Surface waves with high angular momentum: leakage from remote caustics, and tightly coiled streamlines

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Abstract
Outgoing cylindrical waves scattered by a disk, or emerging from a source inside it, are represented by Hankel functions of order \( m \). For large \( m \), these waves decay rapidly outside the disk and resemble radially evanescent surface waves travelling around it. But they eventually leak weakly away, in a manner described accurately by the asymptotics of the Hankel function. The transition occurs at radial distance \( |m| \) (in wavelength units), which constitutes a circular caustic from which the radiation leaking out, described by the streamlines, appears to issue tangentially. In the evanescent region, the streamlines form spirals, whose windings get exponentially closer nearer the disk. These insights are intended to help graduate students demystify mathematics associated with scattering theory.

Keywords: Hankel, outgoing, rays, asymptotic

(Some figures may appear in colour only in the online journal)

1. Introduction

Formalisms in theoretical physics are sometimes presented in ways that fail to bring out ‘the physics of the mathematics’. This is a pity, because underlying the equations there are usually simple pictures that provide physical insight. My purpose here is to illustrate this theme by drawing attention to two physical features associated with a wave commonly encountered in scattering theory, in the hope that this will help graduate students in their first encounter with the subject.

Classical or quantum waves emerging from a disk of radius \( r_0 \), generated either by scattering of an incident wave or from a source inside the disk, can be represented by superpositions of outgoing waves with different angular momenta \( m \). These are solutions \( \Psi \) of
the Helmholtz equation, conveniently written with distances expressed in units of \(1/k = \text{wavelength}/2\pi\):

\[
\nabla^2 \Psi(r) + \Psi(r) = 0 \quad (r = \{r, \theta\}).
\]

(1.1)

The physical features concern the outgoing partial waves with high angular momentum, considered in isolation, namely the solutions

\[
\psi_m(r, \theta) = H^{(1)}_m(r) \exp(i m \theta) \quad (r > r_0, \ m \gg r_0),
\]

in which \(H^{(1)}_m(r)\) is the Hankel function of the first kind \([1]\). For \(m > 0\), this represents an anticlockwise circulation. (The corresponding incoming waves are represented by the Hankel functions of the second kind: \(H^{(2)}_m(r) = [H^{(1)}_m(r)]^*\), multiplying the same azimuthal phase factor.)

For the large \(m\) of interest here, the physics is associated with the behaviour of the Hankel functions of high order. According to the standard Debye asymptotics \([1]\), this is different for \(r < m\) and \(r > m\):

\[
H^{(1)}_m(r) \approx \begin{cases} 
\frac{2i \exp(F)}{\sqrt{2 \pi} (m^2 - r^2)^{1/4}} & (m \gg 1, \ r < m), \\
\frac{\sqrt{2}}{\pi i} (r^2 - m^2)^{1/4} & (m \gg 1, \ r > m).
\end{cases}
\]

(1.3a)

\[
H^{(1)}_m(r) \approx \begin{cases} 
\frac{\sqrt{2}}{\pi i} \exp(i G) (r^2 - m^2)^{1/4} & (m \gg 1, \ r > m).
\end{cases}
\]

(1.3b)

where the positive real functions \(F\) and \(G\) are

\[
F = F(m, r) = m \cosh^{-1}\left(\frac{m}{r}\right) - \sqrt{m^2 - r^2},
\]

\[
G = G(m, r) = \sqrt{r^2 - m^2} - m \cos^{-1}\left(\frac{m}{r}\right).
\]

(1.4)

As \(r\) increases for fixed \(m\), \(F\) decreases and \(G\) increases. Therefore, as \(r\) increases (figure 1), \(H^{(1)}_m(r)\) decreases rapidly without oscillating, according to (1.3a), representing an evanescent wave, until \(r \sim m\), beyond which \(\psi_m\) oscillates while decreasing slowly, according to (1.3b).

The two regimes can be stitched together and encompassed by a ‘uniform approximation’ involving Airy functions \([1]\), of which we will need only the value at \(r = m\):

\[
\text{Figure 1. Decay and oscillation of Hankel function in (1.2) for } m = 20.
\]
The first of the physical implications of (1.3a), (1.3b) concerns surface waves on curved boundaries; it is considered in section 2. Like many examples of the physics of the mathematics, much of this is ‘well known to those who know well’, for example in the theory of evanescent waves outside curved fibres and waveguides [2–4] (see especially [5, 6]). The second implication, less well known, concerns the associated streamlines, along which the wave’s time-averaged energy flows; this is considered in section 3.

