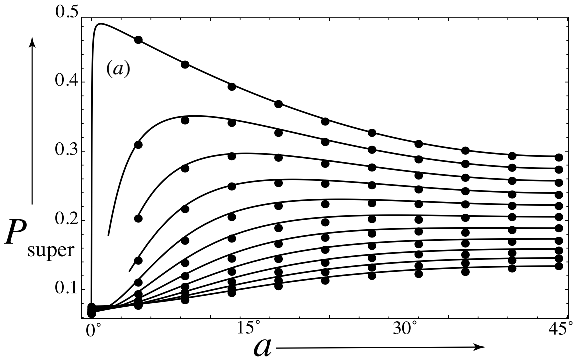


Figure 5. Sample local group velocity (3.5) for Dirac wave with $a = \pi/4$, $b = 0.5$ in the spectrum (6.1) and a choice of random phases and post-selected polarization parameters $\theta = \pi/4$, $\phi = 0$ in (3.4), with the superluminal intervals $|v| > 1$ shaded; (*a*) $t = 0$, (*b*) in spacetime.

have 100 right-moving plane waves, with strengths $\cos^2 a$, and 100 left-moving plane waves, with strengths $\sin^2 a$.

For Klein–Gordon waves, figure 4(*a*) shows a typical position-dependence of $v(x, t)$ for fixed t. 'Typical' means that the 200 phases μ_k were chosen randomly on $0 \leqslant \mu_k \leqslant 2\pi$. The local group velocity is mostly confined to the subluminal interval $-1 < v < +1$ but occasionally strays out into the superluminal regions (shaded) in which we are interested. Similar pictures appeared in calculations of the local wavenumber $k(x, t)$, illustrating 'superweak' values for which $|k(x, t)|$ exceeds the wavenumber k_0 of contributing plane waves in one dimension [17], and for 'backflow' [22], in which $k(x, t)$ can be negative although all contributing plane waves have positive wavenumbers. Over time, these superluminal intervals persist and then disappear and reappear elsewhere, as illustrated in spacetime in figure 4(*b*).

The same phenomena are illustrated for Dirac waves in figures 5(*a*) and (*b*). In this case it is necessary to specify not only the random phases but also the post-selection angles on the Poincaré sphere; the choice $\theta = \pi/2$ gives a democratic distribution of up- and down-spins



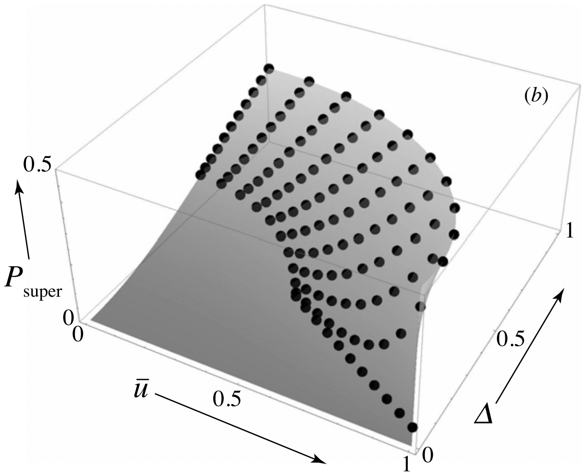


Figure 6. Dots: superluminal probability for Klein–Gordon waves with the spectrum (6.1), averaged over 10 000 sets of random phases, for $b = [0\ (0.1)\ 1]$, compared with the theoretical value (4.7) (*a*) as curves, (*b*) as a surface.

and maximizes the superluminal excursions. (Choices close to $\theta = 0$ or $\theta = \pi$ correspond to post-selection close to eigenvectors of the velocity operator (3.3), so $v(x, t)$ remains close to the light speed ± 1.)

From figures 4(*a*) and 5(*a*) it appears that the spatial dependence of the local group velocity is similar for Klein–Gordon and Dirac waves. But figures 4(*b*) and 5(*b*) indicate that the spacetime behaviour is different in the two cases. For Klein–Gordon waves, the superluminal regions are concentrated and with more erratic boundaries, while for Dirac waves these regions are more sinuous: the superluminality persists longer. But the nature of the superluminal regions depends on the spectrum chosen, so the comparison between Klein–Gordon and Dirac might look different with a spectrum other than (6.1). Our emphasis here is on superluminal statistics, but exploring the morphology of superluminal regions for different spectra would be an interesting and worthwhile project.