2. Surface waves

Just outside the disk, we can express the wave (1.2) in local Cartesian coordinates $x, y$, defined (figure 2) as

$$x \equiv r \sin \theta, \quad y \equiv r \cos \theta - r_0.$$  \hfill (2.1)

Expanding $F$ in (1.3a) now gives

$$\psi_m(r, \theta) \approx u(x, y) = C \exp(iKx - y\sqrt{K^2 - 1}),$$

$$K = \frac{m}{r_0}, \quad C = \frac{\exp(F(m, r_0))}{\sqrt{2\pi (m^2 - r_0^2)^{1/3}}}.$$  \hfill (2.2)

This shows that locally $\psi_m$ behaves like an evanescent wave: decaying into the region $r > r_0$ (increasing $y$), while travelling parallel to the surface towards increasing $x$.

However, the asymptotics (1.3b) shows that this decay of $u(x, y)$ cannot continue indefinitely as $y$ increases. When $r = m$, i.e. $y = m - r_0$, evanescence changes to oscillation: the wave radiates away, the apparent source being not the boundary $r = r_0$ but the more distant circle $r = m$, which we will interpret in section 3 as a ‘remote caustic’. This unavoidable leakage is a feature of any surface wave clinging to a convex curved boundary; for
example, it was noted for surface waves associated with a metal for $r < r_0$ [7], and in the theory of bent waveguides or fibres [3–6] (with an interesting analogy to synchrotron radiation [8]).

The leakage is a very weak effect; the intensity of the wave where leakage starts, compared with its intensity at the disk boundary, is (using (1.5)),

$$\rho(m, r_0) = \frac{|H^{(1)}_m(m)|^2}{|H^{(1)}_m(r_0)|^2} \approx \frac{2^{5/2} \pi(m^2 - r_0^2)^{3/4}}{3^{2/3} \Gamma(\frac{1}{3})^2 m^{2/3}} \exp(-2F(m, r_0)).$$

(2.3)

For example, when $r = 15$, $m = 30$, $\rho = 5.9 \times 10^{-12}$.

What about the wave $\Psi$ inside the disk, i.e. $r < r_0$? This depends on the material there, which determines the boundary condition at $r_0$. For light outside metal, this was calculated in [7]. For light or microwaves outside a dielectric with refractive index $n$, and TE polarisation (where $\Psi$ represents the electric field perpendicular to the disk), or quantum waves with a constant potential inside the disk, the boundary conditions are continuity of $\Psi$ and $\partial_r \Psi$ at $r_0$. These imply the existence of an incoming-wave contribution inside the disk, reflected from its boundary, so we write

$$\Psi(r, \theta) = \exp(i m \theta) \times \begin{cases} H^{(1)}_m(nr) + R H^{(2)}_m(nr) & (r \leq r_0) \\ T H^{(1)}_m(r) & (r \geq r_0), \end{cases}$$

(2.4)

in which the transmission and reflection intensities satisfy $|T|^2 + |R|^2 = 1$ if $n$ is real. Application of the boundary conditions gives the standard formulas

$$T = -\frac{4}{\pi r_0(n H^{(1)}_m(r_0) H^{(2)}_m(nr_0) - H^{(1)\dagger}_m(r_0) H^{(2)\dagger}_m(nr_0))}$$

$$R = \frac{H^{(1)\dagger}_m(r_0) H^{(1)}_m(nr_0) - n H^{(1)}_m(r_0) H^{(1)\dagger}_m(nr_0)}{(n H^{(1)}_m(r_0) H^{(2)}_m(nr_0) - H^{(1)\dagger}_m(r_0) H^{(2)\dagger}_m(nr_0))}.$$  

(2.5)

We are interested in the case $m \gg r_0$ and $m \gg nr_0$, for which it follows from the asymptotics (1.3) that the transmission is exponentially weak and internal reflection close to total: the wave is almost completely confined inside the disk. This is illustrated in figure 3(a).

As an example, for $r_0 = 15$, $m = 30$, and $n = 2$, $|T|^2 = 2.8 \times 10^{-12}$. The approximation (1.3a) gives the decay of the transmission intensity as

$$|T|^2 \sim \exp\left(-2(F(m, r_0) + F(m, nr_0))\right) \quad (m > nr_0).$$

(2.6)

The accuracy of this rough estimate is illustrated in figure 3(b).

This means that for a dielectric cylinder the leakage of the evanescent wave for $r > m$ is doubly weak. The exponential attenuation (2.3) must be multiplied by the exponentially small intensity $|T|^2$ of the wave transmitted outside the disk.

### 3. Streamlines

The oscillatory leakage of an evanescent wave as $r$ passes $m$ is dramatically illustrated by the wavefronts and streamlines of $\psi_m$. Wavefronts are the contours of constant phase $\arg \psi_m \mod 2\pi$. Streamlines are the trajectories (integral curves) determined by the gradient of phase, namely the current vector field (with several interpretations [9], for example lines of current, phase gradient, Poynting vector...).
From (1.2), the components of this vector are
\[ J_{m,r} = \text{Im} \psi_m^* \nabla \psi_m = J_{m,\theta} = \frac{m}{r} |H_m^{(1)}(r)|^2. \] (3.2)

It follows from (1.1) that \( \nabla \cdot J_m = 0 \). Therefore the streamlines are the contours of a scalar stream function \( S_m(r, \theta) \), determined by
\[ J_m = \nabla \times S_m(r, \theta) e_r = \frac{\partial S_m}{\partial \theta} e_r - e_r \partial_r S_m. \] (3.3)

This is easily solved:
\[ S_m(r, \theta) = \frac{2\theta}{\pi} + m \int_r^\infty \frac{dr'}{r'} |H_m^{(1)}(r')|^2. \] (3.4)

Therefore the streamlines, labelled by the direction at \( r = \infty \), are given explicitly by
\[ \theta(r) = \theta_\infty - \frac{\pi m}{2} \int_r^\infty \frac{dr'}{r'} |H_m^{(1)}(r')|^2. \] (3.5)

Figure 4 shows the streamlines, superimposed on the wavefronts, for \( m = 5 \). In the oscillatory region \( r > m \), the streamlines are asymptotically radial, reflecting the fact that \( \psi_m \) in (1.2) is an outgoing wave. In the evanescent region \( r < m \), the streamlines form tight spirals whose coils are almost circular. This requires a comment. In [10] it was wrongly stated that the existence of a stream function implies that the streamlines form closed curves. But this is true only if \( S_m \) is a singlevalued function of position, and the occurrence of \( \theta \) in (3.4) indicates that the stream function for \( \psi_m \) is multivalued: for each positive (anticlockwise) circuit of the disk, \( S_m(r, \theta) \) increases by \( 2\pi \).
The spirals get exponentially tighter as \( r \) decreases from \( r = m \) to \( r = r_0 \), and eventually are impossible to resolve, as figure 4 illustrates. This can be described by the radial separation \( \Delta r \) during each circuit. From (3.5),

\[
\Delta r \approx \frac{4r}{m |H_m^{(1)}(r)|^2} \approx \frac{2\pi r \sqrt{m^2 - r^2}}{m} \exp(-2F(r, m)).
\]  

(3.6)

For \( m = 30, r = 15, \Delta r = 1.4 \times 10^{-10} \). This behaviour is different from \( \Delta r \sim r^3 \), that was calculated in [10] for generic streamlines near a phase singularity (wave vortex) of a smooth wave (i.e. not associated with a source), and for which no stream function exists (i.e. \( \nabla \cdot J \neq 0 \)).

The wavefronts, orthogonal to the streamlines, display the opposite behaviour: radial for \( r < m \) and spiralling approximately azimuthally for \( r > m \).