For the statistics, v was computed for Klein–Gordon and Dirac waves at $x = t = 0$, for a range of parameters a and b in the distribution (6.1). For each a and b, 10 000 sets of random phases were chosen, and, for Dirac waves, 10 000 sets of the post-selection angles θ and ϕ were chosen, uniformly distributed on the Bloch sphere. Those choices for which $|v| > 1$ were selected, and the superluminal fraction calculated, representing the superluminal probability

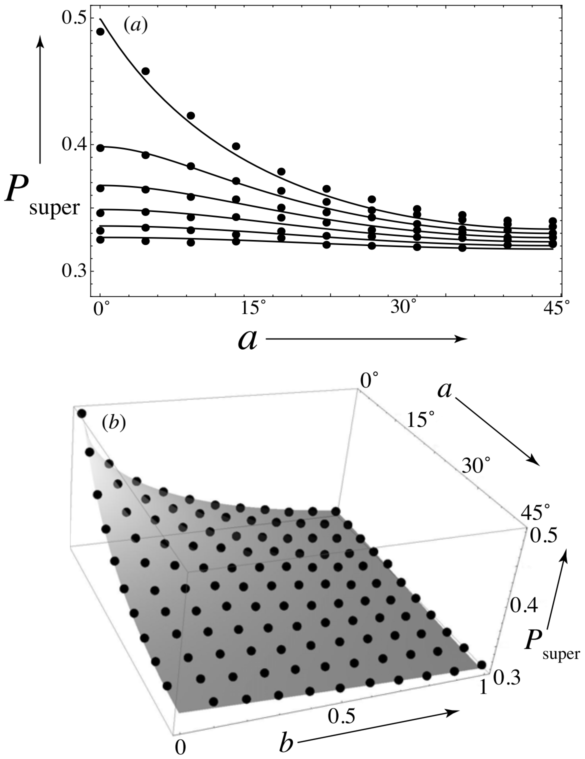


Figure 7. Dots: superluminal probability for Dirac waves with the spectrum (6.1), averaged over 10 000 sets of random phases and 10 000 random post-selection parameters (3.4), compared with the theoretical formula (5.16) (*a*) as curves, with $b = [0\ (0.2)\ 1]$, (*b*) as a surface, with $b = [0\ (0.1)\ 1]$.

over the ensemble. Comparison with theory requires the following averages (cf (4.3) and (5.6)):

$$\bar{u} = \left(1 - \frac{1}{2}b\right)\cos 2a, \quad \Delta = \sqrt{(1-b)\sin^2 2a + \frac{1}{3}b^2\left(1 - \frac{3}{4}\cos^2 2a\right)}$$
$$\langle\langle\sqrt{1-u^2}\rangle\rangle = \frac{1}{2b}\left(\cos^{-1}(1-b) - (1-b)\sqrt{b(2-b)}\right). \tag{6.3}$$

Figures 6 and 7 show the results of the calculations. It is clear that the theory captures the subtle dependence of the superluminal probability on the spectrum parameters a and b. For Klein–Gordon waves, theory and numerics agree to visual accuracy. For Dirac waves there are some small deviations, but these are probably sampling errors, because further numerical experiments (not shown) indicate that these are approximately symmetrically distributed about the theoretical curves.

7. Concluding remarks

The above calculations indicate that local group velocities for quantum waves representing relativistic massive particles have a significant chance of exceeding c: superluminal probabilities for Klein–Gordon and Dirac waves can be as large as $1/2$. Moreover, the interpretation as the weak value of a quantum observable suggests that these superluminal speeds, while not violating relativistic causality, could be experimentally detectable.

The technical calculations are different in the two cases. For Klein–Gordon waves, the result (4.7) is an application of the previously obtained superweak probability [18] for many-state observables, in which the ensemble is over all pre- and post-selected states. For Dirac waves, the calculation is an amalgam of the superweak probability for two-state systems [19] and many-state systems [18], but the result (5.16) differs from both because of the lack of symmetry between the pre-selected state (solution of the Dirac equation with Gaussian random waves) and the post-selected state (uniformly distributed on the Bloch sphere). However, both calculations share a technical feature with other weak value probability calculations [16–19]: they involve averaging over quotients of Gaussian random variables.

There are several natural directions for further study. In the spacetime pictures (figures 4(*b*) and 5(*b*)), the superluminal regions show complicated morphologies, associated with the way in which spatial superluminal intervals develop in time. The dependence of these morphologies on the spectrum of group velocities of the component plane waves is not understood in any systematic way; nor are the differences between the Klein–Gordon and Dirac waves. And here only one space dimension has been considered. In two or more dimensions additional richness is likely to arise, because the superluminal regions will have non-trivial morphology even for fixed time; moreover, the spectrum of component group velocities $\boldsymbol{u}$ can depend on direction (anisotropic Gaussian random waves) as well as on the magnitude $|\boldsymbol{u}|$.

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