Within geometrical-optics or, for quantum waves, within classical mechanics, \( \psi_m(r) \) can be regarded as representing a family of rays or trajectories, directed along the streamlines. These can be understood in terms of the local linear momentum of the wave at \( r \). In our units in which the wavenumber \( k = 1 \), the azimuthal component is \( m/r \); this follows from the azimuthal phase being forced to be \( m\theta \). Therefore the radial component is \( \sqrt{1 - (m/r)^2} \). Rays exist only in the oscillatory region \( r > m \) where the square root is real. In this uniform medium, the rays form a family of straight half-lines, illustrated in figure 5(a), issuing tangentially from a circular caustic with radius \( m \) (see also figure 4 of [11]). It is from this ‘remote caustic’, outside the disk, that the waves from the source inside the disk appear to emerge. Outside the disk and inside the caustic, i.e. for \( r_0 < r < m \), there are no geometrical

\[ \text{Figure 4. Streamlines (3.5) (full curves with arrows) and wavefronts } \text{arg} \psi_m \text{ (the full curves are phases 0 and } \pi, \text{ and the dashed curves are phases } \pi/2 \text{ and } 3\pi/2 \text{ (mod2}) \pi)) \text{, for } m = +5, r_0 = 3 \text{ (for negative } m \text{, the sense of spiralling is reversed).} \]
rays. But when the disk contains a dielectric there are rays inside, totally internally reflected from the surface, and representing ‘whispering-gallery’ modes [12].

Now consider a modification of the wave $\psi_m(r)$ in (1.2), in which the Hankel function in (1.2) is replaced by

$$H_m^{(1)}(r) \Rightarrow \frac{1}{2} (H_m^{(1)}(r) + H_m^{(2)}(r)) = \text{Re} H_m^{(1)}(r) = J_m(r),$$

where $J_m$ is the Bessel function of the first kind. This modified wave includes incoming as well as outgoing waves, and so requires no source. The rays, illustrated in figure 5(b), are doubly infinite straight lines, still touching the caustic at $r = m$ and still with positive (anticlockwise) angular momentum. This caustic is of the familiar type, with the wave rising to an intense maximum close to $r = m$, in contrast to the remote caustic that is our emphasis here, onto which the wave decays.

4. Concluding remarks

My aim has been to expose two aspects of the physics corresponding to the asymptotic formulas (1.3) of the Hankel functions for large $m$. The first is that although the associated waves $\psi_m(r, \theta)$ masquerade as surface waves travelling along the curved boundary of the cylinder, they eventually leak away. The leakage is very weak. The second concerns the streamlines in the evanescent region: they form exponentially tightly coiled spirals, closely approximating the circles associated with the azimuthal phase factor $\exp(im\theta)$.

These phenomena are in principle observable in waves of any kind—not only in light but also in microwaves, in sound, or on water. Since the remote leakage of waves clinging to curved surfaces is already known in waveguide and optical fibre theory, it has probably been detected already, although I have not found precise comparison of measured leakage with the formulas (2.3) and (2.6). Detecting the tightly coiled streamlines in the evanescent region is likely to prove much harder. One possible way is through the nonconservative ‘optical
scattering force’, or ‘curl force’ [13–16] on a small absorbing dielectric particle, though care would be needed to distinguish this force, directed along the phase gradient, from the more familiar trapping force, directed along the intensity gradient [17, 18]; a further difficulty would be discriminating the tiny differences of direction distinguishing the spirals from circles.

Hankel asymptotics provides a simple example of a concept that may be of wider interest. ‘The physics of the mathematics’ provides a counterpoint to the more familiar emphasis on the mathematics of the physics, i.e. mathematics as a language for describing the physical world. There are very many examples. Some, from my own work over the years, include:

- physics of matrix degeneracies, expressed in the ‘conoscopic figures’ seen in a sandwich in which the ‘bread’ consists of two crossed polarising sheets and the filling is a plastic overhead-projector transparency [19, 20];
- physics of the Gauss sums of number theory, expressed in the fractional Talbot effect beyond a diffraction grating [21];
- physics of the Laplace operator, expressed in the images cast by oriental magic mirrors and windows [22, 23];
- physics of singularities of smooth gradient maps, expressed as caustics in direction (rainbows [24, 25]), and in spacetime (tsunamis [26]);
- physics of elliptic integrals, expressed in the pattern of polarisation of the blue sky [27].

This reverse emphasis (‘the arcane in the mundane’ [28]) could help to demystify abstract mathematics for students of physics, and might merit attention by philosophers.

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